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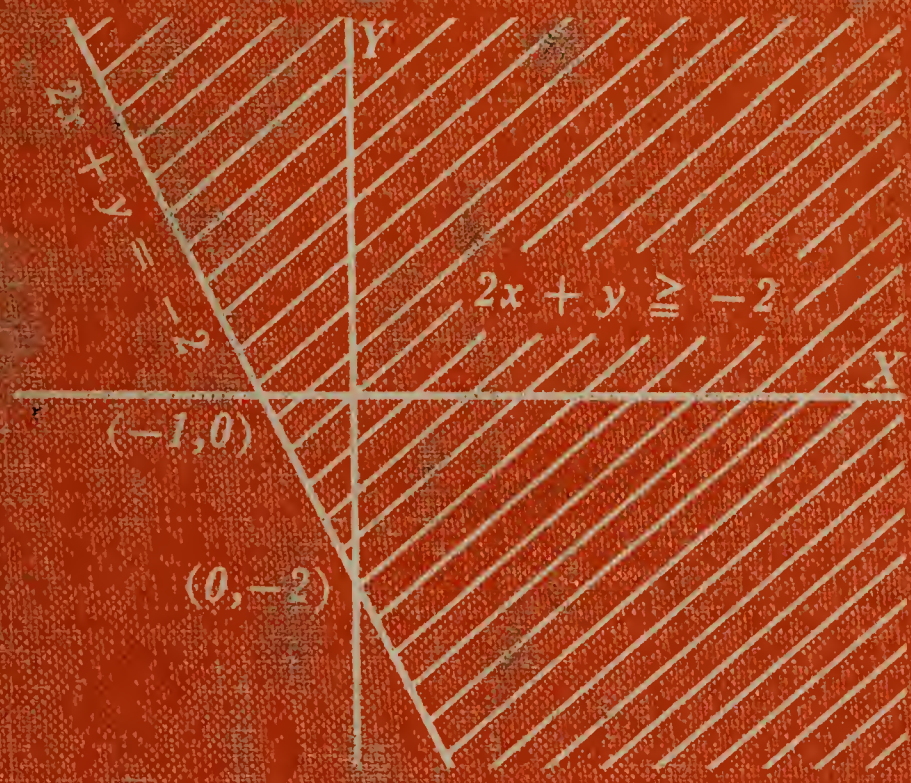
...y School

...ATHEMATICS

Grade Ten



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Secondary School

MATHEMATICS

Grade Ten

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SECONDARY SCHOOL **Mathematics** GRADE TEN

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PREFACE

Secondary School Mathematics Grade Ten is the fourth text published in the Copp Clark Modern Mathematics series from Grade 7 through Grade 12. The preliminary editions of this book have been in trial use for the past two years, and this edition reflects the suggestions and experience gained from this experimental use. It also reflects the recent developments in curriculum planning.

The text has been written to provide a two-stage introduction to the Modern Mathematics program. Classes that have not been taught new mathematics prior to Grade 10 should be able to master quickly the bridging material which provides the background necessary for the new approach. These students should then be able to complete many of the new topics covered in the text.

Classes that have been introduced to the new program at the Grade 9 level, or earlier, should not find it necessary to spend as much time on the review sections and should be able to complete the entire program.

The emphasis in this text is on the understanding of general principles and processes. To give insight, meaning, discovery, and understanding their proper significance, many discovery sections and exercises are included. Well-organized practice examples also provide an opportunity for the student to work independently and then compare his work with models provided at the back of the book. Although more attention has been given to mathematical ideas and thinking than in the past, the authors have kept in mind that students must develop an adequate mastery of the skills of algebra.

Throughout the study of geometry, the emphasis is on the nature of reasoning and on the development of the basic rules of inference employed in logical thinking. Analysis of deductions, both symbolic and geometric, has a key place in this development. A comprehensive section of solutions, which contains the analyses and proofs of many deductions, makes it possible for the student to compare and assess his original discoveries, to develop his power of perception and his ability to think independently. The introduction of the basic rules of inference provides the background necessary for courses in logic in subsequent grades and for understanding the logical organization of mathematics.

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W.B.M.
D.L.M.
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G.A.K.

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MATHEMATICAL SYMBOLS

$\because$	since; because
$\therefore$	therefore
. . .	and so on
.	decimal point
$.\dot{3}$ *or* $.\overline{3}$	repeating decimal
$+$	plus; add; positive
$-$	minus; subtract; negative
$\pm$	plus or minus; positive or negative
$\times$ *or* $\cdot$	multiplied by
$\div$	divided by
$=$	equal to
$\neq$	is not equal to
$\doteq$ *or* $\simeq$	is approximately equal to
$\cong$	is congruent to; is identical to
$\sim$	is similar to (as in $\triangle ABC \sim \triangle DEF$)
$>$	is greater than
$\ngtr$	is not greater than
$<$	is less than
$\nless$	is not less than
$\geqq$	is greater than or equal to
$\ngeqq$	is not greater than or equal to
$\leqq$	is less than or equal to
$\nleqq$	is not less than or equal to
$\cup$	cup; the union of
$\cap$	cap; the intersection of
$\leftrightarrow$	corresponds to; is equivalent to; if and only if
$\rightarrow$	implies; only if
$\sim A$	not A; the negation of A
$A \times B$	the Cartesian product of A and B
$a : b$	the ratio of a to b
$\lvert\ \rvert$	the absolute value of
$\sqrt{\ }$	the square root of
$\sqrt[n]{\ }$	the nth root of

$\|$ is parallel to

$\nparallel$ is not parallel to

$\perp$ is perpendicular to

$\angle$ angle

$\triangle$ triangle

$\|^{gm}$ parallelogram

$^\circ$ degree

$'$ minute; foot

$''$ second; inch

π $\frac{22}{7}$ (3 sig. digits); 3.1416 (5 sig. digits)

$\%$ per cent

( ) round brackets

[ ] square brackets

$\{\ \}$ brace brackets; the set

$\overline{a+b}$ bar bracket

$\{\ |\ \}$ the set builder

$\in$ belongs to; is a member of; is an element of

$\notin$ is not a member of

$\emptyset$ null set

$\subset$ is a proper subset of

$\subseteq$ is a subset of

N the set of natural numbers

N_0 the set of natural numbers and zero

I the set of integers

^+I the set of positive integers

^-I the set of negative integers

Q the set of rational numbers

R the set of real numbers

^+R the set of positive real numbers

^-R the set of negative real numbers

(ClA) Closure Law for Addition

(ClM) Closure Law for Multiplication

(CA) Commutative Law for Addition

(CM) Commutative Law for Multiplication

(AA) Associative Law for Addition

(AM) Associative Law for Multiplication

(D) Distributive Law

Review A

SETS

***A·1* The concept of a set.** From earliest times people have thought in terms of sets or collections of objects: for example,

a collection of books,
a collection of coins,
the members of a family,
the set of fingers on one's hand.

A set is a well-defined collection of objects.

Each object in a set is called a *member* or *element* of the set. The words *well-defined* mean that it is clearly established whether or not an object is included in a particular set. In mathematics the members of sets are generally mathematical objects, such as counting numbers, fractions, points, lines, or angles.

Membership in a set may be indicated in several ways:

(a) by stating a *rule of membership* such as

(i) the set of all odd whole numbers;

(ii) the set of all boys in a school;

(b) by using a *capital letter* which suggests the defining rule of the set: for example, E might be used to represent the set of all even whole numbers;

(c) by *listing* the members of the set in *brace brackets*: for example, the set of all prime numbers between 1 and 20 may be written

$$P = \{2, 3, 5, 7, 11, 13, 17, 19\}$$

which presents a complete list of all the members of the set. The statement is read, "P is the set 2, 3, 5, 7, 11, 13, 17, 19"; the brace brackets symbol $\{\ \ \}$ is read, "the set of".

In the preceding example, P represents the set whose members are 2, 3, 5, 7, 11, 13, 17, 19. The fact that 3 is a member of this set is expressed

$$3 \in P.$$

The symbol $\in$ (the Greek letter epsilon) is read, "is a member of" or "is an element of" or "belongs to".

Write a solution for each of the following problems; compare your solutions with those on page 463.

1. If $A = \{2, 4, 6, 8, 10\}$

(i) describe the set in words;

(ii) express in symbols the fact that 6 is a member of the set.

2. State a defining rule for membership in each of the following sets:

(i) $\{3, 5, 7, 9\}$ (ii) $\left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}\right\}$

3. The concept of a set implies that the members are different from each other. How many members are there in each of the following sets? Rewrite each set listing each element only once.

(i) $\{1, 2, 4, 9\}$ (ii) $\{a, b, c, d, a\}$ (iii) $\{1, 1, 1, 2, 2, 2, 2\}$

(iv) $\{0\}$ (v) $\{0, 2, 3, 4, 3, 2, 0\}$ (vi) $\{a, m, a\}$

***A*·2 Finite sets, infinite sets, null set.** A set may consist of any number of elements. If there is a counting number which represents the number of elements in a set, then the set is a *finite* set ("finite" from the Latin *finitus* meaning "finished" or "ended"). Thus, *a finite set is a set with a limited number of elements.*

The following are examples of finite sets:

(i) $\{x_1, x_2, x_3\}$ is a finite set with three members;

(ii) $\{2, 4, 6, 8, \ldots, 100\}$ is a finite set with fifty members;

(iii) $\{a_1, a_2, a_3, \ldots, a_n\}$, where n represents some counting number, is a finite set with n members.

The three dots ... are used to indicate the continuing nature of the set and in example (ii) imply the even numbers between 8 and 100.

The set of all even numbers between 10 and 12 contains no members. The set of all prime factors of 27 which are greater than 6 also has no members. Sets with no members are said to be *empty* sets, and each is referred to as the *null* set. The null set is represented by the Greek letter phi, $\emptyset$. Thus,

(i) the set of all even numbers between 10 and 12 is $\emptyset$;

(ii) the set of all prime factors of 27 greater than 6 is $\emptyset$.

Sets which are neither finite nor null are called *infinite* sets (not finished, unlimited) and are usually designated by a rule of membership or verbal description. The membership of an infinite set may be implied by listing some members in brace brackets and using three dots to indicate that the membership suggested by the listing is unending. The following are examples of infinite sets:

(i) the set of all counting numbers is $\{1, 2, 3, 4, \ldots\}$;

(ii) the set of all odd numbers is $\{1, 3, 5, 7, \ldots\}$;

(iii) the set of all even numbers is $\{2, 4, 6, 8, \ldots\}$.

***A*•*3* Identical sets and equivalent sets.** Two or more sets are said to be identical if they have exactly the same members. Thus, the sets

$$A = \{x, y, z\} \text{ and } B = \{x, z, y\}$$

are *identical*. The members are not written in the same order, but this is not important since membership is not concerned with the order of listing. To express the idea that set A is identical to set B, we write

$$\{x, y, z\} = \{x, z, y\}$$

or

$$A = B.$$

Each of the sets

$$A = \{a, b, c\} \text{ and } B = \{m, n, p\}$$

has the *same number* of elements, but the elements are different. Such sets are said to be *equivalent*. If the symbol $n\{\ \}$ is defined to mean the number of elements in the set, then we may write

$$n\{a, b, c\} = n\{m, n, p\}$$

or

$$n(A) = n(B)$$

where A and B represent the two sets respectively.

Exercise A-1

(A)

State the members of each of the following sets:

1. The set of all even numbers between 3 and 9.

2. The set of all natural number divisors of 12.

3. The set of all prime numbers between 23 and 29.

4. The set whose members belong to both $\{-3, -2, -1, 0\}$ and $\{0, 1, 2, 3\}$.

State the number of elements in each of the following sets:

5. $\{1, 2, 2, 3, 4, 3, 3, 1, 2\}$ **6.** $\{7, 7, 7, 6\}$ **7.** $\{0\}$

8. $\{a, a, b, b, b, c\}$ **9.** $\{a, a^2, b, b, b, b^2\}$ **10.** $\{\ \}$

State a rule which describes the set in each of the following:

11. $\{2, 4, 6, 8, 10\}$ **12.** $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$ **13.** $\{3, 6, 9, 12, \ldots\}$

14. $\{1, 4, 9, 16, 25, 36, 49, 64, 81\}$ **15.** $\left\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}\right\}$

16. Classify the following sets as finite, infinite, or null:

(i) all odd natural numbers less than 25;

(ii) the natural numbers; (iii) all squares between 50 and 60;

(iv) the set of all factors of 120; (v) the set of all multiples of 5.

(B)

17. List (i) equivalent sets (ii) identical sets, from the following:

$A = \{4, 5, 6, 7, 8\}$ $B = \{a, b\}$ $C = \{2, 4\}$

$D = \{7\}$ $E = \{0\}$ $F = \{100, 101\}$

$G = \emptyset$ $H = \left\{\frac{3}{4}\right\}$ $I = \{6, 4, 7, 5, 8\}$

List the members of the following sets in brace brackets:

18. The set of all prime numbers less than 24.

19. The set of all multiples of 9 between 30 and 35.

20. The set of all natural number divisors of 60.

21. The set of all proper fractions whose numerators and denominators are chosen from $U = \{3, 4, 7, 11\}$.

22. A circle separates all the points on a page into three sets. Describe these point sets in words.

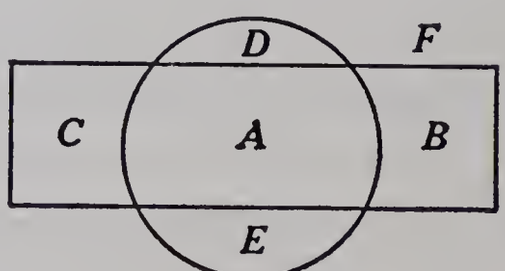
23. In the accompanying figure, the circle and the rectangle separate all points on the page into sets indicated by the letters A, B, C, D, E, and F. Give a description:

(i) for the set A;

(ii) for the set F;

(iii) for the set which includes the sets C and B only;

(iv) for the set which includes the sets D and E only.



***A·4* Subsets.** If $A = \{2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$, then A is said to be a *subset* of B, because *every member of set A is also a member of set B.* This is expressed by the symbols $A \subseteq B$ read, "A is a subset of B".

Using this definition of a subset, do the following problem and compare your solution with the one on page 463.

1. List all the subsets of the set $A = \{a, b, c, d\}$ and thereby determine the number of subsets of A.

***A·5* Venn diagrams.** Sets are sometimes represented pictorially by the type of diagram illustrated in *Figures A-1* to *A-5*. In considering two sets, A and B, all the members of set A may be represented by points within or on the circle marked A, and all the members of set B by points on or within the circle B.

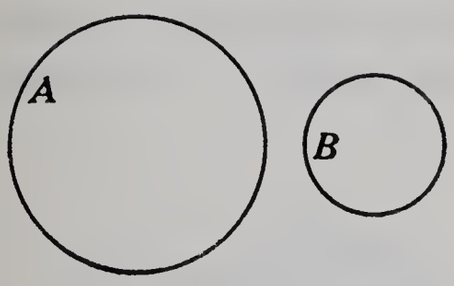


Fig. A-1

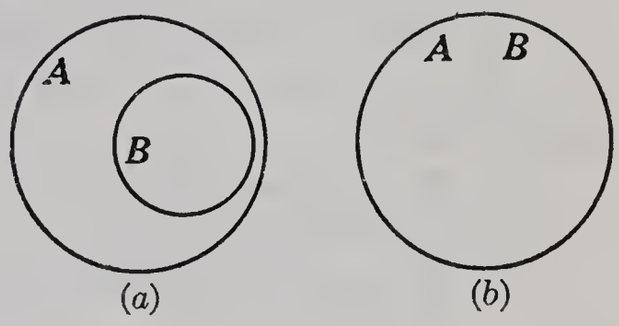


Fig. A-2

Any relationship that may exist between the members of A and B can be indicated by the positioning of the circles with respect to each other. In *Fig. A-1* the two circles are completely separate or distinct from each other. This signifies that the two sets A and B have *no members in common.* Such sets are said to be *disjoint* sets.

In *Fig. A-2* (*a*) the circle B is entirely within circle A, which signifies that B is a *proper subset* of A: that is, $B \subset A$. In *Fig. A-2* (*b*) the circles A and B coincide, which signifies that A and B are identical: that is, $A = B$.

In *Fig. A-3* the circles A and B partially overlap. This signifies that some, but not all, members of B are also members of A, and vice-versa. The shaded portion of the diagram corresponds to those members which belong both to set A and to set B. This set of elements is called the *intersection* or *intersection set* of A and B and is indicated symbolically by $A \cap B$ read, "A intersect B" or "A cap B" or the "intersection of A and B". The word "and" is also used

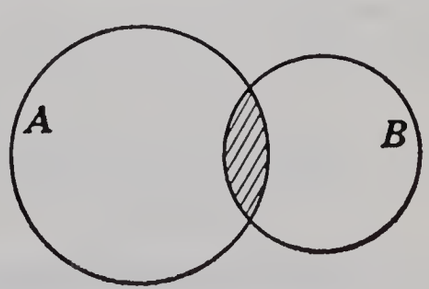


Fig. A-3

to imply the intersection of two sets. Thus, $A \cap B$ may also be written "A and B".

The intersection of two sets A and B is the set of all elements which belong both to set A and set B.

Write a solution to the following problem; compare your solution with that on page 463.

1. $A = \{a, b, c, d, e\}$, $B = \{c, d, e, f, g, h\}$, $C = \{f, g, h\}$

(i) Draw a diagram to illustrate how the sets A and B are related.

(ii) List the elements in $A \cap B$ and shade the diagram to illustrate this set.

(iii) Draw a diagram to illustrate how sets A and C are related.

(iv) What is meant by "A and B"?

(v) List the elements in $B \cap C$.

(vi) How is C related to B? Draw a diagram to illustrate your answer.

In *Fig. A-4* the circles A and B also overlap partially. The shaded portion includes:

(i) the members of set A;
(ii) the members of set B;
(iii) the members belonging both to A and B.

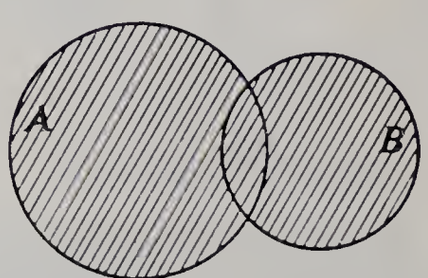


Fig. A-4

This set is called the *union* or *union set* of sets A and B and is designated symbolically by $A \cup B$ which is read, "A union B" or "A cup B" or "the union of A and B". The word "or" is also used to imply the union of two sets. Thus, $A \cup B$ may also be written "A or B".

The union of two sets is the set of all elements which belong to at least one of the two sets.

Write a solution to the following problem; compare your solution with that on page 464.

2. $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{7, 8, 9\}$

(i) Draw a diagram to illustrate how set A and set B are related.

(ii) List the elements in $A \cup B$. (Remember, elements are listed only once.)

(iii) Shade the diagram to indicate A or B.

(iv) Draw a diagram to illustrate how set B and set C are related.

(v) List the elements in B or C.

(vi) Shade the diagram to illustrate $B \cup C$.

(vii) Under what circumstances is $n(B \cup C) = n(B) + n(C)$?
(viii) Find $n(A \cup B)$.

Two sets A and B in some particular situation may be subsets of some general set. For example, the sets

$$A = \{2, 4, 6, 8\} \text{ and } B = \{1, 3, 5, 7, 10\}$$

are subsets of the general set, the set of all counting numbers

$$U = \{1, 2, 3, 4, \ldots\}.$$

The general set contains all the objects being considered in a particular situation and is referred to as the *universal set* U. It is usually represented in a diagram by a rectangle. The sets A and B are usually represented by circles within the rectangle.

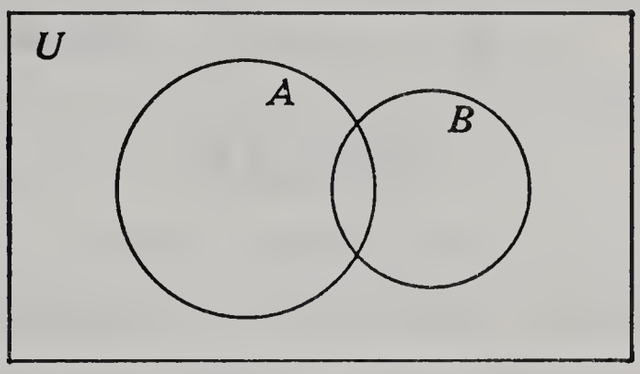


Fig. A-5

For example, in *Fig. A-5*,
U might represent all the boys in your school;
A might represent all the boys in your class;
B might represent all the boys who play hockey in your school.

Since the circles A and B overlap, the figure illustrates that some of the boys in your class play hockey. These boys are the members of the intersection set $A \cap B$.

This use of circles originated with an eighteenth-century Swiss mathematician named Leonhard Euler (1707-1783). The circles are often referred to as *Euler circles*. John Venn, an English mathematician (1834-1923), made use of the same type of figure in his writings on logic. He did not restrict himself to circles but used any closed figure. These diagrams are often referred to as *Venn diagrams*.

Write solutions to the following problems; compare your solutions with those on page 464.

3. Make a Venn diagram to illustrate how the following sets are related; list the members of (i) $A \cap B$ (ii) $A \cup B$.
$U = \{\text{all letters in the alphabet}\}$
$A = \{a, c, e, g, i\}$ $B = \{a, b, c, d, e, f\}$.

4. Given the following sets:

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ $B = \{1, 3, 5, 7, 9\}$

$C = \{2, 4, 6, 8\}$ $D = \{2, 3, 5, 7\}$

(a) Which of the following relations are true?

(i) $B \subset A$ (ii) $C \subseteq A$

(iii) $D \subseteq A$ (iv) $\emptyset \subseteq A$

(b) List the sets described below:

(i) $A \cup B$; $A \cap B$ (ii) $A \cup C$; $A \cap C$

(iii) $B \cup C$; $B \cap C$ (iv) $A \cup D$; $A \cap D$

(v) $C \cup D$; $C \cap D$ (vi) $(B \cup C) \cup D$

(c) State the number of subsets of:

(i) A (ii) B (iii) C (iv) D

Exercise A-2

(B)

Draw Venn diagrams to illustrate how the following sets are related:

1. (i) $A = \{p, q, r, s, t, v, w\}$ $B = \{m, n, o, p, t, w, x, y\}$

$U = \{\text{all letters of the alphabet}\}$

(ii) Shade the area representing A or B.

2. $A = \{\text{all students in Grade 9 classes at Middletown High School}\}$

$B = \{\text{all Grade 9 students at Middletown High School who take French}\}$

$U = \{\text{all students of Middletown High School}\}$

3. Given the following sets

$A = \{\text{all boys taking Grade 10 Latin}\}$,

$B = \{\text{all Grade 10 boys on the football team}\}$,

$U = \{\text{all Grade 10 boys}\}$,

draw a Venn diagram to illustrate each of the following possibilities:

(i) No football player takes Latin.

(ii) All Grade 10 football players also take Latin.

(iii) Some, but not all, boys taking Latin also play football.

4. Given the following sets

U is the set of all students in your high school,

G is the set of all students in your grade,

B is the set of all students who play basketball,

draw Venn diagrams to illustrate each of the following:

(i) All students who play basketball are in your grade.

(ii) No students who play basketball are in your grade.

(iii) Some students who play basketball are in your grade.

(iv) Only those students who are in your grade play basketball.

(v) All students in your grade play basketball.

Use Venn diagrams to solve the following problems:

5. In a class of 32 students, each of whom must take English or History, 20 take English, 9 take both English and History. How many take History?

6. There are 38 cars on a dealer's lot; 25 have automatic transmissions, and 20 have power steering. How many of these cars have both automatic transmission and power steering?

7. At a certain university, all freshmen must take English or Philosophy. If 180 take English and 152 take Philosophy, how many freshmen are there for each of the following situations?

(i) No freshman can take both English and Philosophy.

(ii) Both courses may be taken, and 32 freshmen take both English and Philosophy.

8. If 28 girls belong to the basketball team, 19 girls belong to the swimming team, and, of these, 6 girls belong to both the basketball team and the swimming team, how many box lunches would be required to feed both teams?

9. In one neighbourhood the set of people each owning a sports car is {Jones, Larkin, Smith, Brown, White, Ross, Miller}; the set of people each owning a station wagon is {Larkin, Woods, Reilly, Stephens, Jones}; and the set of people each owning a family car is {Jones, Clark, Anderson, Woods, Smith, Reilly, Brown, Howard, Hughes}. Draw a Venn diagram to show the names of the people owning one car, two cars, or three cars.

10. In preparing a timetable for a class of 38, the following facts are known: 25 pupils take History, 27 take French, 28 take Algebra. Of these, 20 take both History and French, 23 take both French and Algebra, and 21 take both History and Algebra. Among these there are 18 who take all three subjects. How many pupils take (i) only French (ii) only Algebra (iii) none of these three subjects?

11. In a survey of 100 people, 43 said they watched television program A, 55 watched program B, and 45 watched program C. Of these, 15 watched programs A and B, 25 watched programs B and C, and 20 watched programs A and C. This includes 8 who watched all three programs. How many of these people did not watch any of the three programs?

Review B

NATURAL NUMBERS AND ZERO INTEGERS

***B·1* The set of natural numbers.** The set of natural numbers, invented for counting purposes, is represented symbolically by

$$N = \{1, 2, 3, 4, \ldots\} .$$

The following are the properties of the natural number system.

If $a, b, c \in N$, then

	ADDITION	MULTIPLICATION
Closure Law	*$a + b$ represents one and only one natural number* (*ClA*)	*ab represents one and only one natural number* (*ClM*)
Commutative Law	$a + b = b + a$ (*CA*)	$ab = ba$ (*CM*)
Associative Law	$(a+b)+c = a+(b+c)$ (*AA*)	$(ab)c = a(bc)$ (*AM*)
Distributive Law	$a(b + c) = ab + ac$ (*D*)	
Identity Element	*Unity*	

The number 1 (or unity) has the special property with respect to multiplication that

$$\text{if } a \in N, \text{ then } a \times 1 = 1 \times a = a .$$

Thus, 1 is called the *identity* (or *neutral*) *element for multiplication.*

The *inverse* (undoing) operations of subtraction and division are defined in terms of addition and multiplication respectively.

If $a, b, x \in N$, then

$$a - b = x \quad \textit{if and only if} \quad a = b + x$$
$$\textit{and}$$
$$a \div b = x \quad \textit{if and only if} \quad a = b \times x .$$

The set N is not closed under subtraction or division, and these operations are neither commutative nor associative.

***B·2* The set of natural numbers and zero.** The natural numbers together with zero form the set of *whole numbers* (N_0).

$$N_0 = \{0, 1, 2, 3, 4, \ldots\}$$

Operations involving zero are defined as follows:

a. *Addition and subtraction.*

If $a \in N_0$, then

$$a + 0 = 0 + a = a .$$

Since the sum of zero and any whole number is that number itself, zero is called the *identity* (or *neutral*) *element for addition.*

Since subtraction is the inverse of addition,

$$a - 0 = a \quad \textit{and} \quad a - a = 0 .$$

b. *Multiplication.*

If $a, b \in N_0$, then

$$b + 0 = b . \quad \text{(Definition)}$$

If N_0 is to be closed under multiplication, then

$$a(b + 0) = a \cdot b .$$

If multiplication is to be distributive over addition, then

$$a \cdot b + a \cdot 0 = a \cdot b .$$
$$\therefore a \cdot 0 \text{ is the identity for addition.}$$
$$\therefore a \cdot 0 = 0$$

If multiplication is to be commutative, then

$$0 \cdot a = 0 .$$

Thus, $$a \cdot 0 = 0 \cdot a = 0 .$$

Also, $$\textit{if } ab = 0, \textit{ then } a = 0 \textit{ or } b = 0 .$$

c. *Division.* There are three cases of division involving zero. Consider the quotient $a \div b$, where $a, b \in N_0$.

(i) *If $a = 0$, $b \neq 0$, then*

$$0 \div b = x, \quad x \in N_0 \quad \textit{if and only if} \quad 0 = b \times x .$$

This is true if and only if $x = 0$, since $b \neq 0$.
Therefore $0 \div b = 0$ if $b \in N_0$, $b \neq 0$.

(ii) *If* $a \neq 0$, $b = 0$, *then*

$$a \div 0 = x \quad \textit{if and only if} \quad a = 0 \cdot x .$$

This is not true for any whole number represented by x.
Therefore the quotient $a \div 0$, $a \neq 0$ is meaningless.

(iii) *If* $a = 0$, $b = 0$, *then*

$$0 \div 0 = x \quad \textit{if and only if} \quad 0 = 0 \times x .$$

This is true for every whole number represented by x.
Thus, there is no unique whole number which represents $0 \div 0$.
Therefore the quotient $0 \div 0$ is indeterminate.

Division by zero is not defined.

The number 1 (or unity) has the special property with respect to multiplication that

$$\text{if } a \in N_0, \quad \text{then} \quad a \times 1 = 1 \times a = a .$$

Thus, 1 is the *identity* (or *neutral*) *element for multiplication.*

N_0 is closed under the operations of addition and multiplication. These operations are commutative and associative, and multiplication is distributive over addition. N_0 is not closed under the operations of subtraction and division. These operations are neither commutative nor associative, and division by zero is not defined.

***B·3* The *N*-line and the N_0-line.** The set of natural numbers, N, may be represented geometrically on a number line called an *N-line* (*Fig. B-1*). Two points are selected on a line and the numerals 1 and 2 associated with them, as indicated in *Fig. B-1*. Using this line segment as a unit, other points to the right of these are selected by marking off unit distances in succession. The numerals 3, 4, 5, . . . are associated with these points as shown. The numerals are called the *coordinates* (names) of the points. The number line also illustrates the idea of order or magnitude that is inherent in the set of whole numbers. This is discussed in *B·7*.

If the coordinates 0 and 1 are given to the first two points selected, the number line pictures the set of whole numbers and is called an *N_0-line.*

N-line

N_0-line

Fig. B-1

***B·4* The set of integers.** As civilization progressed, the natural numbers and the whole numbers proved to be inadequate:

(i) to represent opposites, such as profit and loss;

(ii) to provide a solution to equations, such as $x + 3 = 0$ (that is, the set of whole numbers is not closed under subtraction).

The *integers*, or signed numbers, were invented to indicate oppositeness in sense or quality. The set of all integers, designated by I, is represented as follows:

$$I = \{\ldots, -3, -2, -1, 0, +1, +2, +3, \ldots\}$$
$$\text{or } I = \{\pm a \mid a \in N\} \cup \{0\} .$$

The $+$ and $-$ signs are called *signs of quality.*

The properties of (i) magnitude and (ii) oppositeness in sense or direction with respect to zero are indicated on an *integer-line*, or *I-line*, in *Fig. B-2.* It should be noted that zero is neither positive nor negative.

The whole number line starts with a point whose coordinate is zero and extends endlessly to the right with certain points, a unit distance apart, having natural number coordinates. If the extension of the line to the left has unit distances marked off in succession from the zero point (origin), then corresponding to each marked point to the right of the origin there is a point an equal distance to the left from the origin. If the points to the right of the origin are given coordinates $+1$ (read, "positive one"), $+2$, $+3$, ..., then, to indicate their oppositeness in sense or direction, the selected points on the left of the origin may be given coordinates -1 (read, "negative one"), -2, -3, ..., as indicated. Note that all distances are measured to the right or left from the origin.

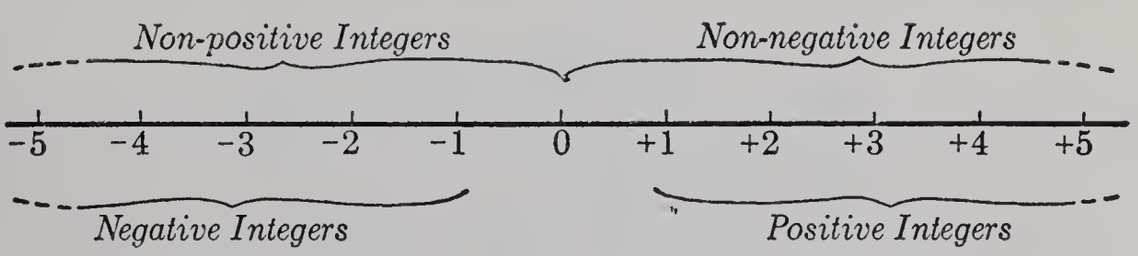


Fig. B-2

Fig. B-2 also indicates the following important subsets of the set of integers:

(i) the set of positive integers, $\{+1, +2, +3, \ldots\}$;

(ii) the set of negative integers, $\{-1, -2, -3, \ldots\}$;

(iii) the set of non-positive integers, $\{0, -1, -2, -3, \ldots\}$;

(iv) the set of non-negative integers, $\{0, +1, +2, +3, \ldots\}$.

B·5 Addition in the set of integers.

a. *Addition of positive integers.* If the positive integers are to correspond to the natural numbers, it is necessary to define

$$(+2) + (+3) = +(2+3) = +5$$

to correspond to $2 + 3 = 5$.

In general, if $a, b \in N$, then

$$(+a) + (+b) = +(a+b).$$

This definition produces a sum which is a unique integer, and therefore the positive integers are closed under addition.

Examples like $(+2) + (+3) = +(2+3) = +5$

and $(+3) + (+2) = +(3+2) = +5$

suggest that this addition is commutative.

In general, if $a, b \in N$ since $(+a) + (+b) = +(a+b)$

and $(+b) + (+a) = +(b+a)$

$= +(a+b)$ *(CA in N)*

$\therefore \quad (+a) + (+b) = (+b) + (+a)$

$= +(a+b)$.

Thus, addition of positive integers is commutative.

Examples such as $[(+1) + (+2)] + (+3) = (+3) + (+3) = +6$

and $(+1) + [(+2) + (+3)] = (+1) + (+5) = +6$

suggest that addition is associative.

Fig. B-3 illustrates the statement $(+2) + (+3) = (+5)$.

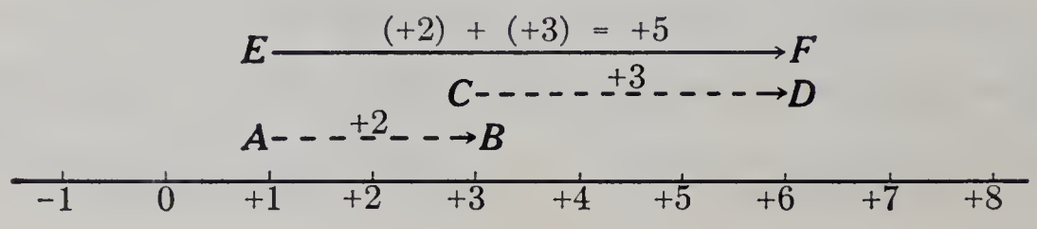


Fig. B-3

In *Fig. B-3* the *directed line segment AB* represents the integer $(+2)$. This directed line segment is called a *vector*. In general, *a vector is a directed line segment used to represent a quantity which has the two properties of (i) magnitude (ii) sense or direction.* The length of the vector, to some scale, represents the magnitude of the quantity, and the arrow on the vector represents its sense or direction. In *Fig. B-3, EF* is the *vector sum*

of the vectors AB and CD. The vector sum of two vectors (AB and CD) is the single vector (EF) which is equivalent to the two vectors. This sum is determined in the manner illustrated in *Fig. B-3.*

b. *Addition of negative integers.*

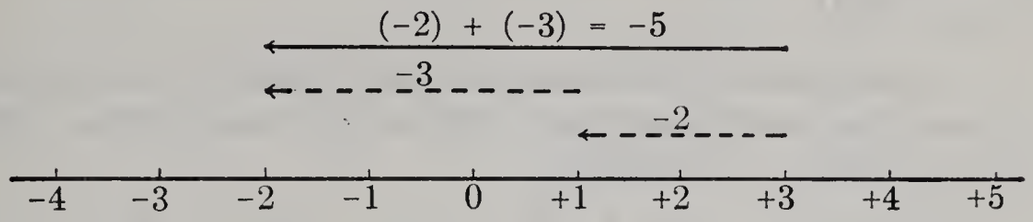


Fig. B-4

Fig. B-4 suggests the following definition for adding negative integers:

$$(-2) + (-3) = -(2 + 3) = -5$$
$$(-3) + (-4) = -(3 + 4) = -7$$

In general, if $a, b \in N$, ***then***

$$(-a) + (-b) = -(a + b).$$

Particular examples suggest that I is closed under addition, and addition is commutative and associative.

c. *Addition of positive and negative integers.*

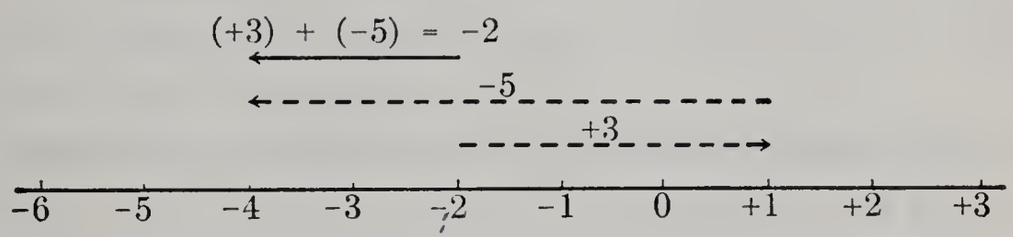


Fig. B-5

Fig. B-5 suggests the following definition for adding positive and negative integers:

$$(+3) + (-5) = -(5 - 3) = (-2)$$

Also,
$$(+5) + (-2) = +(5 - 2) = (+3)$$

In general, if $a, b \in N$, ***then***

$$(+a) + (-b) = +(a - b) \text{ if } a > b$$
$$= -(b - a) \text{ if } a < b.$$

d. *Addition involving zero.* Zero is the identity element for addition in the set of whole numbers. To extend this concept to the set of integers, we define addition involving zero as follows:

If $a \in I$, then

$$a + 0 = 0 + a = a .$$

Zero is the identity (or neutral) element for addition.

Write solutions for each of the following problems and compare them with those on page 465.

1. Using the definition of addition, determine the following sums and use the examples to illustrate that addition is commutative. Illustrate the sums with vector diagrams.

(i) $(+ 3) + (+ 4)$ (ii) $(- 2) + (- 4)$ (iii) $(- 6) + (+ 3)$

2. Show that each of the following sums is associative and illustrate on a vector diagram.

(i) $(+ 2) + (+ 3) + (+ 4)$ (ii) $(- 1) + (- 3) + (- 2)$

(iii) $(+ 2) + (- 1) + (- 4)$

These examples suggest that the set of integers is closed under addition and that addition is commutative and associative.

e. *Summary of addition of integers.*

If $a, b \in N$, then

(i) $(+ a) + (+ b) = (+ b) + (+ a) = + (a + b)$;

(ii) $(- a) + (- b) = (- b) + (- a) = - (a + b)$;

(iii) $(+ a) + (- b) = (- b) + (+ a) = + (a - b)$ if $a > b$

$= - (b - a)$ if $a < b$.

Zero is the *identity* (or *neutral*) *element* for addition of integers.

(iv) If $b = a$, then from

(iii) $(+ a) + (- a) = (- a) + (+ a) = + (a - a) = 0$.

Thus, the integers may be paired off $(+ 1)$ and $(- 1)$, $(+ 2)$ and $(- 2)$, $(+ 3)$ and $(- 3)$, and so on, in such a way that the sum of each pair of integers is zero. The members of these pairs are called *additive inverses* or *negatives* of each other. The additive inverse of zero is zero.

Exercise B-1

(A)

1. Find the following sums:

(i) $+ 3$ / $+ 7$ (ii) $- 4$ / $- 5$ (iii) $+ 8$ / $- 2$ (iv) $- 15$ / $+ 13$

(v) -23	(vi) $+17$	(vii) $+32$	(viii) -45
$\underline{-14}$	$\underline{+24}$	$\underline{-37}$	$\underline{+25}$
(ix) -12	(x) -16	(xi) $+52$	(xii) -14
$\underline{+19}$	$\underline{-8}$	$\underline{+9}$	$\underline{-6}$

2. State the negatives of the following integers:

(i) $+3$ (ii) -35 (iii) $+256$ (iv) -32

(v) $+(4+7)$ (vi) $-(3-2)$ (vii) $-b$ (viii) $-(x+y)$

(B)

3. Simplify, then illustrate each of the following with a vector diagram:

(i) $(+3)+(+7)$ (ii) $(-5)+(-4)$ (iii) $(+7)+(-6)$

4. Simplify, then illustrate each of the following with a vector diagram:

(i) $(+5)+(-7)+(+3)$ (ii) $(-6)+(-4)+(+8)$

5. Simplify:

(i) $(-3)+(+12)+(-4)$

(ii) $(+27)+(-7)+(-13)$

(iii) $(-8)+(-7)+(-9)$

(iv) $(+12)+(-11)+(+13)$

(v) $(-15)+(-14)+(+20)$

(vi) $(+17)+(-3)+(-14)$

(vii) $(+3)+(-2)+(+7)+(-8)$

(viii) $(-3)+(+5)+(-6)+(-2)$

(ix) $(+7)+(+10)+(-13)+(-4)$

(x) $(+32)+(-15)+(-12)+(+7)$

6. Show that $[(+3)+(-4)]+(+9)=(+3)+[(-4)+(+9)]$

(i) by finding the sums (ii) by drawing vector diagrams

***B·6* Subtraction of integers.** Since subtraction is the inverse operation to addition, it is defined as follows:

If $a, b, c \in I$, then

$$a-b=c \quad \textit{if and only if} \quad c+b=a.$$

That is, the difference $a-b$ exists if and only if an integer c can be found whose sum with b is a. The following examples suggest that the integer c always exists and indicate how it may be determined.

ADDITION	SUBTRACTION (INVERSE)	ADDITION PRODUCING THE SAME RESULT
Since $(+6)+(+2)=(+8)$	$\therefore (+8)-(+2)=(+6)$	$(+8)+(-2)=(+6)$
Since $(+6)+(-2)=(+4)$	$\therefore (+4)-(-2)=(+6)$	$(+4)+(+2)=(+6)$
Since $(-6)+(+2)=(-4)$	$\therefore (-4)-(+2)=(-6)$	$(-4)+(-2)=(-6)$
Since $(-6)+(-2)=(-8)$	$\therefore (-8)-(-2)=(-6)$	$(-8)+(+2)=(-6)$

The third column suggests:

To subtract an integer, add its negative (or additive inverse).

***B·7* Order relation in the set of integers.** The idea of order is inherent in the set of natural numbers. Since they were invented to count the number of objects in various sets, it is natural when thinking of any two counting numbers to say that one of these is *equal to*, or is *greater than* ($>$), or is *less than* ($<$) the other and that only one of these can be true. In this sense the natural numbers are ordered, and this order is precisely defined as follows:

If $a, b \in N$, then $a > b$ if $(a - b) \in N$.

To have the ordering of the positive integers correspond to the ordering of the natural numbers it is necessary to say:

If a, b are positive integers, then
$a > b$ if $a - b$ is a positive integer, or
$a > b$ if $a = b + c$, where c is a positive integer.

If we extend this definition to include all integers, then the definition of order in the set of integers is:

If $a, b \in I$, then
$a > b$ if $a - b$ is a positive integer, or
$a > b$ if $a = b + c$, where c is a positive integer.

The consequents of this definition are illustrated in the following examples:

(i) all positive integers are greater than zero,
$(+ 5)$ is greater than zero since $(+ 5) = 0 + (+ 5)$;

(ii) all negative integers are less than zero,
0 is greater than $(- 5)$ since $0 = (- 5) + (+ 5)$;

(iii) all positive integers are greater than all negative integers,
$(+ 5)$ is greater than $(- 5)$ since $(+ 5) = (- 5) + (+ 10)$;

(iv) $(+5)$ is greater than $(+2)$ since $(+5) = (+2) + (+3)$; this corresponds to the ordering of the corresponding natural numbers;

(v) (-2) is greater than (-5) since $(-2) = (-5) + (+3)$.

Since the set of integers is closed under addition and subtraction, the following *trichotomy assumption* of order is made:

If $a, b \in I$, then $a < b$ or $a = b$ or $a > b$.

The geometric interpretation of this definition on a number line is as follows:

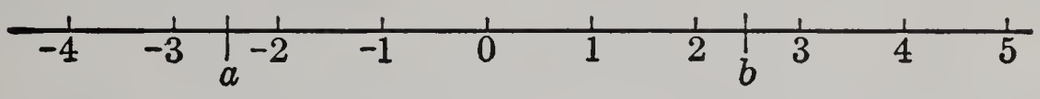


Fig. B-6

if $a > b$, then the point with coordinate a lies to the right of the point with coordinate b;

if $a = b$, then a and b are coordinates of the same point;

if $a < b$, then the point with coordinate a lies to the left of the point with coordinate b (*Fig. B-6*).

A further consequent of the definition of order is the *transitive property* of order:

If $a, b, c \in I$, and $a < b$ and $b < c$, then $a < c$.

Geometrically this means that if a point with coordinate a lies to the left of a point with coordinate b, which lies to the left of a point with coordinate c, then the point with coordinate a lies to the left of the point with coordinate c.

Exercise B-2

(*The variables are integers.*)

(A)

1. Find the following sums:

(i) $+6 + +12$ (ii) $-7 + -13$ (iii) $+7 + -6$

(iv) $-18 + +12$ (v) $-17 + -32$ (vi) $-40 + +20$

(vii) $(-13) + 0$ (viii) $(+35) + (+16)$

(ix) $(-46) + (-64)$ (x) $(-49) + (+57)$

2. State the negative of each of the following:

(i) $(+ 4)$ (ii) $(- 176)$ (iii) $+ (4 + 2)$

(iv) $- (3 - 2)$ (v) negative of $(+ 5)$ (vi) negative of $(- 7)$

(vii) $(- a)$ (viii) $- (a + b)$ (ix) $- (- a - b)$

3. Find the following differences:

(i) $+ 6 - + 3$ (ii) $+ 17 - 14$ (iii) $- 18 - 5$

(iv) $+ 16 - 27$ (v) $+ 52 - + 25$ (vi) $- 10 - 20$

(vii) $(- 15) - (+ 12)$ (viii) $(+ 27) - (- 35)$

(ix) $(- 32) - (- 37)$ (x) $(- 72) - (+ 100)$

(B)

4. Simplify, then illustrate each of the following with a vector diagram:

(i) $(+ 4) + (- 6) + (+ 3)$ (ii) $(- 7) + (+ 3) + (- 1)$

(iii) $(- 1) + (- 2) + (- 2)$ (iv) $(- 5) + (+ 3) + (+ 2)$

5. Simplify:

(i) $(+ 2) + (- 3) + (- 4)$ (ii) $(+ 1) + (+ 4) + (- 6)$

(iii) $(- 1) + (- 5) + (+ 4)$

(iv) $(+ 6) + (- 7) + (+ 12) + (- 15)$

(v) $(- 2) + (- 4) + (+ 3) + (+ 3)$

(vi) $(- 5) + (+ 14) + (0) + (- 6)$

6. Simplify each of the following by first replacing each subtraction operation by addition of a negative:

(i) $(+ 4) - (+ 3)$ (ii) $(+ 5) - (- 2)$

(iii) $(- 2) - (+ 4) - (- 5)$ (iv) $(- 3) - (- 6) - (+ 4)$

7. Simplify:

(i) $(+ 2) - (- 3) - (+ 1)$ (ii) $(+ 7) - (- 2) + (- 9)$

(iii) $0 - (- 3) + (+ 3)$ (iv) $(+ 4) - (- 5) - (- 2)$

(v) $(- 7) + (+ 12) - (- 7) + (- 12)$

(vi) $(+ 9) - (- 12) + (- 8) - (+ 11)$

8. Illustrate graphically that:

(i) if $+ 2 > - 4$ and $- 4 > - 6$, then $+ 2 > - 6$

(ii) if $- 4 < 0$ and $0 < + 3$, then $- 4 < + 3$

(iii) if $a < 1$ and $1 < b$, then $a < b$

(iv) if $a > 0$ and $b < 0$, then $a > b$

B·8 Multiplication in the set of integers.

a. *The product of two positive integers.* Since the integers are to be an extension of the set of natural numbers in which the positive integers correspond to the natural numbers, then the product of two positive integers is defined to correspond to the product of the corresponding natural numbers.

Thus, since $3 \times 4 = 4 \times 3 = 12$

define $(+3) \times (+4) = (+4) \times (+3) = +12$.

In general, if $a, b \in N$, then

$$(+a)(+b) = (+b)(+a) = +(ab).$$

b. *The product of any integer and zero.* A discussion similar to that in section $B \cdot 2$ leads to the statement:

If $a, b \in I$, then

(i) $a \cdot 0 = 0 \cdot a = 0$;

(ii) if $ab = 0$, then $a = 0$ or $b = 0$.

c. *The product of a positive and a negative integer.* To determine the products $(+3)(-4)$ and $(+a)(-b)$, $a, b \in N$, we make use of the fact that the sum of an integer and its negative is zero: that is

$$(+4) + (-4) = 0 \text{ and } (+b) + (-b) = 0.$$

To determine $(+3)(-4)$:

$\because (+4) + (-4) = 0$

$\therefore (+3)[(+4) + (-4)] = 0$

To determine $(+a)(-b)$, $a, b \in N$:

$\because (+b) + (-b) = 0$

$\therefore (+a)[(+b) + (-b)] = 0$

If multiplication is to be distributive over addition, then:

$(+3)(+4) + (+3)(-4) = 0 \quad (D)$

$\therefore (+12) + (+3)(-4) = 0$

$\therefore (+3)(-4)$ is the additive inverse or negative of $(+12)$.

$\therefore (+3)(-4) = -12$

In a similar manner it may be shown that $(-4)(+3) = -12$.

$(+a)(+b) + (+a)(-b) = 0 \quad (D)$

$\therefore +(ab) + (+a)(-b) = 0$

$\therefore (+a)(-b)$ is the additive inverse or negative of $+(ab)$.

$\therefore (+a)(-b) = -(ab)$

In a similar manner it may be shown that $(-b)(+a) = -(ab)$.

In general, if $a, b \in N$, then

$$(+a)(-b) = (-b)(+a) = -(ab).$$

d. *The product of two negative integers.*

To determine $(-4)(-3)$:

$\because (-3) + (+3) = 0$

$\therefore (-4)[(-3) + (+3)] = 0$

To determine $(-a)(-b)$, $a, b \in N$:

$\because (-b) + (+b) = 0$

$\therefore (-a)[(-b) + (+b)] = 0$

If multiplication is to be distributive over addition, then:

$(-4)(-3)+(-4)(+3)=0 \quad (D)$
$\therefore (-4)(-3)+(-12)=0$
$\therefore (-4)(-3)$ is the additive inverse or negative of (-12).
$\therefore (-4)(-3)=(+12)$
In a similar manner it may be shown that $(-3)(-4)=+12$.

$(-a)(-b)+(-a)(+b)=0 \quad (D)$
$\therefore (-a)(-b)+(-ab)=0$
$\therefore (-a)(-b)$ is the additive inverse or negative of $(-ab)$.
$\therefore (-a)(-b)=+(ab)$
In a similar manner it may be shown that $(-b)(-a)=+(ab)$.

In general, if $a, b \in N$, then

$$(-a)(-b)=(-b)(-a)=+(ab).$$

e. *Associative property of multiplication of integers.*

Since $[(+3)(+2)](-4)=(+6)(-4)=-24$

and $(+3)[(+2)(-4)]=(+3)(-8)=-24$

then $[(+3)(+2)](-4)=(+3)[(+2)(-4)]$.

This and similar examples suggest that the multiplication of integers is associative.

In general, if $a, b, c \in I$, then $(ab)c=a(bc)$.

f. *Summary of multiplication in the set of integers.*

If $a, b \in N$, then

(i) $(+a)(+b)=(+b)(+a)=+(ab)$;
(ii) $(-a)(-b)=(-b)(-a)=+(ab)$;
(iii) $(+a)(-b)=(-b)(+a)=-(ab)$.

(iv) If $a, b \in I$, then

$a \cdot 0=0 \cdot a=0$;
if $ab=0$, then $a=0$ or $b=0$.

The product of any integer and zero is zero. If the product of two integers is zero, then at least one of them is zero.

(v) If a be replaced by 1, then
from (i) $(+1)(+b)=(+b)(+1)=+(b \times 1)=+b$;
from (iii) $(+1)(-b)=(-b)(+1)=-(b \times 1)=-b$.
The product of any integer by $(+1)$ is that integer.
$(+1)$ *is the identity* (or *neutral*) *element for multiplication.*

(vi) If b be replaced by 1, then
from (ii) $(-a)(-1)=(-1)(-a)=+(a \times 1)=+a$;
from (iii) $(+a)(-1)=(-1)(+a)=-(a \times 1)=-a$.

The product of any integer and (-1) is the negative of the integer.

***B·9* Division in the set of integers.** Since division is the inverse operation of multiplication, it must be defined as follows:

If $a, b \in N$, then

$$\because (+a)(+b) = +ab \qquad \therefore \frac{+ab}{+b} = +a$$

$$\because (+a)(-b) = -ab \qquad \therefore \frac{-ab}{-b} = +a$$

$$\because (-a)(-b) = +ab \qquad \therefore \frac{+ab}{-b} = -a.$$

Since there is no integer whose product with $(+3)$ is $(+13)$

$\therefore (+13) \div (+3)$ is not an integer.

Therefore the set of integers is not closed with respect to division.

A discussion similar to that of section $B \cdot 2$ leads to the statement:

Division by zero is not defined in the set of integers.

***B·10* Summary of the properties of the integer number system.**

If $a, b, c \in I$, then

	ADDITION	MULTIPLICATION
Closure Law	$(a+b)$ *represents one and only one integer* (*ClA*)	(ab) *represents one and only one integer* (*ClM*)
Commutative Law	$a+b=b+a$ (*CA*)	$ab=ba$ (*CM*)
Associative Law	$(a+b)+c=a+(b+c)$ (*AA*)	$(ab)c=a(bc)$ (*AM*)
Distributive Law	$a(b+c)=ab+ac$	
Identity Element	**0** (*Zero*)	**1** (*Unity*)
Inverse Element	$-a$	

Since the positive integers behave, under addition or multiplication, in exactly the same way as the natural numbers, we use the natural number symbols to represent the positive integers, unless otherwise specified.

Exercise B-3

(*The letter symbols are placeholders for integers for which the expressions are defined.*)

(A)

1. Simplify:

(i) $(+4)(-7)$ (ii) $(-4)(+7)$ (iii) $(-3)(+5)$
(iv) $(-10)(-1)$ (v) $(-3)(-2)$ (vi) $(+6)(+13)$

Simplify:

(vii) $(-2)(x)$ (viii) $(-x)(-x)$ (ix) $(-y)(-1)$
(x) $(x)(-y)$ (xi) $(-m)(-m^3)$ (xii) $(-4)(3y^2)$
(xiii) $(-2a)(-3a^2)$ (xiv) $(-5xy)(2xy)$ (xv) $(-2a^2b)(-3ab^2)$

2. State the sign of the following products:

(i) $(-6)(-7)(+6)$ (ii) $7(2)(-3)$ (iii) $(3)(-4)(-6)$
(iv) $(-8)(-9)(-10)$ (v) $(-1)^4$ (vi) $(-2)^7$

3. Simplify each part of question 2.

4. Simplify:

(i) $(+16) \div (+4)$ (ii) $(+16) \div (-4)$
(iii) $(-16) \div (-4)$ (iv) $(25a) \div (-5a)$
(v) $(-225b) \div (-25)$ (vi) $(3ab) \div (3b)$
(vii) $(-2x) \div (-1)$ (viii) $(0) \div (-3a)$

(B)

5. Simplify:

(i) $(a)(-b)(-c)$ (ii) $(-2x)(6x)(x)$
(iii) $(-2ab)(-4ab)(-ab)$ (iv) $(+3mn)(-2mn)(0)(-4mn)$
(v) $(-1)(-x^2a)(ax)$ (vi) $(-a^3)(-a^4)$

6. Simplify:

(i) $(-1)^{10}$ (ii) $(-1)^{19}$
(iii) $(-1)^7 \times (-1)^{15}$ (iv) $(-1)^{2n}$, $n \in N$
(v) $(-1)^{2n+1}$, $n \in N$

7. Simplify:

(i) $\dfrac{49x^3}{7x^2}$ (ii) $\dfrac{-25ab}{5a}$

(iii) $\dfrac{-16x^2y}{-8xy}$ (iv) $\dfrac{(-3)(-8)}{(-2)(-12)}$

(v) $\dfrac{42a^5b^4c^3}{-7a^4b^3c^3}$ (vi) $\dfrac{(-1)^7(-a^2)b^3}{(-1)^{14}a^3(-b)}$

(vii) $\dfrac{(+6a)(0)(-3a^3)}{(-2a^2)(-9a^4)}$ (viii) $\dfrac{(-10b)(-5c)(-2a)}{(-25a)}$

8. Expand, using the distributive property:

(i) $(+5)[(+x)+(+y)]$ (ii) $(+6)[(-m)+(-n)]$
(iii) $(-2)[(+a)+(+b)]$ (iv) $(-3)[(-c)+(+d)]$
(v) $(-1)[(-3)+(+2)]$ (vi) $(-1)[(-a)+(+b)]$

9. Expand:

(i) $(+a)[(+b)+(-c)]$ (ii) $(-m)[(-2)+(+n)]$

(iii) $(+2x)[(-a)+(-3y)]$ (iv) $(-x^2)[(-2x)+(+3y)]$

(v) $(-3y)[(-2y)+(-3z)]$ (vi) $(+2m)[(-n)+(-5p)]$

***B·11* Polynomials.** The symbols 2, 3, 0, $4+7$, $\frac{8+2}{5}$, 6×7 are *numerals* or *expressions* which represent number ideas. In the expression $4+7$ the numerals 4 and 7 are called the *terms* of the expression. In the expression 4×7 the numerals 4 and 7 are called *factors* of the expression.

Similarly, if the symbols x, y, z represent integers, then

$$3x,\ \ 4y,\ \ \frac{x}{y},\ \ 3x+4y,\ \ 5x-z+3,\ \ \frac{x+z}{y+z},\ \ x^2+2x-7$$

are expressions which represent number ideas. These expressions which involve variables are called *algebraic expressions*. In these expressions, the set of integers I is the *replacement set* for the symbols x, y, z which represent variables, since they are placeholders for the integer number ideas. An expression like $3x+4y$ is meaningless without a given replacement set for x and y.

Algebraic expressions of the form

$$\tfrac{3}{2}x,\ \ 3x+4y,\ \ 5x-y+2z+3,\ \ \tfrac{4}{3}x+\tfrac{1}{2}y$$

in which the variables do not appear as denominators are called *polynomial expressions* or simply *polynomials*.

The expression $(+1)+(-2)+(+3)$ is an expression whose three terms are the integers $(+1)$, (-2), $(+3)$. Because it is cumbersome to write the expression in the manner indicated, including parentheses to separate the $+$ and $-$ signs of operation and quality, it is customary to write the expression

$$1-2+3$$

leaving out the operation signs of addition. Although the addition signs are not written, they are understood, and the terms of the expression are considered to be $(+1)$, (-2), and $(+3)$. This expression may be read, "positive one plus negative two plus positive three" or "one plus negative two plus three".

Also, $(+2)-(+5)=(+2)+(-5)$

$\qquad\qquad\qquad\quad =2-5$

In the expression $2-5$, the addition sign of operation is understood. The expression $2-5$ may be read, "two minus five" which is understood to mean "positive two minus positive five" or "positive two plus negative five".

Similarly, in the algebraic expression

$$3a - 4b + 2c - 4d, \text{ where } a, b, c, d \in I,$$

the terms of the expression are understood to be

$$+ 3a, \ - 4b, \ + 2c, \ - 4d,$$

and it is understood that these terms are to be added. Thus, for the purpose of applying the laws of addition and multiplication, the expression is considered to be (or mean)

$$(+ 3a) + (- 4b) + (+ 2c) + (- 4d) .$$

This means that every expression is considered to be a sum, and thus the commutative law of addition and the distributive law may always be applied.

For ease in reference, polynomials with one, two, and three terms are referred to as *monomials*, *binomials*, and *trinomials*, respectively.

***B·12* Addition and subtraction of polynomials.** In the following examples the variables are integers.

Example 1. (i) Simplify $(5a + 3b - c) + (4a + 5b - 6c)$.

This expression is read, "the expression $5a$ plus $3b$ minus c plus the expression $4a$ plus $5b$ minus $6c$".

(ii) Add $5a + 3b - c$ and $4a + 5b - 6c$.

Solution.

(i)
$$\begin{aligned}
&(5a + 3b - c) + (4a + 5b - 6c) \\
&= 5a + 3b - c + 4a + 5b - 6c \\
&= 5a + 4a + 3b + 5b - c - 6c && (CA) \\
&= (5 + 4)a + (3 + 5)b + (- 1 - 6)c && (D) \\
&= 9a + 8b + (- 7)c \\
&= 9a + 8b - 7c
\end{aligned}$$

(ii)
$$\begin{array}{r} 5a + 3b - \ c \\ 4a + 5b - 6c \\ \hline 9a + 8b - 7c \end{array}$$

It should be realized that in (ii) the term $+ 9a$ in the sum is determined by applying the distributive property to $5a + 4a$, and the term $+ 8b$ by applying the distributive property to $3b + 5b$, and so on.

With practice the commutative and distributive properties may be applied to examples like (i) as follows:

$$\begin{aligned}
(5a + 3b - c) + (4a + 5b - 6c) &= 5a + 3b - c + 4a + 5b - 6c \\
&= 9a + 8b - 7c
\end{aligned}$$

In the worked examples of the text the authorities which are most significant in the development are given in brackets. When writing a solution you should think the authorities for the steps, but it is not necessary to include them in your solution.

The study of the subtraction of integers leads to the working rule:

To subtract an integer, add its negative.

The following example illustrates how the negative of an integer represented by a polynomial may be determined.

Example 2. Determine the negative of the following algebraic expressions:
(i) $3a + 4b$ (ii) $-4x + 3y - 2$ (iii) $-(2a - 3b + 4c)$

Solution. Since the negative of an integer may be obtained by multiplying the integer by (-1):

(i) the negative of $3a + 4b$ is $(-1)(3a + 4b) = -3a - 4b$; (D)

(ii) the negative of $-4x+3y-2$ is $(-1)(-4x+3y-2)=4x-3y+2$; (D)

(iii) the negative of $-(2a - 3b + 4c)$ is $(-1)(-1)(2a - 3b + 4c)$
$= 2a - 3b + 4c$. (D)

The negative of an algebraic expression may be obtained by replacing each term by its negative.

Example 3. (i) Simplify $(5x - 2y + 6x^2) - (4x^2 - 3x + 2y)$.
(ii) Subtract $4x^2 - 3x + 2y$ from $5x - 2y + 6x^2$.

Solution.

(i)
$$\begin{aligned}
&(5x - 2y + 6x^2) - (4x^2 - 3x + 2y)\\
&= 5x - 2y + 6x^2 + (-4x^2 + 3x - 2y)\\
&= 5x - 2y + 6x^2 - 4x^2 + 3x - 2y\\
&= 6x^2 - 4x^2 - 2y - 2y + 5x + 3x \quad (CA)\\
&= (6 - 4)x^2 + (-2 - 2)y + (5 + 3)x \quad (D)\\
&= 2x^2 + (-4)y + 8x\\
&= 2x^2 - 4y + 8x
\end{aligned}$$

(ii)
$$\begin{array}{r}
5x - 2y + 6x^2\\
-3x + 2y + 4x^2\\
\hline
8x - 4y + 2x^2
\end{array}$$

In the column solution of (ii) the distributive law is applied to $5x$ and $3x$ (after noting mentally that $3x$ is the negative of $-3x$) to obtain $8x$, and so on.

With practice the solution of (i) may be abbreviated as follows:

$$\begin{aligned}
(5x - 2y + 6x^2) - (4x^2 - 3x + 2y) &= 5x-2y+6x^2-4x^2+3x-2y\\
&= 8x - 4y + 2x^2
\end{aligned}$$

The simplification of a polynomial is sometimes referred to as the *collection of like terms* (terms containing the same variable or literal coefficient). By means of the distributive law the specific numerical coefficient of this variable in the simplified expression is obtained.

Write a solution for each of the following and compare your solutions with those on page 466.

1. Add $7x^2 - x + 2$, $3x - 10x^2 - 5$, $x^2 - 2x + 7$.

2. Simplify $(5x + 3y - 2) + (3x - 4y + 3) - (6x + 3y - 6)$.

Exercise B-4

(The variables are integers.)

(A)

1. State the negatives of each of the following:

(i) $-3a$ (ii) $2a - 3b + 4c$

(iii) $-5y + 4z - 2c$ (iv) $-(3a - 2b + c)$

(v) $-3n + 4p - 5c$ (vi) $3a + 5m - 2q$

(B)

Simplify:

2. $2a - 3b + (-a + 2b)$ **3.** $(x - 4y) - (2y - x)$

4. $6x^2 + x - 2 - (4x^2 - 3x + 1) + (x^2 - 7x + 2)$

5. $(3a - b + 2c) - (2a + 5b - 4c)$

6. $(6n - 2m + 4p) - (-m - 2n + 3p) + (3m + 2p - 4n)$

7. $(a + 2b - 3c) - (3a - 5b + c) - (4a + 6b - 2c)$

If $x = -3$, $y = -2$, find the number represented by each of the following:

8. $6x - 3y + (-6x + y) - (3x - 5y)$

9. $(x^2 - 6xy + y^2) - (-3x^2 + xy - y^2)$

10. $(x - 2y) - (2x + 3y) - (-2x - 3y)$

Simplify:

11. $0 - (a - 5b) - [(5b - a) + (2a - 3b)]$

12. $[(-x - 2y) - (-3x + 4y)] - [(x + 4y) - (2x + y)]$

13. $[-(m + n) - (3m - 4n)] - [(5m - n) - (9m - 4n)]$

14. If $x = 2a - b$, and $y = a - 3b$, express $(x - 2y) - (2x - y)$ in terms of a and b.

Write as expressions in x and y:

15. $a(x + y) + b(3x + 2y) - c(2y - 3x)$

16. $p(3x - y) - q(2y - x) - r(x - y)$

17. $x(m - n) + y(n - m) - p(x - y)$

Arrange each of the following in descending powers of x:

18. $ax - dx^2 + cx^3 + bx^2 - 2x + 8 - 3x^3$

19. $px^2 - hx + ax^2 - kx^3 + x^3 - bx$

20. $p(x^2 - x) - qx^2 - 2(x - x^3) + mx^3 - n(x^3 - 1)$

21. Write the coefficient of x^2 in $6x^3 + 4ax + bx^2 - cx - dx^2 + x^2$.

22. Write the coefficient of x in $mx^2 - 8 + ax - x^2 - bx - 4cx$.

23. Write the coefficient of y in $3a(1 - y) - 2b(y - 2) - y$.

***B·13* Multiplication of polynomials.** In each of the following examples the variables are integers.

Example 1. Expand: (i) $3(a + 2b)$ (ii) $3a(2a - 4b)$
(iii) $3a(2a + 3b - 4c + 5d)$

Solution. (i) $3(a + 2b)$
$= 3a + 6b$ (D)

(ii) $3a(2a - 4b)$
$= 3a[(2a) + (-4b)]$
$= 6a^2 + (3a)(-4b)$ (D)
$= 6a^2 - 12ab$

(iii) $3a(2a + 3b - 4c + 5d)$
$= 3a[(2a + 3b) + (-4c + 5d)]$ (AA)
$= 3a(2a + 3b) + 3a(-4c + 5d)$ (D)
$= 6a^2 + 9ab - 12ac + 15ad$ (D)

In practice the intermediate steps shown in (ii) and (iii) may be omitted.

Example 2. If $a = -3$, $b = +2$, $c = -2$, factor $8ab - 12ac$ and evaluate.

Solution.
$$\begin{aligned} 8ab - 12ac &= 4a(2b - 3c) \quad (D) \\ &= 4(-3)[2(+2) - 3(-2)] \\ &= (-12)(+4 + 6) \\ &= (-12)(10) \\ &= -120 \end{aligned}$$

Example 3. Expand the product $(2x + 3y)(x - 2y)$.

Solution.
$$\begin{aligned} (2x+3y)(x-2y) &= (2x+3y)x+(2x+3y)(-2y) \quad (D) \\ &= 2x^2 + 3xy + 4xy - 6y^2 \quad (D) \\ &= 2x^2 + (3 + 4)xy - 6y^2 \quad (D) \\ &= 2x^2 - xy - 6y^2 \end{aligned}$$

$$\begin{array}{rl} 2x + 3y & \\ x - 2y & \\ \hline 2x^2 + 3xy & (D) \\ -4xy - 6y^2 & (D) \\ \hline 2x^2 - xy - 6y^2 & (D) \end{array}$$

Exercise B-5

(*Variables are integers.*)

(A)

1. Expand the following:

(i) $3(2a + 3b)$ (ii) $(-4)(2x - 5y)$ (iii) $(-1)(-3m - 5n)$
(iv) $5(3a - 2b + 4c - 5d)$ (v) $(-6)(2x - 4y + 6z - 2q)$

2. Factor the following:

(i) $4m + 8p$ (ii) $6ab - 4ac$ (iii) $-3a - 12b$
(iv) $16a^2 - 4ab - 8ac$ (v) $-7mp - 14mq - 21mr$

3. State the negative of each of the following:

(i) $4m + 5n$ (ii) $3a - 2b$ (iii) $-(2x + y)$
(iv) $-4m - 2n + p$ (v) $x^2 - 5x + 6$
(vi) $-(-2x^2 - x + 1)$

(B)

4. Simplify:

(i) $2(a + b - c) + 3(2a - b + 3c)$
(ii) $(3x + 2y) - 2(x + 4y) + 3(2x - 3y)$
(iii) $0 - 2(3x + 2y) + (-4)(2x - 8y)$
(iv) $(-3)(2a - 4b) + 2(3a - 6b) - 4(a + b - 3c)$
(v) $4m(2m + 3n) - 4n(3m - 2n) + 3(m^2 - n^2)$

5. If $a = 2$, $b = -4$, $c = -1$, compare:

(i) $a^2 - b^2$ and $(a - b)(a + b)$
(ii) $a^3 + b^3 + c^3 - 3abc$ and
$(a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc)$

6. If $x = 3$, $y = -2$, $z = 4$, find $\dfrac{2x + 3y}{z} + \dfrac{4x - 5z}{-2y}$.

7. Factor:

(i) $a^2 + ab$ (ii) $3a + 6b - 12c$
(iii) $-4mp - 8p - 12pq$ (iv) $-12x + 3a - 15b$

8. Expand and simplify:

(i) $m - n[(x + y) - 5z]$ (ii) $(x + y)(3a + 2b)$
(iii) $(4a + b)(a - 3b)$ (iv) $(x - 2y)(2x - y)$

9. Expand each of the following:

(i) $(3x + 2)(2x + 3)$ (ii) $(5m + 2)(5m + 2)$
(iii) $(9a + 5b)(9a - 5b)$ (iv) $(2a + b)(3a - b + 2c)$
(v) $(m - 3n)(6m - 2n + 7)$ (vi) $(2x + 3)(4x^2 + 12x + 9)$

10. Expand and simplify:

(i) $(3a - 2b)(3a + 2b) + (4a - 3b)(2a + b) - (a - 7b)(a + b)$
(ii) $(2m-1)(m^2+m+1)-(m-2)(m^2+2m+4)+(2m-3)(m^2-1)$

Chapter I

RATIONAL NUMBERS

IRRATIONAL NUMBERS

REAL NUMBERS

***1·1* The set of rational numbers (Q).** The extension of the number system to the set of integers provides a set of numbers closed under the operations of addition, subtraction, and multiplication, but *not closed under division.* Since there is no integer whose product with $(+ 2)$ is $(+ 7)$, the quotient $(+ 7) \div (+ 2)$ does not exist in this set. To overcome this deficiency a further extension of the number system has been made, and a set of numerals of the form

$$\frac{-3}{+2}, \qquad \frac{-4}{-7}, \qquad \frac{+5}{-11}$$

has been invented to represent quotients of every integer divided by a non-zero integer.

In general, for every pair of integers represented by a and b, $b \neq 0$, there is a numeral of the form $\frac{a}{b}$ (read, "a over b") which represents the quotient $a \div b$. The set of numbers thus represented is called the set of *rational* (ratio) numbers. It is designated by Q (quotient) and is defined to be the set of all numbers represented by $\frac{a}{b}$ such that a and b represent integers and $b \neq 0$.

In set-builder notation,

$$Q = \left\{ \frac{a}{b} \mid a, b \in I, b \neq 0 \right\}.$$

The symbol $\frac{a}{b}, b \neq 0$ is called a *fraction*; a is the *numerator*, and b the *denominator* of the fraction; a and b are the *terms* of the fraction.

If $a, b \in I, b \neq 0$

$$\because \frac{a}{b} = a \div b \quad \therefore (a \div b)(b) = a \quad \text{(Definition of division)}$$

$$\therefore \frac{a}{b} \times b = a .$$

Since multiplication is to be commutative, this property is expressed as follows:

If $a, b \in I, b \neq 0$, *then*

$$\frac{a}{b} \times b = b \times \frac{a}{b} = a .$$

1·2 **Multiplication in the set of rational numbers.** If the set of rational numbers is to be an extension of the set of integers, it is necessary that the operations of addition and multiplication be consistent with those of the integers. The set should be closed under the operations, and the operations should be commutative and associative, and multiplication should be distributive over addition.

a. *Product of any two rational numbers.*

If $\frac{a}{b}, \frac{c}{d} \in Q$, then

$$\because \frac{a}{b} \times b = a$$

$$\text{and} \quad \frac{c}{d} \times d = c$$

$$\therefore \left(\frac{a}{b} \times b\right)\left(\frac{c}{d} \times d\right) = a \times c .$$

If multiplication is to be commutative and associative, then

$$\left(\frac{a}{b} \times \frac{c}{d}\right)\left(b \times d\right) = a \times c$$

$$\therefore \frac{a}{b} \times \frac{c}{d} = (a \times c) \div (b \times d) \quad \text{(Definition of division)}$$

$$\text{or } \frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} .$$

Thus, if $\frac{a}{b}, \frac{c}{d} \in Q$ ***, then*** $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

Since $a, c, b, d \in I$, therefore ac and $bd \in I$, and therefore $\frac{ac}{bd} \in Q$.

Thus, the set of rational numbers is closed under multiplication. The definition was developed on the assumption that multiplication be commutative and associative.

b. *Identity element for multiplication.*

If $\frac{a}{b} \in Q$, then

(i) $\frac{a}{b} \times \frac{(+1)}{(+1)} = \frac{a(+1)}{b(+1)} = \frac{a}{b}$

(ii) $\frac{-a}{b} \times \frac{(+1)}{(+1)} = \frac{(-a)(+1)}{b(+1)} = \frac{-a}{b}$.

These examples illustrate that $\frac{+1}{+1}$ or 1 is the *identity element* for multiplication.

c. *Reciprocal (multiplicative inverse).*

If $a, b \in I$, $a, b \neq 0$, then

$$\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ba}$$

$$= \frac{ab}{ab} = 1 . \qquad (CM)$$

Two numbers whose product is 1 are called *reciprocals* or *multiplicative inverses* of each other. Division by zero is not defined; zero has no reciprocal.

Example 1. State the reciprocals of:

(i) $\frac{3}{-2}$ (ii) $\frac{-4}{-5}$ (iii) $\frac{-6}{7}$ (iv) 7 (v) 1 (vi) 0

Solution.

(i) $\frac{-2}{3}$ (ii) $\frac{-5}{-4}$ (iii) $\frac{7}{-6}$ (iv) $\frac{1}{7}$ (v) 1

(vi) Zero has no reciprocal.

1·3 **The principle of equivalent fractions.** Since 1 is the identity element for multiplication and

$$1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{-1}{-1} = \frac{-2}{-2} = \frac{-3}{-3} = \frac{x}{x},\ x \in I,\ x \neq 0,$$

then if $\frac{a}{b} \in Q$,

$$\begin{aligned}\frac{a}{b} &= \frac{a}{b} \times \frac{1}{1} = \frac{a \times 1}{b \times 1} \\ &= \frac{a}{b} \times \frac{2}{2} = \frac{a \times 2}{b \times 2} \\ &= \frac{a}{b} \times \frac{-3}{-3} = \frac{a(-3)}{b(-3)} \\ &= \frac{a}{b} \times \frac{x}{x} = \frac{ax}{bx},\ x \in I,\ x \neq 0.\end{aligned}$$

Thus, there are many fractions which represent the same rational number. This fact is expressed in the

Principle of equivalent fractions (PEF):

If $\frac{a}{b} \in Q$ ***and*** $x \in I$, $x \neq 0$, ***then*** $\frac{a}{b} = \frac{ax}{bx}$.

In the following exercises, symbols of the form $\frac{a}{b}$ *represent rational numbers.*

Exercise 1-1

(A)

1. State two fractions equivalent to each of the following:

(i) $\frac{3}{-4}$ (ii) $\frac{-2}{5}$ (iii) $\frac{-4}{-7}$ (iv) $\frac{+3}{+8}$

2. Using the principle of equivalent fractions, reduce the following to lowest terms:

(i) $\frac{-10}{44}$ (ii) $\frac{120}{-15}$ (iii) $\frac{-34}{-85}$ (iv) $\frac{+36}{+180}$

(v) $\frac{-66}{21}$ (vi) $\frac{14ab}{91a}$ (vii) $\frac{0}{-2}$ (viii) $\frac{2}{10}$

3. Express the following products in simplest form:

(i) $\frac{-4}{15} \times \frac{9}{-16}$ (ii) $\frac{-3}{14} \times 3$ (iii) $\frac{-3}{5} \times \frac{4}{7}$

(iv) $5 \times \left(\frac{4}{-5}\right)$ (v) $\left(\frac{-1}{2}\right) \times \left(\frac{3}{-5}\right) \times \frac{10}{3}$ (vi) $\frac{x^2}{a} \times \frac{b}{c} \times \frac{y}{d}$

(vii) $\frac{-6}{5} \times \frac{10a}{15b}$ (viii) $\frac{-5}{7} \times \frac{0}{-5} \times \frac{7}{8}$ (ix) $\frac{-y^2}{10} \times \frac{x^3}{y}$

4. State the reciprocals of:

(i) $\frac{5}{-3}$ (ii) $\frac{3}{-4}$ (iii) 4 (iv) -4

(v) $\frac{-7}{-8}$ (vi) -1 (vii) $\frac{0}{-2}$ (viii) $\frac{a}{-a}$

(B)

5. Reduce to lowest terms:

(i) $\frac{-2a}{6a}$ (ii) $\frac{15pqr}{-r}$ (iii) $\frac{5x+5y}{5a}$

(iv) $\frac{3 \times 0}{-4}$ (v) $\frac{-3}{24 \times 10}$ (vi) $\frac{a}{ab+ac}$

6. List three rational numerals equivalent to $\frac{-p}{q}$.

7. Write the reciprocals of:

(i) $\frac{-15a}{30b}$ (ii) $\frac{0}{-4}$ (iii) 1 (iv) $\frac{-9x}{3x}$

8. Write the negative reciprocal of:

(i) $\frac{2}{3}$ (ii) $\frac{3}{-4}$ (iii) $\frac{-a}{b}$

(iv) -1 (v) 0 (vi) $\frac{-6}{-8}$

9. Express the following products in simplest form:

(i) $\frac{-6}{7} \times \frac{4}{3}$ (ii) $\frac{3}{2} \times \frac{2a+3b}{3}$ (iii) $\frac{6a}{2(-b)} \times \frac{xy}{-3a}$

Express the following products in simplest form:

(iv) $6(-a) \times \frac{1}{12ac}$ (v) $\frac{1}{2} \times (-8) \times \frac{1}{-a}$ (vi) $\frac{3ab}{-c} \times \frac{ac}{a^2}$

(vii) $\frac{4xy}{-3xz} \times \frac{9yz}{2xyz}$ (viii) $\frac{a}{-b} \times \frac{-b}{c} \times \frac{-c}{d}$

(ix) $\frac{1}{-2} \times \frac{-2}{3} \times \frac{3}{-4}$

10. If $a = \frac{-2}{3}$ and $b = \frac{3}{4}$, find:

(i) a^2 (ii) ab (iii) b^2

11. If $x = -1$, $y = -2$, $z = -3$, find:

(i) $x^3 + y^3 + z^3 - 3xyz$

(ii) $(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

12. If $p = -\frac{5}{6}$, $q = 0$, $r = 0$, find $pq + qr + rp$.

13. Show that $\left(\frac{2}{-3} \times \frac{-5}{-7}\right) \times \frac{-6}{17} = \frac{2}{-3}\left(\frac{-5}{-7} \times \frac{-6}{17}\right)$.

14. Show that $\left(\frac{-x}{y} \times \frac{p}{-q}\right) \times \frac{-m}{-n} = \left(\frac{m}{n} \times \frac{p}{-q}\right)\left(\frac{-x}{y}\right)$.

1·4 Positive and negative rational numbers.

If $\frac{-2}{+3} = x$, then $x(+3) = -2$.

$\therefore$ x represents a negative rational number.

That is, $\frac{-2}{+3}$ represents a negative rational number.

In general, a rational number is negative if the product of its terms is negative.

If $\frac{-2}{-3} = y$, then $y(-3) = -2$.

$\therefore$ y represents a positive rational number.

That is, $\frac{-2}{-3}$ represents a positive rational number.

In general, a rational number is positive if the product of its terms is positive.

1·5 **Alternate form of numerals for rational numbers.** Since

$$\frac{-2}{+3} = \frac{(-1)}{(+1)} \times \frac{(+2)}{(+3)} = (-1) \times \frac{(+2)}{(+3)}$$

$$\therefore \frac{-2}{+3} \text{ is equivalent to } -\frac{+2}{+3} \text{ or } -\frac{2}{3}.$$

The form $-\frac{2}{3}$ emphasizes that $\frac{-2}{+3}$ represents a *negative* rational number.

Similarly,

$$\frac{-2}{-3} = \frac{(-1)}{(-1)} \times \frac{(+2)}{(+3)}$$

$$= (+1) \times \frac{(+2)}{(+3)}$$

$$\therefore \frac{-2}{-3} \text{ is equivalent to } +\frac{2}{3} \text{ or } \frac{2}{3}.$$

The form $+\frac{2}{3}$ emphasizes that $\frac{-2}{-3}$ represents a *positive* rational number.

Also,

$$\frac{+2}{-3} = \frac{(+1)}{(-1)} \times \frac{(+2)}{(+3)}$$

$$= (-1) \times \frac{(+2)}{(+3)}$$

$$\therefore \frac{+2}{-3} \text{ is equivalent to } -\frac{2}{3}.$$

In general, if $a, b \in {}^{+}I$, *then*

$$\frac{-a}{+b} = -\frac{a}{b}$$

$$\frac{+a}{-b} = -\frac{a}{b}$$

$$\frac{-a}{-b} = \frac{a}{b}.$$

1·6 **Division in the set of rational numbers.** Since division is the inverse operation of multiplication,

if $a, b, c \in Q$, then $a \div b = c$ if and only if $bc = a$.

The following examples illustrate how the quotient may be determined.

MULTIPLICATION	DIVISION	MULTIPLICATION WHICH PRODUCES THE SAME RESULT
$\frac{-2}{+3}\times\frac{-5}{+7}=\frac{(-2)(-5)}{(+3)(+7)}$	$\frac{(-2)(-5)}{(+3)(+7)}\div\frac{-5}{+7}=\frac{-2}{+3}$	$\frac{(-2)(-5)}{(+3)(+7)}\times\frac{+7}{-5}$ $=\frac{(-2)(-5)(+7)}{(+3)(+7)(-5)}$ $=\frac{-2}{+3}$
If $\frac{a}{b}, \frac{c}{d} \in Q$, then $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.	$\frac{ac}{bd}\div\frac{c}{d}=\frac{a}{b}$	$\frac{ac}{bd}\times\frac{d}{c}=\frac{acd}{bdc}$ $=\frac{a}{b}$

To divide by a fraction, multiply by its reciprocal.

In general, if $\frac{a}{b}, \frac{c}{d} \in Q$, ***then***

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc},\ c\neq 0.$$

Since $\frac{ad}{bc}$ $(b, c \neq 0)$ is a rational number, the set of rational numbers is closed under division, except for division by zero which is not defined.

Example 1. Find:

(i) $\frac{-2}{5}\div\frac{8}{-3}$ (ii) $\frac{16}{25}\div(-4)$ (iii) $-3\div\frac{1}{5}$

(iv) $-\frac{35}{8}\div\frac{7}{6}$ (v) $\frac{\frac{-2}{+5}}{\frac{-3}{-7}}$

Solution.

(i) $\frac{-2}{5}\div\frac{8}{-3}=\frac{-2}{5}\times\frac{-3}{8}=\frac{2\times 3}{5\times 8}=\frac{3}{20}$

(ii) $\frac{16}{25}\div(-4)=\frac{16}{25}\times\frac{1}{-4}=-\frac{16\times 1}{25\times 4}=-\frac{4}{25}$

(iii) $-3 \div \frac{1}{5} = -3 \times \frac{5}{1} = -15$

(iv) $-\frac{35}{8} \div \frac{7}{6} = -\frac{35}{8} \times \frac{6}{7} = -\frac{35 \times 6}{8 \times 7} = -\frac{15}{4}$

(v) $\dfrac{\dfrac{-2}{+5}}{\dfrac{-3}{-7}} = \frac{-2}{+5} \div \frac{-3}{-7} = \frac{-2}{+5} \times \frac{-7}{-3} = \frac{14}{-15} = -\frac{14}{15}$

Exercise 1-2

(A)

Simplify:

1. $\frac{4}{3} \div \frac{12}{5}$ **2.** $\frac{-4}{3} \div \frac{12}{5}$ **3.** $\left(\frac{4}{3} \div \frac{12}{-5}\right)$

4. $\frac{-4}{-3} \div \left(-\frac{12}{5}\right)$ **5.** $-\frac{-4}{-3} \div \frac{-12}{-5}$ **6.** $\frac{x}{y} \div \frac{p}{q}$

7. $\dfrac{\frac{3}{4}}{-\frac{7}{8}}$ **8.** $\dfrac{\frac{1}{-3}}{\frac{1}{-5}}$ **9.** $\dfrac{-\frac{1}{x}}{\frac{1}{y}}$ **10.** $\dfrac{\frac{1}{-x}}{y}$

(B)

11. $-\frac{2}{3} \div \frac{5}{-6}$ **12.** $-\frac{1}{4} \div \frac{-5}{-28}$ **13.** $\frac{-5}{7} \div \left(-\frac{6}{13}\right)$

14. $\frac{-6}{-17} \div \frac{-2}{-34}$ **15.** $\frac{2a}{-5} \div \frac{a^3}{8}$ **16.** $\frac{2}{-p} \div \frac{6p}{-5}$

17. $x \div \frac{3x}{-5}$ **18.** $\frac{r^2}{4} \div \frac{5rs}{-12}$ **19.** $\frac{-1}{xy^2z} \div \frac{-1}{xyz}$

20. $-3 \div \frac{-12}{-ab}$ **21.** $\frac{x}{-y} \div \frac{x^3}{-y^2}$ **22.** $\frac{-2x^2}{3xy} \div \frac{-8xy^2}{x^3y}$

23. $\dfrac{\frac{-3}{4}}{\frac{5}{-8}}$ **24.** $\dfrac{\frac{1}{x}}{\frac{-2}{-x}}$ **25.** $\dfrac{\frac{3}{mn}}{\frac{-2m}{n}}$

1·7 **Addition in the set of rational numbers.**

a. *Addition of rational numbers with the same denominator.*

$$\text{If } \frac{a}{b},\ \frac{c}{b} \in Q, \text{ then } \frac{a}{b} + \frac{c}{b} = \frac{a}{1} \times \frac{1}{b} + \frac{c}{1} \times \frac{1}{b}$$

$$= a \times \frac{1}{b} + c \times \frac{1}{b}$$

$$= (a + c)\frac{1}{b} \qquad (D)$$

$$= \frac{a + c}{b}.$$

b. *Addition of any two rational numbers.* The sum of any two rational numbers may be related to the sum of two rational numbers with the same denominator by means of the principle of equivalent fractions.

To find the sum $\frac{2}{3} + \frac{-5}{4}$

note that:

$$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12} \qquad (PEF)$$

$$\frac{-5}{4} = \frac{(-5) \times 3}{4 \times 3} = \frac{-15}{12} \qquad (PEF)$$

$$\therefore \frac{2}{3} + \frac{-5}{4} = \frac{8}{12} + \frac{-5}{12}$$

$$= \frac{8 - 5}{12}$$

$$= \frac{1}{4} \qquad (PEF)$$

To find the sum $\frac{a}{b} + \frac{c}{d}$ if $\frac{a}{b}, \frac{c}{d} \in Q$

note that:

$$\frac{a}{b} = \frac{ad}{bd} \qquad (PEF)$$

$$\frac{c}{d} = \frac{cb}{db} = \frac{bc}{bd} \qquad (PEF)\ (CM)$$

$$\therefore \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd}$$

$$= \frac{ad + bc}{bd}$$

In general, if $\frac{a}{b}, \frac{c}{d} \in Q$, *then*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}.$$

The set Q is closed under addition; addition is commutative and associative, and multiplication is distributive over addition.

c. *Identity element for addition.*

$$\frac{a}{b} + 0 = \frac{a}{b} + \frac{0}{b} = \frac{a + 0}{b} = \frac{a}{b}$$

Thus, zero is the identity element for addition.

d. *Additive inverse or negative of a rational number.*

Since
$$\frac{a}{b}+\frac{-a}{b}=\frac{a+(-a)}{b}$$
$$=\frac{0}{b}$$
$$=0$$

$\therefore \frac{-a}{b}$ is the additive inverse or negative of $\frac{a}{b}$, and

$\frac{a}{b}$ is the additive inverse or negative of $\frac{-a}{b}$.

$\therefore$ *the product of any rational number (except zero) and* (-1) *is the negative of the rational number.*

$$\because \frac{-a}{b}=-\frac{a}{b}=\frac{a}{-b}$$

$\therefore$ the negative of $\frac{a}{b}$ may be written $\frac{-a}{b}$ or $-\frac{a}{b}$ or $\frac{a}{-b}$.

1·8 **Subtraction in the set of rational numbers.** Since subtraction is the inverse operation of addition, therefore if $a, b, c \in Q$, then

$$a-b=c \text{ if and only if } c+b=a.$$

The following examples illustrate subtraction of rational numbers.

ADDITION	SUBTRACTION	ADDITION WHICH PRODUCES THE SAME RESULT
$\frac{1}{2}+\frac{3}{4}=\frac{2+3}{4}=\frac{5}{4}$	$\therefore \frac{5}{4}-\frac{3}{4}=\frac{1}{2}$	$\frac{5}{4}+\frac{(-3)}{4}=\frac{5+(-3)}{4}=\frac{2}{4}=\frac{1}{2}$
If $\frac{a}{b}, \frac{c}{d} \in Q$, then		
$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$.	$\therefore \frac{ad+bc}{bd}-\frac{c}{d}$	$\frac{ad+bc}{bd}+\frac{-c}{d}=\frac{ad+bc+(-bc)}{bd}$
	$=\frac{a}{b}$	$=\frac{ad}{bd}$
		$=\frac{a}{b}$

To subtract a rational number, add its negative.

If $\frac{a}{b}, \frac{c}{d} \in Q$, then

$$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d} = \frac{ad - bc}{bd}.$$

Since $a, b, c, d \in I$,

$$\therefore\ ad, bc, bd, ad - bc \in I.$$

$$\therefore\ \frac{ad - bc}{bd} \in Q$$

Thus, the set of rational numbers is closed under subtraction.

Write a solution to the following problem and compare it with that on page 466.

1. Simplify the following:

(i) $\frac{2}{3} + \frac{-5}{4} - \frac{7}{-2}$ (ii) $\frac{5}{xy} - \frac{3}{xy} + \frac{-2}{xy}$, $x, y \in I$

Exercise 1-3

(A)

State the negative of:

1. $\frac{-5}{+7}$ **2.** $\frac{+3}{-10}$ **3.** $\frac{-5}{-11}$

4. $\frac{-6}{5ab}$ **5.** $\frac{+2x}{-3y}$ **6.** $-\frac{a}{3c}$

7. $-\frac{2}{-5}$ **8.** $-\frac{3}{2a}$ **9.** $-\frac{-5b}{-11c}$

Simplify:

10. $\frac{2}{3} + \frac{2}{3}$ **11.** $\frac{2}{3} + \frac{2}{-3}$ **12.** $\frac{2}{-3} + \frac{-2}{3}$

13. $\left(-\frac{2}{5}\right) + \left(-\frac{2}{5}\right)$ **14.** $\frac{5}{6} - \frac{1}{3}$ **15.** $2 + \frac{-5}{6}$

16. $\frac{2}{5} + \frac{3}{7}$ **17.** $\frac{2}{-5} + \frac{-3}{7}$ **18.** $\frac{2x}{3} - \frac{-x}{3}$

(B)

Simplify:

19. $\frac{4}{5}+\frac{4}{-5}$ **20.** $\frac{3}{4}+\left(-\frac{1}{4}\right)$ **21.** $\frac{7}{8}-\frac{3}{4}$

22. $\frac{9}{16}-\frac{1}{-4}$ **23.** $\left(\frac{-3}{5}\right)+\left(\frac{-2}{5}\right)$ **24.** $\frac{-3}{8}-\frac{-5}{8}$

25. $\frac{-3}{10}+\frac{1}{5}$ **26.** $-\frac{5}{x}+\frac{4}{x}$ **27.** $\frac{-5}{x}-\frac{4}{x}$

28. $\frac{-5}{x}+\frac{3}{-x}$ **29.** $\frac{5}{x}-\frac{8}{2x}$ **30.** $\frac{5}{x}+\frac{4}{-2x}$

31. $\frac{2}{-b}-\frac{6}{-2b}$ **32.** $\left(\frac{-3}{c}\right)+\left(\frac{-5}{c}\right)$ **33.** $\frac{3}{5x}-\left(-\frac{2}{x}\right)$

34. $\frac{5}{ab}-\frac{6}{2ab}+\frac{-6}{3ab}$ **35.** $\frac{1}{3a}-\frac{2}{2b}+\frac{-1}{ab}$ **36.** $\frac{a^2}{b^2c^2}-\frac{2b^2}{a^2c^2}+\frac{3c^2}{a^2b^2}$

37. $\frac{5a}{2bc}-\frac{6c}{12ca}$ **38.** $1-\frac{a}{2b}$ **39.** $\frac{a}{b}-\frac{b}{a}-\frac{1}{-2}$

***1·9* Order relations in the set of rational numbers.** If the same idea of order as that established for the integers is to be extended to the rational numbers, it is necessary to make the following definition:

If $a,\ b \in Q$, then

$a > b$ if $a - b$ is a positive rational number, or
$a > b$ if $a = b + c$, where c is a positive rational number.

It follows from this definition that:

(i) all positive rational numbers are greater than zero;

(ii) all negative rational numbers are less than zero;

(iii) all positive rational numbers are greater than all negative rational numbers;

(iv) $-\frac{1}{4}$ is greater than $-\frac{3}{2}$

since $-\frac{1}{4}-\left(-\frac{3}{2}\right)=\frac{-1}{4}+\frac{3}{2}=\frac{-1+6}{4}$

$=\frac{5}{4}$ (a positive rational number).

The following are the two fundamental properties of this order relation.

(i) *The trichotomy property:*
If $a, b \in Q$, then $a < b$ or $a = b$ or $a > b$.

(ii) *The transitive property:*
If $a, b \in Q$ and $a > b$ and $b > c$, then $a > c$.

Write a solution for the following problem and compare it with that on page 466.

1. Arrange the elements of the following set of rational numbers in increasing order of magnitude:

$$\left\{-2, \frac{-2}{-3}, \frac{-4}{3}, \frac{-5}{-5}, -1\tfrac{2}{3}, \frac{4}{-1}\right\}$$

Exercise 1-4

(B)

Determine the order relation for the following pairs of rational numbers:

1. $\frac{2}{3}, \frac{3}{5}$ 2. $-\frac{1}{2}, -\frac{3}{4}$ 3. $-1, \frac{2}{3}$

4. $-\frac{8}{9}, -\frac{9}{10}$ 5. $\frac{5}{13}, \frac{7}{17}$ 6. $-\frac{8}{3}, -\frac{12}{5}$

Arrange the elements of the following sets in increasing order of magnitude:

7. $\left\{\frac{-2}{5}, \frac{-2}{-5}, \frac{3}{-5}, -\frac{-3}{5}\right\}$ 8. $\left\{-\frac{7}{-3}, \frac{-7}{3}, \frac{-8}{3}, \frac{-8}{-3}\right\}$

9. $\left\{-\frac{3}{-8}, -\frac{-3}{4}, \frac{-3}{4}, \frac{3}{-8}\right\}$ 10. $\left\{-\frac{1}{3}, \frac{1}{4}, -\frac{5}{6}, -\frac{-2}{3}, \frac{5}{12}\right\}$

1·10 **Rational numbers in decimal notation.** In decimal notation

$$\frac{3}{4} = .75, \quad \frac{3}{8} = .375, \quad \frac{5}{2} = 2.5.$$

In changing the rational numerals $\frac{3}{4}, \frac{3}{8}, \frac{5}{2}$ to their decimal equivalents, the division process is complete, or terminates. The decimals are called *terminating decimals.*

In decimal notation,

$$\frac{1}{3} = .3333\ldots \quad \text{or} \quad \frac{1}{3} = .\dot{3}$$

$$\frac{2}{7} = .285714285714\ldots \quad \text{or} \quad \frac{2}{7} = .\dot{2}85714\dot{}$$

In attempting to change the rational numerals $\frac{1}{3}$, $\frac{2}{7}$ to their decimal equivalents, the division does not terminate but has recurring sets of digits. These decimals are referred to as *non-terminating recurring decimals.*

It is also customary to write the repeating decimals $.\dot{3}$ and $.\dot{2}85714\dot{}$ in the form $.\overline{3}$ and $.\overline{285714}$.

THEOREM A. *Any rational number may be represented by either a terminating or a non-terminating recurring decimal.*

Two particular examples illustrate the plausibility of this theorem. (i) To find the decimal equivalent of a fraction which represents a rational number, the numerator is divided by the denominator. At each step of the division a remainder occurs. If at one stage of the division the remainder is zero, the division is complete and the decimal terminates.

(ii) If 2 is divided by 7, the only possible remainders which can occur are 0, 1, 2, 3, 4, 5, 6. If the remainder zero does not occur, then, as the division continues, some remainder must occur again. The division then repeats itself.

Either (i) or (ii) is true for all rational numbers. The division of 2 by 7 has been completed by the long division method only to display the remainders.

```
   .375          .2857142
8 | 3.0000    7 | 2.0000000
    2 4           1 4
    ---           ---
     60            60
     56            56
     --            --
     40            40
     40            35
     --            --
      0             50
                    49
                    --
                    10
                     7
                     -
                    30
                    28
                    --
                     20
                     14
                     --
                     60
```

THEOREM B. *Any terminating or non-terminating recurring decimal represents a rational number.*

The following examples illustrate the plausibility of this theorem.

(i) Any terminating decimal may be written as a fraction with denominator 10 or a power of 10 and is, therefore, a rational number.

$$.5 = \frac{5}{10} \qquad .34 = \frac{34}{10^2}$$

$$= \frac{1}{2} \qquad = \frac{17}{50}$$

(ii) Any non-terminating recurring decimal may be expressed as a fraction by employing the method shown in the following example.

Example 1. Show that the following non-terminating recurring decimals represent rational numbers:

(i) $.\dot{6}$ (ii) $.\dot{1}\dot{8}$ (iii) $.\dot{2}3\dot{5}$ (iv) $.2\dot{3}\dot{5}$

Solution.

(i) Let $x = .\dot{6} = .6666\ldots$

$10x = 6.66666\ldots$

$x = \quad .66666\ldots$

$\therefore 9x = 6$

$\therefore x = \frac{6}{9}$

$\therefore .\dot{6} = \frac{2}{3}$

(ii) Let $y = .\dot{1}\dot{8} = .181818\ldots$

$100y = 18.18181818\ldots$

$y = \quad .18181818\ldots$

$\therefore 99y = 18$

$\therefore y = \frac{18}{99}$

$\therefore .\dot{1}\dot{8} = \frac{2}{11}$

(iii) Let $m = .\dot{2}3\dot{5} = .235235\ldots$

$1000m = 235.235235\ldots$

$m = \quad .235235\ldots$

$\therefore 999m = 235$

$\therefore m = \frac{235}{999}$

(iv) Let $n = .2\dot{3}\dot{5} = .2353535\ldots$

$1000n = 235.353535\ldots$

$10n = \quad 2.353535\ldots$

$\therefore 990n = 233$

$\therefore n = \frac{233}{990}$

Each of the above results may be checked by division. The decimal fraction .375 is often referred to as a non-terminating recurring decimal, the repeating digit being zero, that is, $.375\dot{0}$. Thus, all rational numbers may be represented by non-terminating recurring decimals which are referred to as *periodic decimals.*

It is interesting to note that some rational numbers have *two decimal equivalents.* This is illustrated in the following example.

Example 2. Show that (i) .5 and (ii) $.4\dot{9}$ both represent the rational number $\frac{1}{2}$.

Solution. (i) $.5 = \frac{5}{10} = \frac{1}{2}$

(ii) Let $x = .499\ldots$

then $100x = 49.999\ldots$

$10x = \quad 4.999\ldots$

$\therefore 90x = 45$

$\therefore x = \frac{1}{2}$

Similarly it may be shown that the numbers 5 and $4.\dot{9}$ represent the same rational number.

Write a solution for the following problem and compare it with that on page 467.

1. Show that (i) $.3749\dot{9}$ and (ii) .375 are equivalent to the rational numeral $\frac{3}{8}$.

It should be noted that the decimal part of any periodic decimal can be expressed in fraction form by the following rules.

The *numerator* is obtained by subtracting the number formed by the non-repeating digits from the number formed by all the digits.

The *denominator* is formed by writing a 9 for each repeating digit, followed by a zero for each non-repeating digit.

For example: $.2\dot{3}\dot{5} = \dfrac{235 - 2}{990} = \dfrac{233}{990}$

Exercise 1-5

(A)

State each of the following periodic decimals in fraction form:

1.	(i) .6	(ii) .75	(iii) .72	(iv) .15
2.	(i) $.\dot{2}$	(ii) $.2\dot{4}$	(iii) $.\dot{3}0\dot{5}$	(iv) $.\dot{4}7\dot{2}$
3.	(i) $.2\dot{7}$	(ii) $.1\dot{3}$	(iii) $.3\dot{2}\dot{4}$	(iv) $.09\dot{2}$
4.	(i) $.2\dot{3}\dot{3}$	(ii) $.4\dot{5}6\dot{2}$	(iii) $3.\dot{3}\dot{2}$	(iv) $3.4\dot{2}$

(B)

Show by the process of division that the following are equivalent to periodic decimals:

5. $3\frac{1}{4}$	**6.** $8\frac{2}{3}$	**7.** $\dfrac{1}{13}$	**8.** $\dfrac{13}{27}$
9. $\dfrac{28}{9}$	**10.** $\dfrac{130}{7}$	**11.** $-33\frac{1}{5}$	**12.** $-\dfrac{27}{31}$

Show that the following represent rational numbers:

13. $.\dot{3}$	**14.** $.\dot{2}\dot{3}$	**15.** $3.\dot{2}$	**16.** 1.345
17. $2.\dot{7}82\dot{5}$	**18.** $10.23\dot{2}\dot{7}$	**19.** $0.3524\dot{1}$	**20.** $1.4\dot{8}3\dot{7}$
21. $32.\dot{6}21\dot{3}$			

Show that the following pairs of periodic decimals represent the same number:

22. .7 and $.6\dot{9}$ **23.** .64 and $.63\dot{9}$

24. Copy the table below and check off the properties possessed by the indicated number systems.

PROPERTY		N	N_0	I	Q
Addition:	(1) Closed				
	(2) Commutative				
	(3) Associative				
	(4) Identity				
	(5) Inverse				
Multiplication:	(6) Closed				
	(7) Commutative				
	(8) Associative				
	(9) Identity				
	(10) Inverse				
	(11) Distributive				

***1·11* The density of rational numbers.** The rational numbers may be considered as coordinates of points on a number line. Such a number line may be called the *rational number line.* It is both interesting and essential to note that between any two rational numbers there is another rational number, and hence between any two points on the rational number line there is another point with a rational coordinate. This is suggested by the following examples.

Between 1 and 2 is the rational number $\frac{1+2}{2} = \frac{3}{2}$.

If $a, b \in Q$ and $a < b$, then between a and b lies the rational number $\frac{a+b}{2}$.

Between $\frac{3}{2}$ and 2 is the rational number $\frac{\frac{3}{2}+2}{2}=\frac{7}{4}$, and so on, without limit.

Between $\frac{a+b}{2}$ and b lies the rational number $\frac{\frac{a+b}{2}+b}{2}=\frac{a+3b}{4}$, and so on, without limit.

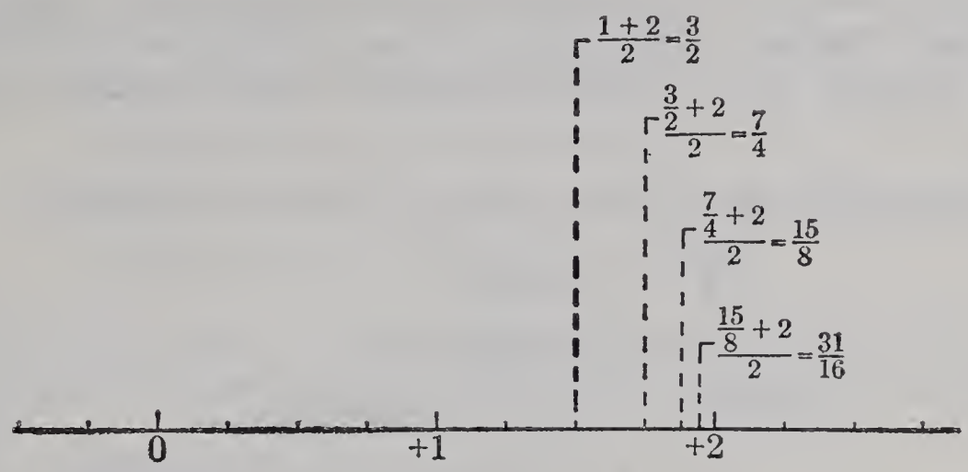


Fig. 1-1

To describe mathematically the fact that between any two rational numbers there is another rational number, we say the rational numbers are *dense*.

It might appear from this discussion that not only is there a distinct point on a line corresponding to each rational number, but that to each point on that line there corresponds a rational number coordinate. In other words, it appears that there is a one-to-one correspondence between the points on a line and the rational numbers. But this is not so. The points on a line which have rational coordinates are only a small portion of all the points on that line.

***1·12* The set of irrational numbers.** In section 1·11 it was stated that although there is a point on a line which corresponds to each rational number, there is not a rational coordinate for each point on that line. The rational number system is not complete in this sense. It was also observed, although not proved, that each rational number could be represented in the form of a periodic decimal and that each periodic decimal represents a rational number.

It is easy, however, to make up a decimal form which is not periodic in nature. For example, such a number form may be constructed by starting with the periodic decimal:

.32323232 . . .

If the digit 5 is inserted after the first period .32, 55 after the second period, 555 after the third period, 5555 after the fourth period, and so on, the resulting number

$$.325325532555325555\ldots$$

neither terminates nor repeats and is called a *non-periodic decimal.*

Similarly, by inserting different digits in this way, any periodic decimal may be changed into any number of non-periodic decimals. Thus, there is an unlimited number of non-periodic decimals that can be formed from each periodic decimal. Since the numbers represented by these non-periodic decimals are not rational numbers, they are called *irrational numbers.*

Numerals such as the following represent irrational numbers:

$$\sqrt{2} = 1.4142136\ldots$$
$$\sqrt{3} = 1.73205080\ldots$$
$$\sqrt[3]{1001} = 10.003333222283909464\ldots$$
$$\pi = 3.14159265358979323846264338 32\ldots$$

The following example illustrates that the irrational numbers are coordinates of definite points on the number line.

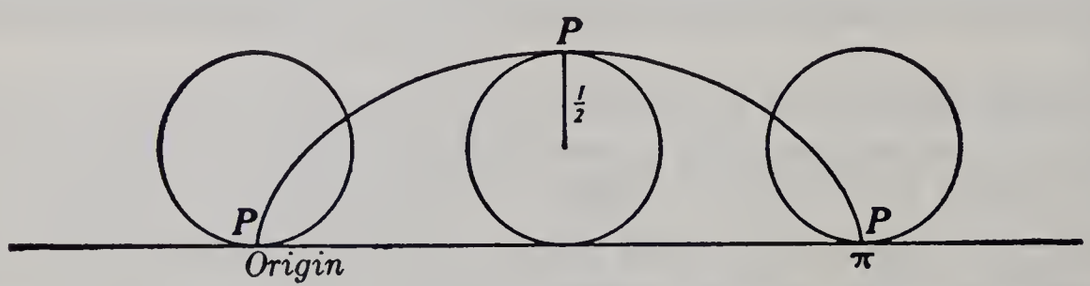


Fig. 1-2

A circle with radius $\frac{1}{2}$ unit (*Fig. 1-2*) has a circumference of $(2\pi \times \frac{1}{2})$ units $= \pi$ units. If such a circle is rolled, without slipping, for one complete revolution along a number line starting with a point P on the circumference coinciding with the zero point on the line, then a point corresponding to π units is determined. This point has the irrational coordinate π.

1·13 **Finding the point corresponding to an infinite decimal.** Theoretically, we may determine the point corresponding to

$$\pi \text{ or } 3.1415926535\ldots$$

by considering rational approximations to this number. For example, 3.14 is a *rational approximation* to the infinite decimal because:

(i) it is a rational number (a terminating decimal);

(ii) it is an approximation obtained by rounding off the infinite decimal to two decimal places.

Evidently π lies between 3.1 and 3.2, both of which represent rational numbers and are the coordinates of points which can be determined on a rational number line.

Similarly, π lies between 3.14 and 3.15,

π lies between 3.141 and 3.142,

π lies between 3.1415 and 3.1416,

. . .

π lies between 3.141592 and 3.141593.

Each of these pairs of numbers determines a segment of the number line, and each successive segment lies within the previous one and is $\frac{1}{10}$ as long.

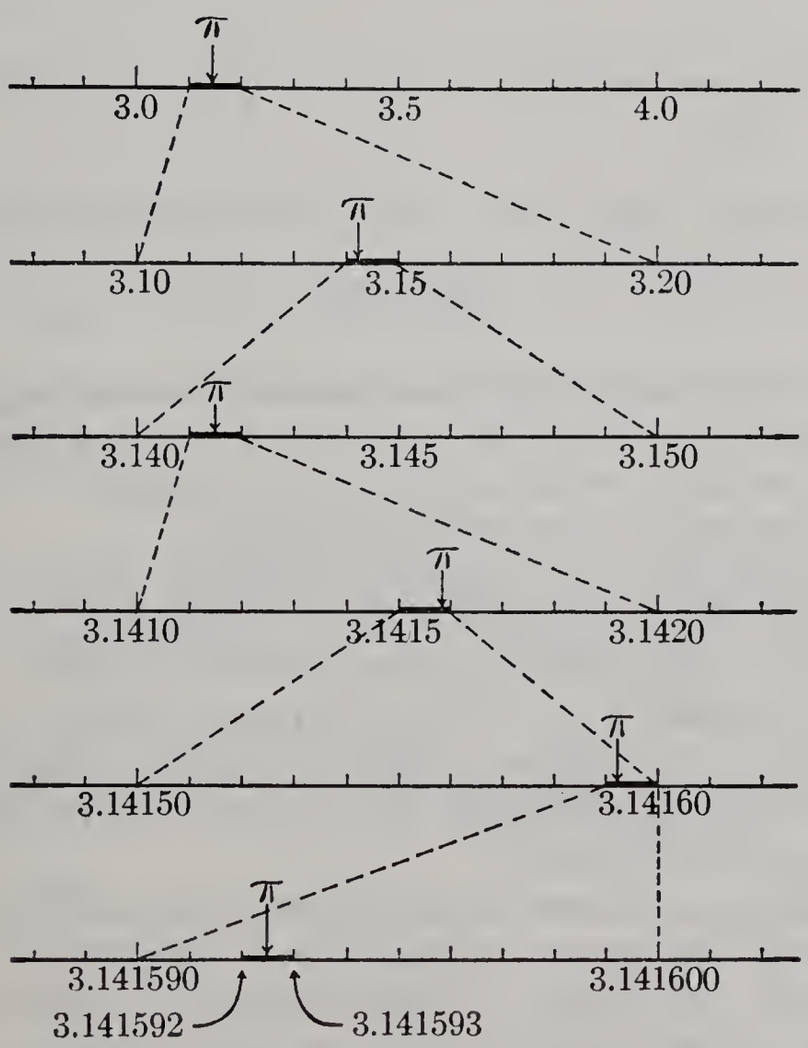


Fig. 1-3

This is illustrated in *Fig. 1-3*. It seems likely that if this process is continued without limit there is just *one point common to all these segments, and this point corresponds to the number* π. The same procedure could be carried out for any infinite decimal. The nature of the method makes us feel certain there is a definite point corresponding to each infinite decimal, and to each point there corresponds an infinite decimal. It will be necessary for us to assume that this is true.

1·14 **The set of real numbers (R).** The set of all decimals (the union of the periodic and non-periodic decimals) represents a set of numbers which is called the *set of real numbers*. This set is complete in the sense that there is a one-to-one correspondence between its elements and the points on a number line. Its composition is shown in *Fig. 1-4*.

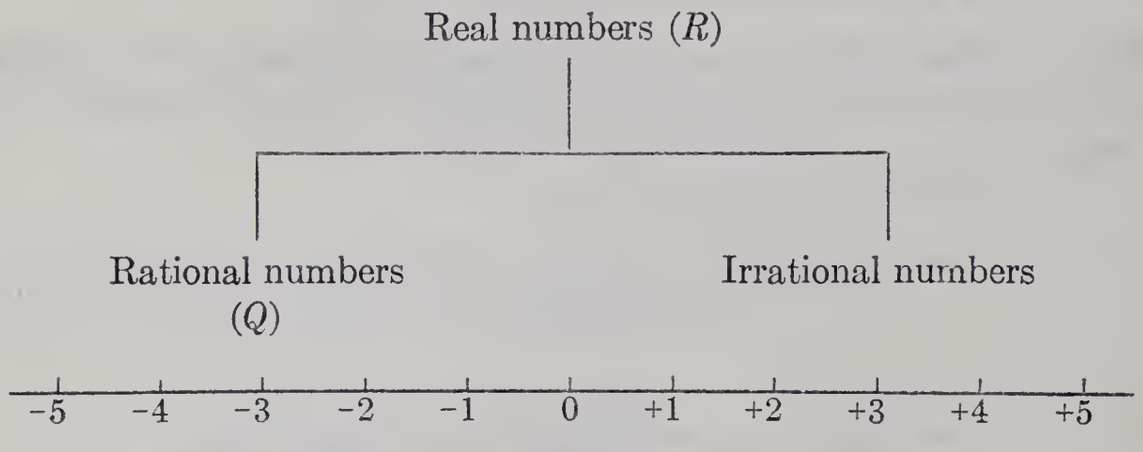


Fig. 1-4

The real number line in *Fig. 1-4* draws attention to the fact that the set of real numbers contains all positive and negative decimals. Each decimal may be paired with its negative: for example, the real number represented by a may be paired with its negative, $-a$.

Whenever operations with irrational numbers in decimal form are required, rational approximations are used. For example, to find the area of a circle (πr^2), useful rational approximations to the area of the circle are obtained by using $\frac{22}{7}$, or 3.14, or 3.142, or 3.1416 as rational approximations for π. Similarly, any infinite decimal is rounded off to provide a suitable rational approximation for use under given circumstances, according to the following rule.

> *In rounding off, discard all unnecessary digits and add* 1 *to the last retained digit whenever the next digit on its right is* 5 *or more, except when a single digit* 5 *is being discarded, in which case it is customary to round off to the nearest even digit.*

It is assumed that the definitions and properties of addition and multiplication of rational numbers are extended to the real numbers.

***1·15* Summary.** The set of real numbers, R, which consists of the set of rational numbers, Q, and the set of irrational numbers, has subsets which, under addition and multiplication, behave in exactly the same manner as the integers and the natural numbers. In this sense, the set of real numbers is an extension of each of these sets, retaining the basic properties of them and providing a set of numbers closed with respect to addition, multiplication, subtraction, and division.

The set of real numbers has:

(i) an identity element for addition (zero);

(ii) an identity element for multiplication (unity);

(iii) a negative or additive inverse for each element;

(iv) a reciprocal or multiplicative inverse for each element, except zero.

The steps in the development from the natural numbers to the real numbers are listed below:

(i) $N = \{1, 2, 3, \ldots\}$

(ii) $N_0 = \{0, 1, 2, 3, \ldots\}$

(iii) $I = \{\ldots, -2, -1, 0, +1, +2, \ldots\}$
$= \{\pm a \mid a \in N\} \cup \{0\}$

(iv) $Q = \left\{\frac{a}{b} \mid a, b \in I, b \neq 0\right\}$

(v) R is the union set of Q and the set of irrational numbers, that is, the set of all decimals.

The *laws of the real number system* are listed below:

If $a, b, c \in R$, then

	ADDITION	MULTIPLICATION
Closure Law	*$a + b$ represents one and only one real number* (*ClA*)	*ab represents one and only one real number* (*ClM*)
Commutative Law	$a + b = b + a$ (*CA*)	$ab = ba$ (*CM*)
Associative Law	$(a+b)+c = a+(b+c)$ (*AA*)	$(ab)c = a(bc)$ (*AM*)
Distributive Law	$a(b + c) = ab + ac$ (*D*)	
Identity Elements	*Zero*	*Unity*
Inverse Elements	$-a$	$\frac{1}{a}$, $a \neq 0$

Order: If $a, b \in R$, then

$a > b$ if $a - b$ is a positive real number, or

$a > b$ if $a = b + c$, where c is a positive real number.

Trichotomy property: $a < b$ or $a = b$ or $a > b$.

Transitive property: If $a > b$ and $b > c$, then $a > c$.

Completeness property: The real number system is complete, that is, each point on a number line corresponds to a real number, and each real number corresponds to a point on the number line.

Unless otherwise specified, it will be understood that the variables are real numbers in the remainder of this text.

Exercise 1-6

(A)

1. Describe in set notation the sets N, N_0, I, Q, R .
2. Name two important subsets of R .
3. From the periodic decimal $2.\dot{3}\dot{4}$ form four different non-periodic decimals. Discuss the statement: "There are infinitely many more irrational numbers than rational numbers."
4. Of what elements is the set of real numbers composed?
5. What is a periodic decimal?
6. What assumption was made concerning (i) the real numbers and points on a number line (ii) points on a number line and real numbers?
7. How are the real numbers ordered? What is meant by saying $a > b$ where a, b represent real numbers?
8. State one rational number and one irrational number between the irrational numbers: $a = 6.3274108\ldots$ and $b = 5.3725038\ldots$.
9. State one rational number and one irrational number between the rational numbers: $c = 0.32323232\ldots$ and $d = 0.65372372\ldots$.

(B)

State rational approximations for the following real numbers correct to 2, 3, 4 decimal places:

10. $3.14159\ldots$	**11.** $1.41426\ldots$	**12.** $15.345726\ldots$
13. $17.432768\ldots$	**14.** $2.1\dot{5}$	**15.** $18.\dot{2}\dot{3}$
16. $15.2\dot{3}\dot{5}$	**17.** $3.\dot{6}72\dot{5}$	

***1·16* Powers, definition of a^m, $m \in {}^+I$.** It is convenient shorthand to write

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5$$

where the *exponent* 5 indicates that the *base* 2 is to be taken as a factor

5 times. 2^5 or 32 is called the *fifth power of* 2. 2^5 is read, "2 (raised) to the exponent five".

This example leads to the general definition:

If $a \in R$ ***and*** $m \in {}^+I$, ***then***

$$a^m \text{ means } a \cdot a \cdot a \ldots \text{ to } m \text{ factors.}$$

a^m is called a *power* and is referred to as the *mth power of a.*

1·17 **The law of exponents for multiplication for a positive integral exponent.** The product

$$\begin{aligned} x^3 \times x^4 &= (x \cdot x \cdot x)(x \cdot x \cdot x \cdot x) && \text{(Definition)} \\ &= x^7 && \text{(Definition)} \\ &= x^{3+4}. \end{aligned}$$

This and similar examples suggest the general law:

$$a^m \times a^n = a^{m+n}, \quad m, n \in {}^+I$$

which is called the *law of exponents for multiplication.*

Example 1. Simplify: (i) $a^7 \cdot a^8$ (ii) $x^4 \cdot x^2 \cdot x^5$

Solution. (i) $a^7 \cdot a^8 = a^{7+8} = a^{15}$ (ii) $x^4 \cdot x^2 \cdot x^5 = x^{4+2+5} = x^{11}$

Example 2. Simplify the product $(x^2y^3)(x^4y^5)$.

Solution.

$$\begin{aligned} (x^2y^3)(x^4y^5) &= x^2 \cdot y^3 \cdot x^4 \cdot y^5 \\ &= x^2 \cdot x^4 \cdot y^3 \cdot y^5 && (CM) \\ &= (x^2 \cdot x^4)(y^3 \cdot y^5) && (AM) \\ &= x^{2+4} \cdot y^{3+5} && \text{(Multiplication law)} \\ &= x^6y^8 \end{aligned}$$

Example 3. Find the 4th power of x^2.

Solution.

$$\begin{aligned} (x^2)^4 &= x^2 \cdot x^2 \cdot x^2 \cdot x^2 && \text{(Definition)} \\ &= x^8 && \text{(Multiplication law)} \\ &= x^{2 \cdot 4} \end{aligned}$$

This and similar examples suggest the general law:

$$(a^n)^m = a^{nm}, \quad n, m \in {}^+I$$

which is called the *power law.*

Example 4. Find the 3rd power of $3xy$.

Solution.

$$\begin{aligned} (3xy)^3 &= (3xy)(3xy)(3xy) && \text{(Definition)} \\ &= 3^3x^3y^3 && \text{(Multiplication law)} \\ &= 27x^3y^3 \end{aligned}$$

This and similar examples suggest the general law:

$$(a \cdot b)^m = a^m \cdot b^m, \quad m \in {}^+I$$

which is called the *law of exponents for the power of a product.*

Write equivalent expressions for each of the following and check your solutions with those on page 467.

1. x^4 **2.** $(a \cdot b)^5$ **3.** $(a^2b^3)(a^4b^5)$ **4.** $(d^3)^5$

5. $(a^2b^3)^4$ **6.** $(3abcd)^2$ **7.** $32a^5b^{10}c^{15}$ **8.** $(-a^2)^3$

1·18 Summary of the laws of exponents involving multiplication.

If $a, b \in R$ and $m, n \in {}^+I$, then

(i) multiplication law: $a^m \times a^n = a^{m+n}$

(ii) power law: $(a^m)^n = a^{mn}$

(iii) power of a product: $(a \cdot b)^m = a^m b^m$.

Exercise 1-7

(Variables are real numbers for which the expressions are defined.)

(A)

1. State the meaning of:

(i) m^5 (ii) $(2a)^4$ (iii) $(3 \times 2)^3$

(iv) $(ab)^2$ (v) a^2b^2 (vi) $(a^2)^2$

(vii) $(b^2c^2)^2$ (viii) $3a^2b^3$ (ix) $a^4 \times a^6$

2. Simplify or state an equivalent form for:

(i) $3a^2 \times 4a^2$ (ii) $x^2y^3 \times x^3y^2$ (iii) $3m^4y^5 \times 2 \times m^2y^2$

(iv) $a^2 \times (2a)^3b^2$ (v) $5p^2q^3r^4 \times 3p^3qr^2$ (vi) $(ap)^5$

(vii) $b^4c^4d^4$ (viii) $(a^2b^3)^2$ (ix) $7p^3 \times 8p^2 \times p^4$

3. State the second and third power of each of the following:

(i) 4 (ii) $5a$ (iii) $3a^4$

(iv) $6ab^2$ (v) $3xy^3$ (vi) $2x^2y^5$

(B)

4. Simplify:

(i) $3a^2b \times 4a^3b^2$ (ii) $3(a^2b^3)^2$ (iii) $3a^2(b^3)^3$

(iv) $5p^3q^4 \times 6p^7q^{11}$ (v) $(2x^4y^6)^3$ (vi) $5m^5(p^4)^5$

5. Find:

(i) the fifth power of $2x^2y^3$ (ii) the 7th power of $3xy^2z^3$

6. Supply the missing factor in each of the following:

(i) $36x^8y^4z^6 = (9x^2y^2z^2)(\quad)$ (ii) $42a^4b^2c^3 = (7a^4bc^3)(\quad)$

(iii) $27a^3b^6c^9 = (3ab^2c^3)(\quad)$ (iv) $35a^2bc^3 = (7ac)(\quad)$

(v) $18x^4y^3z = (-6x^2y)(\quad)$ (vi) $24m^5p^3q^4 = (4m^3p^3q^3)(\quad)$

1·19 The law of exponents for division. Zero exponent.

a. Since division is the inverse operation of multiplication,

then $\because\ x^2 \times x^3 = x^5$ (Multiplication law)

$\therefore\ x^5 \div x^3 = x^2, \quad x \neq 0$

$= x^{5-3}$.

This and similar examples suggest the general law:

$$a^m \div a^n = a^{m-n}, \quad m > n,\ m, n \in {}^+I$$

b. $\because\ 1 \times x^3 = x^3$

$$\therefore\ \frac{x^3}{x^3} = 1,\ x \neq 0$$

This and similar examples suggest the general law:

$$a^m \div a^m = 1, \quad m \in {}^+I$$

However, if the division law of **a** is applied to this example, then

$$x^3 \div x^3 = x^{3-3},\ x \neq 0$$
$$= x^0, \quad x \neq 0.$$

This and similar examples suggest the definition:

$$a^0 = 1, \quad a \neq 0$$

The following example indicates that the law of exponents for multiplication leads to the same conclusion:

$$x^4 \times x^0 = x^{4+0} \quad \text{(Multiplication law)}$$
$$= x^4$$

which can be true only if $x^0 = 1$.

In deriving the definition from examples involving division, the base could not be zero; therefore this restriction is placed on the definition: 0^0 is not defined.

This definition extends the laws of exponents to allow the exponent to be any non-negative integer.

Example 1. Simplify $\dfrac{x^3}{x^5}$.

Solution. $\dfrac{x^3}{x^5} = \dfrac{x^3 \div x^3}{x^5 \div x^3} = \dfrac{1}{x^2},\ x \neq 0$

Simplify or write equivalent expressions for each of the following and compare your solutions with those on page 467. Consider that the variables are not zero.

1. $a^7 \div a^4$ **2.** $b^7 \times b^0$ **3.** $b^7 \div b^0$

4. $b^0 \times b^0$ **5.** $b^0 \div b^0$ **6.** $c^5 \div c^7$

7. $(x^2y^3) \div (xy^2)$ **8.** $\dfrac{6p^4q^2}{3p^2q^4}$ **9.** $\dfrac{6a^8b^6c^7}{2a^2b^4c^9}$

Example 1. Find an equivalent expression for (i) $\left(\dfrac{a}{b}\right)^3$ (ii) $\dfrac{m^2}{n^2}$ if $b, n \neq 0$.

Solution. (i) $\left(\dfrac{a}{b}\right)^3 = \dfrac{a}{b} \cdot \dfrac{a}{b} \cdot \dfrac{a}{b}$ (Definition)

$= \dfrac{a^3}{b^3}$

(ii) $\dfrac{m^2}{n^2} = \dfrac{m \cdot m}{n \cdot n}$ (Definition)

$= \dfrac{m}{n} \cdot \dfrac{m}{n} = \left(\dfrac{m}{n}\right)^2$

This and similar examples suggest the general law:

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, \quad b \neq 0, \quad \textit{m is a non-negative integer,}$$

which is called the *law of exponents for the power of a quotient.*

Simplify or write an equivalent expression for each of the following and compare your solutions with those on page 467. Consider that the variables are real numbers for which the expressions are defined.

10. $\left(\dfrac{3}{2}\right)^5$ **11.** $\dfrac{4^3}{7^3}$ **12.** $\dfrac{a^2b^2}{c^2}$

13. $\dfrac{x^3y^3}{a^3b^3}$ **14.** $\left(\dfrac{abc}{mnp}\right)^4$ **15.** $\dfrac{8a^3}{125b^3}$

1·20 **Summary of laws of exponents involving division.**

If $a, b \in R$, $a, b \neq 0$, m, n are non-negative integers, then

(i) $a^m \div a^n = a^{m-n}, \quad m \geqq n$

(ii) $a^0 = 1$

(iii) $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$.

Exercise 1-8

(Variables are real numbers for which the expressions are defined.)

(A)

1. State the meaning of:

(i) $a^4 \div a^3$ (ii) $(ab)^0$ (iii) $\left(\frac{m}{p}\right)^3$

(iv) $\frac{m^4}{p^4}$ (v) 0^0 (vi) $\left(\frac{a}{b}\right)^0$

2. Simplify or state an equivalent form for:

(i) $a^{17} \div a^{14}$ (ii) $(6p^5) \div (p^5)$ (iii) $x^5 \div x^8$

(iv) $\left(\frac{m}{n}\right)^3$ (v) $\frac{m^4}{r^4}$ (vi) $\left(\frac{2x}{3y}\right)^0$

(vii) $\frac{r^4s^5}{r^2s^3}$ (viii) $\frac{x^5y^3}{x^4y^4}$ (ix) $\frac{3x^0y^4z^5}{2xy^2z^6}$

(x) $\frac{6x^4y^2}{3x^2y^2}$ (xi) $\frac{2a^3}{2a^2}$ (xii) $\frac{24a^7b^{16}}{2a^5b^{12}}$

(xiii) $\frac{15x^2y^4z^0}{3xy^2}$ (xiv) $\frac{49a^9b^7}{7a^{10}b^7}$ (xv) $\frac{125x^0y^4z^6}{250x^2y^3z^7}$

(B)

3. Find:

(i) $(35a^3b^2) \div (5ab)$ (ii) $x^8 \div x$ (iii) $\frac{15x^4y^3z^5}{3xyz}$

(iv) $\frac{250abcd^2}{25abcd}$ (v) $\frac{121a^4b^2c^0d^5}{11a^2b^2c^2d^2}$ (vi) $\frac{72m^0n^5y^6z^4}{6m^2n^3y^8z^{10}}$

4. Why is the quotient of $(15a^2b^3c) \div (3abc)$ equal to $5ab^2$?

5. Find the following quotients:

(i) $(25a^2) \div (5a^2)$ (ii) $(16x^2y^4) \div (8xy)$ (iii) $(15y^4p^5) \div (3y^2p)$

(iv) $(25a^4y^0) \div (5a^2)$ (v) $(20a^3b^4) \div ((2ab^2)^2)$ (vi) $(0) \div (3abc)$

(vii) $\frac{36x^4y^6z^{12}}{9xyz^{10}}$ (viii) $\frac{10a^0b^2c^4}{2a^0b^0c}$ (ix) $\frac{3x^2y^3 \times 4xyz}{2xy}$

6. What information is obtained by division in the following? Perform the division and interpret the results.
   (i) A rectangle of area $10x^2y^2$ square inches has width $2xy$ inches.
   (ii) A trip of $30m^2n^3$ miles is made at $5m^2$ miles per hour.
   (iii) At $\$7m^2s$ per ton, a quantity of coal costs $\$42m^2s^2$.
   (iv) The cost of x dozen oranges is $\$x$.
   (v) The perimeter of a square is $20m$ inches.

1·21 **The laws of exponents extended to negative integral exponents.** The law of exponents for multiplication for non-negative exponents states

$$a^m \times a^n = a^{m+n}, \qquad a \neq 0.$$

If we wish to extend this law to permit m and n to be any exponents, then it must be true, for example, that:

$$4^6 \times 4^{-6} = 4^{6+(-6)} = 4^0 = 1$$

or, in general, $a^m \times a^{-m} = a^{m+(-m)} = a^0 = 1, \quad m \in I.$

This suggests the definition $a^{-m} = \dfrac{1}{a^m}$.

Then a^m and a^{-m} are reciprocals since their product is 1.

Also, if the division law $a^m \div a^n = a^{m-n}$ for non-negative integral exponents is to be extended to include integral exponents, then it must be true, for example, that

$$(-2)^4 \div (-2)^7 = (-2)^{4-7} = (-2)^{-3}.$$

But we know that $(-2)^4 \div (-2)^7 = \dfrac{(-2)^4}{(-2)^7} = \dfrac{1}{(-2)^3}$.

That is, $(-2)^{-3}$ must equal $\dfrac{1}{(-2)^3}$, and $(-2)^{-3}$ and $(-2)^3$ must be reciprocals.

Thus, the laws

$$a^m \times a^n = a^{m+n},$$
$$a^m \div a^n = a^{m-n}, \quad a \neq 0$$

are extended to include all integral exponents by the definition:

$$a^{-m} = \frac{1}{a^m}, \; a \neq 0, \quad m \in I.$$

Simplify or write an equivalent expression for each of the following and compare your solutions with those on page 468. Consider that the variables are real numbers for which the expressions are defined.

1. a^{-4} 2. $\dfrac{1}{b^{-6}}$ 3. $\dfrac{a^{-2}}{b^{-3}}$ 4. $a^{-2}b^{-3}$

5. $\dfrac{a^2b^3}{a^{-2}b^{-4}}$ **6.** $a^2 \times a^{-3}$ **7.** $(6x^4y^{-5}) \div (2x^2y^{-3})$

8. $4m^5n^{-3} \times 3m^{-2}n^3$ **9.** $\dfrac{12a^{-4}b^{-6}}{(3a^{-1}b^{-3})(4a^{-3}b^{-3})}$

1·22 Summary of the laws of exponents for integral exponents.

If $a, b \in R$, $a, b \neq 0$, $m, n \in I$, then

(i) multiplication law: $a^m \times a^n = a^{m+n}$

(ii) division law: $a^m \div a^n = a^{m-n}$

(iii) zero exponent: $a^0 = 1$

(iv) negative exponent: $a^{-m} = \dfrac{1}{a^m}$

(v) power law: $(a^m)^n = a^{mn}$

(vi) power of a product: $(a \cdot b)^m = a^m \cdot b^m$

(vii) power of a quotient: $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$.

Exercise 1-9

(The variables are real numbers for which the expressions are defined.)

(A)

1. State the definition for a^x, $x \in {}^+I$.

2. State the law of exponents for multiplication.

3. State the law of exponents for division. Why must the base not be zero?

4. State the power law of exponents.

5. State the law of exponents for the power of a product.

6. State the law of exponents for the power of a quotient.

7. State the meaning of:

(i) $(-2)^3$ (ii) $(-2)^4$ (iii) $(3 \times (-2))^5$

(iv) $(-a)^2$ (v) $-(ab)^2$ (vi) $2^{-1}x^2y^{-1}$

(vii) x^{-2} (viii) $(xy)^{-1}$ (ix) $2a^{-3}b^{-2}$

8. Simplify or write in an equivalent form:

(i) $(3a^2)(-4a^3)$ (ii) $(x^2y^3)(x^3y^2)$ (iii) $(-3)^3$

(iv) $(-1)^4a^2 \times (-2)^3b^2$ (v) m^0 (vi) $a^0b^0c^0$

(vii) $m^{-1}n^{-2}p^0$ (viii) $\dfrac{1}{m}$ (ix) $\dfrac{1}{m^{-2}m^{-3}}$

(x) $\frac{1}{x^m}$ (xi) $\frac{1}{x^{-3}y^{-4}z^0}$ (xii) $\frac{6x^4y^{-2}}{3x^2y^{-2}}$

(xiii) $\frac{2^2 \times 2^{-3}}{2^{-5}}$ (xiv) $3^{-2} + 2^{-3}$ (xv) $2^{-4} - 3^{-5}$

(B)

Simplify:

9. $\frac{3x^2y^4}{x^2y^4}$ **10.** $(12a^0b^2c^{-4})(3a^{-1}b^0c^2)$

11. $\frac{35a^4b^2}{7a^5b^{-2}}$ **12.** $\frac{49x^{-5}y^{-6}}{7x^{-6}y^{-5}}$

13. $(15x^4y^0z^6)(3x^{-2}y^{-3}z^{-4})$ **14.** $(3a^{-4}y^0b^6)(4a^4y^0b^{-6})$

15. $(3a^2b^0c^{-2})^2$ **16.** $(2x^4y^{-3}z^0)^3$

17. $(25a^4y^0) \div (-5a^{-2}y^{-2})$ **18.** $(-15x^4z^{-5}y^0) \div (-3x^{-4}y^6)$

19. $(-72m^0n^5y^6) \div (-6m^2n^{-3}y^2)$ **20.** $(121a^4b^{-2}c^0d^{-5}) \div (-11a^2b^2c^2d^2)$

21. Rewrite and supply the missing factor in each of the following:

(i) $35a^2b^0c^{-2} = (7abc)(\quad)$ (ii) $-18x^4y^{-3}z^0 = (6x^2yz)(\quad)$

(iii) $24m^5p^{-3}q^4 = (-4m^3p^2q^{-2})(\quad)$

(iv) $56x^2y^{-4}z^5 = (28xy^{-4}z^{-5})(\quad)$

22. Simplify: $\frac{1}{5^{-2}} - \frac{1}{2^{-3}}$ **23.** Simplify: $3^{-2} + 2^{-3}$

1·23 **Scientific notation; numerals in standard form.** It is often necessary, particularly in scientific calculation, to use very large or very small positive numbers.

The mean distance of the earth from the sun is 93,000,000 miles. For convenience in expressing such numbers, and in computing with them, this distance is usually expressed in the form 9.3×10^7 miles.

The nucleus of the hydrogen atom is approximately 0.000,000,000,000,1 cm. in diameter. This small number may be expressed in the form

$$1.0 \times 10^{-13}.$$

Thus, powers of ten may be used to help express large and small positive numbers in a convenient form.

If a positive number is expressed as the product of a decimal numeral equal to, or greater than, 1 and less than 10 and a power of 10, it is said to be in standard form.

If a positive number is

greater than 1, the exponent of the base 10 is positive;
equal to 1, the exponent of the base 10 is zero;
less than 1, the exponent of the base 10 is negative.

Exercise 1-10

(B)

The following is a list of objects the approximate lengths of which are given in centimetres. Express these lengths in standard form:

1.	Diameter of the hydrogen atom	0.000,000,01 cm.
2.	Distance between molecules of ferrous sulphide	0.000,000,1 cm.
3.	Diameter of molecular ring of nitrocellulose	0.000,01 cm.
4.	Length of diphtheria bacterium	0.000,3 cm.
5.	Diameter of particles of fine printer's ink	0.000,4 cm.
6.	Diameter of human red blood cell	0.000,8 cm.
7.	Thickness of the thinnest plastic	0.001 cm.
8.	Length of a plant cell	0.01 cm.
9.	Diameter of a snow-flake	0.25 cm.
10.	Diameter of a ten-cent piece	1.8 cm.
11.	Height of an average man	175 cm.
12.	Height of the Empire State Building	33,000 cm.
13.	Diameter of the earth	506,880,000 cm.
14.	Diameter of the sun	553,836,800,000 cm.
15.	Diameter of the solar system	130,000,000,000,000 cm.

Express the following magnitudes in standard form:

16.	Time for an electron to revolve around the nucleus of a hydrogen atom	0.000,000,000,000,000,1 sec.
17.	Time of the fastest camera exposure	0.000,000,1 sec.
18.	Time of atom bomb explosion	0.000,01 sec.
19.	Time for a house-fly to beat his wings once	0.001 sec.
20.	Time for a human heart beat	1.0 sec.
21.	Human life span	2,000,000,000 sec.
22.	Age of the earth's oldest rocks	100,000,000,000,000,000 sec.

Express the following magnitudes in expanded form:

23.	Mass of the earth	6.6×10^{21} tons
24.	Mass of the sun	2.1×10^{27} tons
25.	Mass of the moon	8.1×10^{19} tons
26.	Mean distance from the earth to the moon	2.4×10^{5} miles
27.	Coefficient of linear expansion of copper	1.7×10^{-5}
28.	Mass of an electron	9×10^{-28} grams
29.	Electrical charge of one electron	1.6×10^{-19} coulombs
30.	Number of electrons in 1 coulomb	6.3×10^{18}
31.	Radio station CBL	
	(i) AM frequency	7.40×10^{5} cycles/sec.
	(ii) FM frequency	9.1×10^{7} cycles/sec.
32.	Velocity of light	3×10^{8} metres/sec.
33.	Unit of light wave length (angstrom unit)	10^{-8} cm.

In the following practice exercises, the variables are real numbers for which the expressions are defined.

Practice Exercise 1-11

(B)

Simplify:

1. $(-5)(-ax^2)(-ax)^2$

2. $(-3m)^2(5m)(-2m)$

3. $\dfrac{-18a^7b^8}{-3a^7}$

4. $\dfrac{-32m^5a}{16m^2a}$

5. $\dfrac{0}{-abc}$

6. $\dfrac{-10^5a^6b^4}{-10^2a^5b^4}$

7. $\dfrac{-27x^6bc}{-3x}$

8. $\dfrac{25a^5b^7}{50a^2b^2}$

9. $(-39x^4y^5) \div (-13x^2y)$

10. $(-45x^4y^5z^6) \div (9x^2y^2z^2)$

11. $(162a^{16}b^{12}) \div (-27a^{12}b^6)$

12. $(-343x^7y^3) \div (-49x^2y^3)$

Expand:

13. $-3[(-5) + (-4)]$

14. $(3a + 2b)(-4c)$

15. $(4a)(3a - 2b + 3c)$ **16.** $(-4b)[6m + (-3p) - 4c]$

17. If $a = -2$, $b = 3$, find:

(i) $3a - 2b + (5a + b) - (6a - 4b)$

(ii) $(5a^2 + ab + b^2) - (a^2 - 2ab + 3b^2)$

18. Show that $\left(\frac{-x}{y} \times \frac{a}{-b}\right) \times \frac{-ay}{-xb} = \frac{-x}{y} \times \left(\frac{a}{-b} \times \frac{-ay}{-xb}\right)$.

19. Arrange the elements of the following set in increasing order of magnitude: $\left\{\frac{5}{-16}, \frac{-3}{8}, \frac{3}{-4}, \frac{2}{3}, \frac{-7}{-8}\right\}$.

Practice Exercise 1-12

(B)

Simplify:

1. $\frac{15}{-45}$ **2.** $\frac{-24abc}{8c}$ **3.** $\frac{8a}{6a} + \frac{-6a}{24a}$

4. $\frac{(-5)(-12)}{-30}$ **5.** $\frac{-5}{-35} \div \frac{-15}{21}$ **6.** $\frac{ab}{cd} \div \frac{-ac}{fd}$

7. $\frac{(-a)(-b)(-1)}{3a}$ **8.** $\frac{1}{xy} + \frac{-5}{x} - \frac{4}{y}$ **9.** $m \div \frac{-m^2n}{n^2m}$

10. $\left(\frac{1}{-a}\right)\left(\frac{-ab}{c}\right)\left(\frac{-c}{cb}\right)$ **11.** $\frac{-3 \div 4}{27 \div (-3)}$

12. $\frac{(-6) - (-12)(20)}{(-3)(15) \div (-2)}$ **13.** $\frac{-\frac{5}{6}}{-\frac{25}{24}}$

14. $\frac{\frac{-1}{xy}}{(x^2y^3) \div (x^2y)}$ **15.** $\frac{\frac{2}{bc} + \frac{-1}{bc}}{-\frac{4}{abc}}$

16. Prove $\frac{(-m^2)(-a)^2}{m^3(-a)^3} = \frac{m(-a)^3}{(a^2)^2(-m^2)}$.

17. Prove $\left(\frac{-5}{16}\right)\left(\frac{56}{-15}\right)\left(\frac{-10}{7}\right) = \frac{8bc}{-3ba} \times \frac{10ax}{16cx}$, $a, b, c, x \neq 0$.

18. Arrange the elements of the following set in decreasing order of magnitude: $\left\{\frac{-5}{6}, \frac{4}{3}, \frac{-3}{-2}, \frac{0}{2}, \frac{1}{-3}, \frac{-3}{8}\right\}$.

Practice Exercise 1-13

(B)

1. Find each of the following when $a = -2$, $b = -3$:

(i) a^2b^{-3} (ii) $2a^{-2}$ (iii) $\frac{a^{-3}}{b^{-2}}$

(iv) $\frac{ab^{-2}}{a^{-1}b}$ (v) $\frac{9a^{-2}}{2b^2}$ (vi) $\frac{8a^{-3}}{-9b^{-2}}$

(vii) $\frac{1}{a^{-1}} + \frac{1}{b^{-1}}$ (viii) $\frac{a}{a^{-1}} + \frac{b}{b^{-1}}$ (ix) $\frac{1}{a^{-2}} + \frac{b^3}{a^{-1}}$

2. Find each of the following when $x = -2$, $y = -1$, $z = 2$:

(i) x^y (ii) $(xy)^z$ (iii) x^zy^{-z}

(iv) $\left(\frac{x}{y}\right)^z$ (v) $(y^{-1}z^{-1})^x$ (vi) $(x^2y^2z^2)^{-x}$

(vii) $(xy)^{yz}$ (viii) $(x^{-1}y^{-1}z^{-1})^y$ (ix) $(x^0y^0)^{-z}$

(x) $(x^{-z}y^{-z})^0$ (xi) $\left(\frac{1}{x^{-1}}\right)^0 + \left(\frac{1}{y^{-1}}\right)^0$ (xii) $\frac{x^0}{y^0} + \frac{y^0}{x^0}$

3. Express each of the following in expanded form:

(i) 5.54×10^{11} (ii) 1.3×10^4 (iii) 7.2×10^{-2}

(iv) 3.3×10^2 (v) 1.0×10^{-4} (vi) 8.2×10^{-1}

(vii) 2.5×10^1 (viii) 6.5×10^{-8} (ix) $\frac{6.3 \times 10^{-5}}{2.1 \times 10^{-5}}$

(x) $\frac{6.8 \times 10^4}{3.4 \times 10^{-2}}$ (xi) $\frac{(2.0 \times 10^{-3})(3.2 \times 10^2)}{(1.6 \times 10^{-2})(5.0 \times 10^3)}$

4. Express each of the following in standard form:

(i) 6,600,000,000 (ii) 93,000,000

(iii) 1,000,000,000 (iv) 0.000,075

(v) 0.000,000,000,23 (vi) 280

(vii) 0.28 (viii) 0.0372

(ix) 428.6 (x) $7{,}250.0 \times 10^3$

(xi) $18{,}000 \times 10^{-4}$ (xii) 0.005×10^{-6}

Chapter II

EQUATIONS AND INEQUATIONS

2·1 Solving equations of the first degree in one variable. Sentences like $x - 4 = -2$ are called *equations of the first degree in one variable*, because the exponent of the variable is unity. The variable, represented by x, is an element from a replacement set which is always specified (R, unless otherwise indicated). To solve such an equation means to determine the number (or numbers) from the replacement set which makes the sentence true: that is, makes the number represented by $x - 4$ (Left Side, L.S.) equal to -2 (Right Side, R.S.). In this example, the replacement for x which makes the sentence true is 2. 2 is called the *root* of the equation and $\{2\}$ the *solution set*.

It may be observed by inspection that in the following set of equations

(i) $x = 2$ (ii) $4x = 8$ (iii) $x + 3 = 5$

(iv) $x - 4 = -2$ (v) $\frac{x}{2} = 1$

each has the same root 2 and the same solution set $\{2\}$.

Since each equation has the same root 2, we should be able to obtain one equation from the other using the number properties.

The Closure Law for Multiplication in the set of real numbers states:

If $a, b \in R$, then $a \cdot b$ is a *unique* real number.

Thus, from equation (ii), if $x \in R$, then $4x$ is a *unique* real number. Also, since $2 \in R$, then $4 \cdot 2$ is the unique real number 8. But equation (i) states that x equals 2, and, hence, by the uniqueness property of the closure law for multiplication, $4x$ and 8 represent the same real number.

Thus, if $x = 2$, then $4x = 8$. *(ClM)*

Similarly, the closure law for addition, subtraction, and division in the set of real numbers allows (iii), (iv), and (v) to be obtained from equation (i).

Thus, if $x = 2$, then $x + 3 = 2 + 3$, or $x + 3 = 5$ *(ClA)*

if $x = 2$, then $x - 4 = 2 - 4$, or $x - 4 = -2$ *(ClS)*

if $x = 2$, then $\frac{x}{2} = \frac{2}{2}$, or $\frac{x}{2} = 1$. *(ClD)*

The same laws permit the reverse steps to be taken:

if $4x = 8$, then $\frac{4x}{4} = \frac{8}{4}$, or $x = 2$ *(ClD)*

if $x + 3 = 5$, then $x + 3 - 3 = 5 - 3$, or $x = 2$ *(ClS)*

if $x - 4 = -2$, then $x - 4 + 4 = -2 + 4$, or $x = 2$ *(ClA)*

if $\frac{x}{2} = 1$, then $\frac{x}{2} \times 2 = 2 \cdot 1$, or $x = 2$ *(ClM)*

Thus, it can be said, for example,

if $x = 2$, then $4x = 8$, and

if $4x = 8$, then $x = 2$.

Since both *if . . . then* statements can be made, we say the two statements $x = 2$ and $4x = 8$ are *equivalent.* This implies, in the case of equations, that they have the same solution set.

Similarly, by applying the uniqueness property of closure for the four operations in the set of real numbers, each of the equations (i) to (v) may be shown to be equivalent. Such equations are called *equivalent equations.* They have the same solution set.

In solving equations, the solution set may be obtained by inspection in simple cases, but usually the solution set is obtained by forming a succession of equivalent equations leading to an equation, such as $x = 2$, from which the solution set is obtained.

To solve the equation $2x - 3 = 9$, $x \in R$, we note

	if	$2x - 3 = 9$	
(add 3 to each side)	then	$2x = 12$	*(ClA)*
(divide each side by 2)	then	$x = 6$.	*(ClD)*

Thus (i) if $2x - 3 = 9$, then $x = 6$.

It remains to be shown that

(ii) if $x = 6$, then $2x - 3 = 9$.

Statement (ii) is proved by showing that the steps taken in proving (i) are reversible as follows:

	if	$x = 6$	
(multiply each side by 2)	then	$2x = 12$	(*ClM*)
(subtract 3 from each side)	then	$2x - 3 = 9$.	(*ClS*)

Thus, if $x = 6$, then $2x - 3 = 9$.

It has now been shown that $x = 6$ and $2x - 3 = 9$ are equivalent equations. Therefore 6 is the root and $\{6\}$ the solution set of the equation $2x - 3 = 9$, $x \in R$.

When the closure property for addition, subtraction, multiplication, and division in the set of real numbers is used to write a sequence of equations, the steps are always reversible, and thus the equations formed are always equivalent. Thus, it is not necessary to write the second part of the solution. It is customary to verify the number (or numbers) obtained as follows:

Verification. L.S. $= 2 \times 6 - 3$ R.S. $= 9$
$= 9$

The verification shows that 6 *satisfies* the equation.

In general, given an equation

$$a = b,$$

equivalent equations may be formed by applying the closure property for addition, subtraction, multiplication, and division in the set of real numbers.

Thus, $a = b$ and

(i)	$a + c = b + c$	(Addition property)
(ii)	$a - c = b - c$	(Subtraction property)
(iii)	$a \times c = b \times c$	(Multiplication property)
(iv)	$a \div c = b \div c, c \neq 0$	(Division property)

are equivalent equations. These properties are properties of the equality relation between two real numbers.

Example. Solve $7x - 4 = 2x + 21$, $x \in R$ and verify.

Solution.		$7x - 4 = 2x + 21$	
(add 4 to each side)		$7x = 2x + 25$	(Addition property)
(subtract $2x$ from each side)	$7x - 2x$	$= 25$	(Subtraction property)
(by distributive property)		$5x = 25$	(*D*)
(divide each side by 5)		$x = 5$	(Division property)

The root is 5 and the solution set is $\{5\}$.

Verification. L.S. $= 7 \times 5 - 4$ R.S. $= 2 \times 5 + 21$
$= 31$ $= 31$

In the worked examples of the text, the authorities which are most significant in the development are given in brackets. When writing a solution you should think the authorities for the steps, but it is not necessary to include them in your solution.

It is sometimes useful and convenient to describe a set using *set-builder notation.* For example, the set of all real numbers which makes the equation $3x + 2 = 8$ true may be represented by

$$\{x \mid 3x + 2 = 8,\ x \in R\}$$

which is read, "the set of all x such that $3x + 2 = 8$ and $x \in R$".

The symbol $\{\ \}$ is read, "the set of";
the solidus $\mid$ is read, "such that" or "defined by";
the comma , means "and";
the sentence $3x + 2 = 8,\ x \in R$ is called the *defining sentence* of the set.

In this text it is understood that the variables are real numbers unless otherwise specified. Thus, $\{x \mid 3x + 2 = 8,\ x \in R\}$ is written $\{x \mid 3x + 2 = 8\}$. This notation is used in some of the examples in the text and in the exercises.

Write solutions to the following problems and compare them with those on page 468. *In question* 1 *show that the steps are reversible, and in question* 2 *and* 3 *verify by substitution.*

1. Solve $(10x + 6) - (11 - 15x) = 20x$.

2. Find $\{x \mid 8x + (x - 7) + 3 = 2x - (3 + x)\}$.

3. Find the solution set defined by
$3(x - 5) - 2(3 + 4x) = 8 - 6(2x - 1)$.

Unless otherwise stated, the number symbols in the exercises which follow represent real numbers.

Exercise 2-1

(A)

In each of the following, the given equations are equivalent. State how each equation has been obtained from the preceding equation:

1. $5x - 4 = 7, \quad 5x = 11, \quad x = 2\frac{1}{5}$

2. $\frac{x}{2} + 3 = 5, \quad \frac{x}{2} = 2, \quad x = 4$

3. $5x = 2x - 3, \quad 5x - 2x = -3, \quad 3x = -3, \quad x = -1$

4. $\frac{x}{3} - 2 = 4, \quad \frac{x}{2} = 6, \quad x = 12$

Form equivalent equations for each of the following by performing the indicated operation:

5. $3x = 6$, division by 3

6. $6x - 4 = 7$, addition of 4

7. $6x + 2x = 4x - 3$, by the distributive property

8. $x = 3x - 7$, multiplication by 2

9. $7x + 4 = 2x - 3$, subtraction of $2x$

Solve the following equations orally, stating the equivalent equations formed in the process and the authority for each statement:

10. $5x - 5 = 50$

11. $3y = y + 36$

12. $2x = 18 - x$

13. $4n = 5n - 81$

14. $5w = 80 - 3w$

15. $x + 5 = 20 - 4x$

16. $11y + 2 = 26 + 7y$

17. $6a - 3a = a + 10$

18. $18z - 5 = 2z + 27$

19. $5(x - 2) = 25$

(B)

Solve and verify:

20. $2x - 5 - (x - 3) = 7$

21. $5x + 3(8 - x) = 36$

22. $x - 5 = 8 - (2x + 4)$

23. $(x^2 + x + 4) - (x^2 - 1) = 0$

Solve:

24. $(6x + 1) - 3 = 12x - (8x - 4)$

25. $7(4y + 5) + 9 = 8(3y + 5)$

26. $5(3y - 4) = -14 - 3(y - 10)$

27. $2x^2 + x - 16 = 2(x^2 - 4x + 4) - 2(x + 1)$

28. $6(x + 1) - 3 = 12x - 4(2x - 1)$

29. $6x - 4(5x + 9) = 4(2x - 9)$

Find the solution set defined by each of the following:

30. $8(y - 1) + 17(y - 3) = 4(4y - 9) + 4$

31. $8(n + 3) + (6 + 2n) = 2(n - 2) + 5(5 + n)$

32. Find $\{x \mid 5x - (3x - 7) + 2x + (2x - 3) = 14\}$

33. Find $\{x \mid 352 - 11(x - 9) = 4x - 44\}$

34. Find $\{m \mid 12 + 3(m + 4) = 5m - 2\}$

35. Find $\{n \mid 3n - (6 - 2n) = 5 - 2(2 - 4n)\}$

2·2 Equations with fractional coefficients. The method of solution is illustrated in the following example.

Example. Solve $\frac{1}{2}x + \frac{1}{3}x = 10$ and verify.

Solution.

METHOD 1

$$\frac{1}{2}x + \frac{1}{3}x = 10$$

(add the fractions) $$\therefore \frac{3x + 2x}{6} = 10$$

$$\therefore \frac{5x}{6} = 10 \qquad (D)$$

$$\therefore x = 10 \times \frac{6}{5} \qquad \text{(Multiplication property)}$$

$$\therefore x = 12$$

METHOD 2

$$\frac{1}{2}x + \frac{1}{3}x = 10$$

Clear of fractions by multiplying each side by 6, the lowest common denominator (L.C.D.).

$$\therefore 6\left(\frac{1}{2}x + \frac{1}{3}x\right) = 6 \times 10 \qquad \text{(Multiplication property)}$$

$$\therefore 6\left(\frac{1}{2}x\right) + 6\left(\frac{1}{3}x\right) = 60 \qquad (D)$$

$$\therefore 3x + 2x = 60$$

$$\therefore 5x = 60 \qquad (D)$$

$$\therefore x = 12 \qquad \text{(Division property)}$$

Verification. L.S. $= \frac{1}{2}(12) + \frac{1}{3}(12)$ R.S. $= 10$

$= 6 + 4$

$= 10$

Write solutions to the following problems; compare them with those on page 469.

1. Solve $\frac{1}{3}(2x - 1) - \frac{1}{5}(4x - 6) = \frac{7}{15}$.

2. Find $\left\{x \mid \frac{x-2}{2} + \frac{x+3}{3} = \frac{3x+4}{4} - 2\right\}$ and verify.

Exercise 2-2

(A)

Solve:

1. $\frac{1}{2}x = 5$
2. $\frac{2}{3}y = 4$
3. $\frac{1}{2}m = 7$
4. $-\frac{1}{5}x = -5$
5. $\frac{1}{6}a = 0$
6. $\frac{1}{6}a = 1$
7. $\frac{3}{4}z = 1$
8. $\frac{3}{5}x = 9$
9. $\frac{x}{4} = -\frac{1}{2}$

(B)

Solve:

10. $\frac{3x}{4} = \frac{x}{3} + \frac{5}{6}$
11. $\frac{x}{2} + \frac{2x}{3} = \frac{7}{2}$
12. $\frac{x}{3} + \frac{2x}{5} = \frac{11}{15}$
13. $\frac{1}{5}x + \frac{2}{5} = \frac{5}{6}x - \frac{3}{2}$
14. $\frac{1}{2}x - \frac{1}{4} = \frac{x}{4} + \frac{3}{2}$
15. $\frac{x+2}{2} - \frac{x}{3} = 2$

Solve and verify:

16. $\frac{1}{3}(x+1) + \frac{1}{2}(x-1) = \frac{2}{3}$
17. $\frac{x-3}{2} = \frac{x+1}{4}$
18. $\frac{x+5}{2} + \frac{3x-3}{8} + 4 = 0$
19. $\frac{1}{4}(2x-2) - \frac{1}{2}(3x+1) = \frac{1}{6}(x+1)$

Find the solution set for each of the following:

20. $\frac{1}{10}(7x+1) - \frac{1}{4}(x-9) = 1$
21. $\frac{1}{2}(5x-3) - \frac{1}{3}(6x-1) = \frac{1}{4}(3x-2)$
22. $\frac{3}{5}(2y-1) - \frac{2}{3}(3y+5) = \frac{1}{15}$
23. $\frac{x+2}{2} - \frac{5-x}{4} = \frac{x+18}{8}$

24. $\frac{x-2}{3} - \frac{x+2}{2} - \frac{x-1}{4} = 0$ **25.** $4 - \frac{4x-3}{2} = \frac{x+10}{3}$

26. $\frac{1}{3}(x+1) - \frac{1}{2}(x-1) = \frac{1}{4}(2x-1) - \frac{1}{6}(3x-1)$

Find:

27. $\left\{x \mid \frac{2(x+2)}{3} - \frac{3(x-7)}{7} = 6\right\}$

28. $\left\{y \mid \frac{3}{16}(y-1) - \frac{5}{12}(y-4) = \frac{2}{5}(y-6) + \frac{5}{48}\right\}$

29. $\left\{x \mid \frac{(x+1)^2}{3} - \frac{(x+2)^2}{2} = \frac{2-x^2}{6}\right\}$

30. $\left\{x \mid \frac{2x-7}{11} - \frac{x-2}{7} = \frac{5x-3}{7} - 6\right\}$

2·3 Equations involving products.

Example 1. Solve $(2x-3)(x+4) - (x+3)(2x+1) = -17$.

Solution.

$$(2x-3)(x+4) - (x+3)(2x+1) = -17$$
$$\therefore 2x^2 + 5x - 12 - (2x^2 + 7x + 3) = -17 \qquad (D)$$
$$\therefore \quad 2x^2 + 5x - 12 - 2x^2 - 7x - 3 = -17$$
$$\therefore -2x = -2$$
$$\therefore x = 1$$

Write a solution to the following problem and compare it with that on page 469.

1. Find $\{x \mid 3(x+2)(x-1) - 2(x-4)^2 = x^2 + 19,\ x \in I\}$; verify.

Exercise 2-3

(A)

Expand:

1. $(2a-1)(4a+1)$ **2.** $(2x-3)(2x-5)$

3. $(3x+y)(2x-y)$ **4.** $(3b-5)(3b+5)$

5. $(xy+4)(xy+3)$ **6.** $(9-2x)(9+2x)$

7. $(3a-2)^2$ **8.** $(ab+3)^2$

9. $(6x+5y)^2$ **10.** $(2x-10y)(2x+10y)$

11. $(1-5n)^2$ **12.** $(1-6n)(1-7n)$

(B)

Solve and verify:

13. $2x + 3 = 16 - (2x - 3)$

14. $7(2y - 3) - 11(3 - 2y) = 18$

15. $5x - 6(x - 5) = 2(x + 5) + 5(x - 4)$

16. $15(x - 1) + 4(x + 3) = 2(7 + x)$

17. $2x(x + 1) - (x - 1)^2 = (x + 3)^2$

18. $(3x + 1)(2x - 7) = 6(x - 3)^2 + 7$

Solve:

19. $(x - 1)(x + 2) = (x - 3)(x - 2)$

20. $2(x - 1)(x - 3) - 5 = (2x - 1)(x + 3)$

21. $2(x - 1)^2 - 3(x - 2)^2 = 6 - (x - 1)^2$

22. $(y - 5)^2 + 3(y + 2)^2 = 4(y^2 - 1) - 8$

23. $3(x - 1)^2 - 3(x^2 - 1) = x - 15$

Find the solution set defined by each of the following:

24. $x(2x + 1) - 2(x + 2)(x - 4) = 21$

25. $(x - 3)(x - 4) - 2(x - 4)(x - 5) = -(x - 5)(x - 2)$

26. $(x + 1)^2 + 2(x + 3)^2 = 3x(x + 2) + 35$

27. $6(y - 1)^2 - 3(y + 2)^2 - (3y - 1)(y + 4) = 0$

28. $r + 7 + 2(2r - 1)^2 = (1 - 2r)(1 - 4r)$

Find:

29. $\{a \mid (3 - a)^2 = 2(3 + a)(3 - a) + 3(1 + a^2)\}$

30. $\{x \mid (x - 3)(x + 15) - (x - 3)^2 = 30 - 15(x - 1)\}$

31. $\{x \mid (2x - 5)(x - 3) - 5 = x^2 + (x - 1)(x + 1)\}$

32. $\{y \mid (2y - 3)^2 + 3(y + 5) = 4(y + 6)(y + 1)\}$

2·4 Graphs of solution sets.

Example 1. Find the solution set defined by

$$(2x - 7)(3x + 1) = 7 + 6(x - 3)^2;$$

verify and draw the graph of the solution set.

Solution. $(2x - 7)(3x + 1) = 7 + 6(x - 3)^2$

$\therefore 6x^2 - 19x - 7 = 7 + 6(x^2 - 6x + 9)$ (D)

$\therefore 6x^2 - 19x - 7 = 7 + 6x^2 - 36x + 54$ (D)

$\therefore 17x = 68$

$\therefore x = 4$

The solution set is $\{4\}$.

Verification. L.S. $= (2x - 7)(3x + 1)$ R.S. $= 7 + 6(x - 3)^2$

$= (1)(13)$ $= 7 + 6(1)^2$

$= 13$ $= 13$

The graph is the point with coordinate 4 in *Fig. 2-1.*

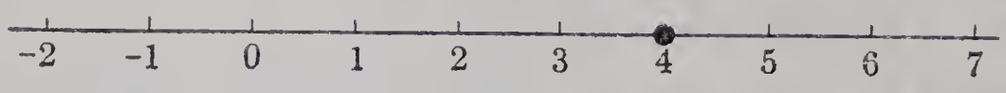


Fig. 2-1

Example 2. Determine the solution set defined by $(2x - 7)(3x + 1) \neq 7 + 6(x - 3)^2$ and draw its graph.

Solution. $\{x \mid (2x - 7)(3x + 1) = 7 + 6(x - 3)^2\} = \{4\}$

$\therefore \{x \mid (2x - 7)(3x + 1) \neq 7 + 6(x - 3)^2\} = \{x \mid x \neq 4\}$

Thus the graph of $\{x \mid (2x - 7)(3x + 1) \neq 7 + 6(x - 3)^2\}$ includes all points on a real number line, except the point with coordinate 4, *Fig. 2-2.*

Fig. 2-2

The continuous nature of the graph is indicated by the solid line. The arrows indicate that the graph extends indefinitely.

Example 3. Find $\left\{x \mid \frac{3x - 4}{2} - \frac{x - 6}{4} \neq \frac{5x}{6} - 3, x \in I\right\}$ and draw its graph.

Solution. To solve this inequation, first solve the equality

$$\frac{3x - 4}{2} - \frac{x - 6}{4} = \frac{5x}{6} - 3 .$$

Multiply each term by 12 (L.C.D.)

$\therefore 6(3x - 4) - 3(x - 6) = 2(5x) - 36$ (Multiplication property)

$\therefore 18x - 24 - 3x + 18 = 10x - 36$ (*D*)

$\therefore 5x = -30$

$\therefore x = -6$ (Division property)

$$\therefore \left\{x \mid \frac{3x - 4}{2} - \frac{x - 6}{4} = \frac{5x}{6} - 3, \ x \in I\right\} = \{-6\}$$

$$\therefore \left\{x \mid \frac{3x - 4}{2} - \frac{x - 6}{4} \neq \frac{5x}{6} - 3, \ x \in I\right\}$$

$$= \{\ldots, -8, -7, -5, -4, \ldots\}$$

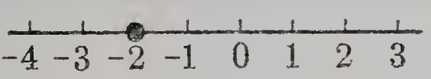
The graph is shown in *Fig. 2-3.*

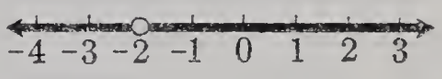
Fig. 2-3

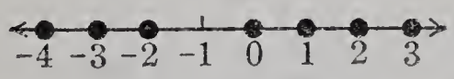
Exercise 2-4

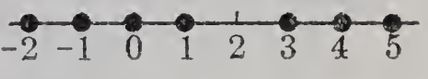
(A)

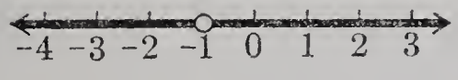
Use set-builder notation to define the sets illustrated by each of the following graphs:

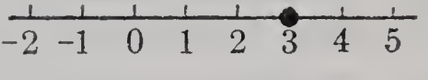
1. 

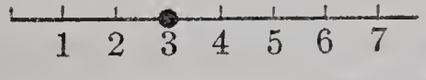
2. 

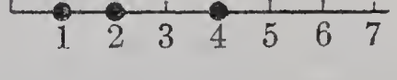
3. 

4. 

5. 

6. 

7. 

8. 

(B)

Find the solution set defined by each of the following and draw its graph:

9. $5(5 - 2x) = 3(8 - 2x)$

10. $5(x - 3) = 21 - 7(x - 6) - 3(8 - x),\ x \in N$

11. $\dfrac{x + 11}{6} \neq \dfrac{4 - x}{3} - 1,\ x \in I$

12. $\dfrac{x - 1}{10} - \dfrac{2 - x}{5} = \dfrac{5 - x}{2}$

13. $5x + 12 - [(x - 3) - (5x - 4)] \neq -7,\ x \in I$

14. $(3x + 8)(x - 5) - (x - 4)(3x - 2) \neq 8,\ x \in I$

15. $\dfrac{1}{8}x - x + \dfrac{1}{6}x \neq \dfrac{5}{6} - \dfrac{1}{2}x$

16. $2(x + 3)(x + 1) - (2x + 1)(x + 5) \neq -8,\ x \in N$

17. $\frac{4-3x}{8}+2 \neq \frac{x-5}{4}-x,\ x \in N$

18. $3(x-4)^2-2(x-2)^2 \neq x^2-8$

(i) for $x \in R$ (ii) for $x \in I$ (iii) for $x \in N$

***2·5* The inequality relation.** The solving of equations is based on the properties of the equality relation outlined in section 2·1.

The solving of inequations, such as

$$3x-12>0,\ x \in R,$$

is based on the properties of the *inequality relation*, that is, *greater than*, $>$, or *less than*, $<$, for the set of numbers involved.

The definition of inequality (order) for real numbers and the trichotomy and transitive properties are as follows.

(i) *Definition of inequality:*
If $a,\ b \in R$ and $a > b$, then
$a-b$ is a positive real number, or
$a = b+c$, where c is a positive real number (i.e. $c \in {}^+R$).

(ii) *Trichotomy property:*
If $a,\ b \in R$, then $a < b$ or $a = b$ or $a > b$.

(iii) *Transitive property:*
If $a,\ b,\ c \in R$ and $a > b$ and $b > c$, then $a > c$.

Other properties of the inequality relation for real numbers may be obtained by examining the following three cases.

a. *Addition (or subtraction) of the same real number to (or from) each member of an inequality.*

Examples.

(i) $5 > 3$
If 3 is added to each side:
L.S. $= 5+3$ R.S. $= 3+3$
$= 8$ $= 6$
But $8 > 6$
$\therefore\ 5+3 > 3+3$

(ii) $-2 > -6$
If -3 is added to each side:
L.S. $= -2+(-3)$ R.S. $= -6+(-3)$
$= -5$ $= -9$
But $-5 > -9$
$\therefore\ -2+(-3) > -6+(-3)$

(iii) $8 > 4$
If 3 is subtracted from each side:
L.S. $= 8-3$ R.S. $= 4-3$
$= 5$ $= 1$
But $5 > 1$
$\therefore\ 8-3 > 4-3$

(iv) $6 > -2$
If -3 is subtracted from each side:
L.S. $= 6-(-3)$ R.S. $= (-2)-(-3)$
$= 9$ $= 1$
But $9 > 1$
$\therefore\ 6-(-3) > -2-(-3)$

These examples suggest the following theorem.

THEOREM. *The sense of an inequality is not changed if the same real number is added to or subtracted from each member.*

That is, if $a, b, c \in R$ and $a > b$, then

(i) $a + c > b + c$ (Addition property)

(ii) $a - c > b - c$. (Subtraction property)

Proof of the theorem (supplementary).

Given: $a, b, c \in R$ and $a > b$

Prove: (i) $a + c > b + c$ (Addition property)

(ii) $a - c > b - c$ (Subtraction property)

Proof:

$a > b$

$\therefore a = b + d,\ d \in {}^{+}R$ (Definition)

$\therefore a + c = b + d + c$ (Equality addition property)

$\therefore a + c = (b + c) + d$ (*CA*) (*AA*)

$\therefore a + c > b + c$ (Definition)

Similarly, $a - c = b + d - c$ (Equality subtraction property)

$\therefore a - c = (b - c) + d$ (*CA*) (*AA*)

$\therefore a - c > b - c$ (Definition)

b. *Multiplication (or division) of each member of an inequality by a positive real number.*

Examples.

(i) $5 > 3$

If each side is multiplied by 3:

L.S. $= 5 \times 3 = 15$ R.S. $= 3 \times 3 = 9$

But $15 > 9$

$\therefore\ 5 \times 3 > 3 \times 3$

(ii) $-2 > -6$

If each side is multiplied by 3:

L.S. $= (-2)(3) = -6$ R.S. $= (-6)(3) = -18$

But $-6 > -18$

$\therefore\ (-2)(3) > (-6)(3)$

(iii) $8 > 4$

If each side is divided by 3:

L.S. $= \frac{8}{3}$ R.S. $= \frac{4}{3}$

But $\frac{8}{3} > \frac{4}{3}$

$\therefore\ 8 \div 3 > 4 \div 3$

(iv) $6 > -2$

If each side is divided by 3:

L.S. $= \frac{6}{3}$ R.S. $= \frac{-2}{3}$

But $\frac{6}{3} > -\frac{2}{3}$

$\therefore\ 6 \div 3 > (-2) \div 3$

These examples suggest the following theorem.

THEOREM. *The sense of an inequality is not changed if each member is multiplied or divided by the same positive real number.*

That is, if $a, b, c \in R$ and $c > 0$ (i.e. $c \in {}^+R$) and $a > b$, then

(i) $ac > bc$ (Multiplication property)

(ii) $a \div c > b \div c$. (Division property)

Proof of the theorem (supplementary).

Given: $a, b \in R$, $c \in {}^+R$ and $a > b$

Prove: (i) $ac > bc$

(ii) $\frac{a}{c} > \frac{b}{c}$

Proof: $\because a > b$

$\therefore a = b + d,\ d \in {}^+R$ (Definition)

$\therefore ac = (b + d)c$ (Equality multiplication property)

$\therefore ac = bc + dc$ (D)

But $dc \in {}^+R$

$\therefore ac > bc$ (Definition)

Similarly, $\frac{a}{c} = \frac{b+d}{c}$ (Equality division property)

$\therefore \frac{a}{c} = \frac{b}{c} + \frac{d}{c}$ (Definition)

But $\frac{d}{c} \in {}^+R$

$\therefore \frac{a}{c} > \frac{b}{c}$ (Definition)

c. *Multiplication (or division) of each member of an inequality by a negative real number.*

Examples.

(i) $5 > 3$

If each side is multiplied by -3:

L.S. $= 5(-3)$ R.S. $= 3(-3)$

$= -15$ $= -9$

But $-15 < -9$

$\therefore 5(-3) < 3(-3)$

(ii) $-2 > -6$

If each side is multiplied by -3:

L.S. $= (-2)(-3)$ R.S. $= (-6)(-3)$

$= +6$ $= +18$

But $(+6) < (+18)$

$\therefore (-2)(-3) < (-6)(-18)$

(iii) $8 > 4$

If each side is divided by -3:

$$\text{L.S.} = \frac{8}{-3} \qquad \text{R.S.} = \frac{4}{-3}$$

$$= -\frac{8}{3} \qquad = -\frac{4}{3}$$

But $-\frac{8}{3} < -\frac{4}{3}$

$\therefore \quad 8 \div (-3) < 4 \div (-3)$

(iv) $6 > -2$

If each side is divided by -3:

$$\text{L.S.} = \frac{6}{-3} \qquad \text{R.S.} = \frac{-2}{-3}$$

$$= -\frac{6}{3} \qquad = \frac{2}{3}$$

But $-\frac{6}{3} < \frac{2}{3}$

$\therefore \quad 6 \div (-3) < (-2) \div (-3)$

These examples suggest the following theorem.

THEOREM. *The sense of an inequality is reversed if each member is multiplied or divided by the same negative real number.*

That is, if $a, b, c \in R$ and $c < 0$ (i.e. $c \in {}^{-}R$), and $a > b$, then

(i) $ac < bc$ (Multiplication property)

(ii) $a \div c < b \div c$. (Division property)

Proof of the theorem (supplementary).

Given: $a, b \in R$, $c \in {}^{-}R$ and $a > b$

Prove: (i) $ac < bc$

(ii) $\frac{a}{c} < \frac{b}{c}$

Proof: $\because \quad a > b$

$\therefore \quad a = b + d,\ d \in {}^{+}R$ (Definition)

$\therefore \quad ac = (b + d)c$ (Equality multiplication property)

$\therefore \quad ac = bc + dc$ (D)

But $dc \in {}^{-}R$

$\therefore \quad ac < bc$ (Definition)

Similarly, $\frac{a}{c} = \frac{b + d}{c}$ (Equality division property)

$\therefore \quad \frac{a}{c} = \frac{b}{c} + \frac{d}{c}$ (Definition)

But $\frac{d}{c} \in {}^{-}R$

$\therefore \quad \frac{a}{c} < \frac{b}{c}$ (Definition)

2·6 Summary of the properties of the inequality relation for real numbers used in forming equivalent inequations.

(a) If $a, b \in R$ and $a > b$, then

$$a = b + d,\ d \in {}^{+}R.$$

(b) If $a, b, c \in R$ and $a > b$, then

(i) $a + c > b + c$ (Addition property)

(ii) $a - c > b - c$ (Subtraction property)

(iii) $ac > bc$ if $c \in {}^{+}R$; $ac < bc$ if $c \in {}^{-}R$ (Multiplication property)

(iv) $\frac{a}{c} > \frac{b}{c}$ if $c \in {}^{+}R$; $\frac{a}{c} < \frac{b}{c}$ if $c \in {}^{-}R$ (Division property)

(c) If $a, b, c \in R$, then $a < b$ or $a = b$ or $a > b$.
(Trichotomy property)

(d) If $a, b \in R$, then if $a > b$ and $b > c$, then $a > c$.
(Transitive property)

2·7 Solution of inequations. The pattern of solution of inequations is parallel to that of equations, except that the properties of the inequality relation must be used. It is a process of writing simpler and simpler equivalent inequations based on the proper number properties. For example:

$$\text{if} \quad 3x - 12 > 0,\ x \in R$$

(add 12 to each side)	$3x > 12$	(Addition property)
(divide each side by 3)	$x > 4$	(Division property)

This discussion discloses the following:

$$\text{if } 3x - 12 > 0, \text{ then } x > 4.$$

The problem,

$$\text{if } x > 4, \text{ then } 3x - 12 > 0,$$

is proved by showing that the steps of the first development are reversible as follows:

$$\text{if} \quad x > 4,\ x \in R$$

(multiply each side by 3)	$3x > 12$	(Multiplication property)
(subtract 12 from each side)	$3x - 12 > 0$	(Subtraction property)

It has now been shown that $3x - 12 > 0$ and $x > 4$ are *equivalent inequations*, and therefore the solution set is $\{x \mid x > 4,\ x \in R\}$.

When the properties of the inequality relation listed in section 2·b are

applied to determine roots or the solution set of an inequation of the first degree in one variable, the steps are always reversible. Thus, it is not necessary to write the second part of the solution. The solution set may be checked or verified by substitution as follows.

Verification.

Since $x > 4$, let $x = 4 + d$, where d is a positive real number.

L.S. $= 3(4 + d) - 12$ R.S. $= 0$

$= 12 + 3d - 12$

$= 3d$ (a positive number)

$\therefore$ for all $x > 4$ L.S. is greater than zero, or $3x - 12 > 0$.

$\therefore$ the solution set is $\{x \mid x > 4,\ x \in R\}$.

Example 1. Find the solution set defined by $-12 - 3x > 2x + 8$ and draw its graph.

Solution. $-12 - 3x > 2x + 8$

$\therefore -3x - 2x > 8 + 12$ (Addition and subtraction properties)

$\therefore -5x > 20$ (D)

$\therefore x < -4$ (Division property)

The solution set is $\{x \mid x < -4,\ x \in R\}$.

Note that in applying the division property that division by -5 has changed the sense of the inequality, and the final inequality reads $x < -4$.

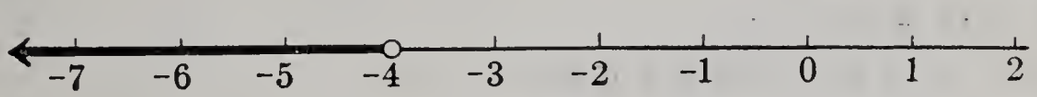


Fig. 2-4

Write solutions to the following problems and compare your solutions with those on page 470.

1. Find $\{x \mid 21 - 8x \leqq 9 - 5x\}$ and draw its graph.
2. Find the solution set defined by $3(-4 - y) \geqq 2(2y + 3) - 11,\ y \in I$ and draw its graph.

Exercise 2-5

(A)

Solve:

1. $x + 2 > 5$
2. $x + 2 < 5$
3. $2x < 6$
4. $3x > 12$
5. $3x < -9$
6. $-2x > 6$

Solve:

7. $x - 2 \leqq 4$ **8.** $x + 3 \geqq -1$ **9.** $2x + 1 < 3$

10. $3x > 2x - 2$ **11.** $2x > 3x - 2$ **12.** $3x < -2 + 3x$

Find:

13. $\{x \mid 2x > 10\}$ **14.** $\{x \mid 4x < 11\}$

15. $\{x \mid x + 5 \geqq 8\}$ **16.** $\{x \mid 2x - 5 \leqq x\}$

17. $\{x \mid 2x > 3x - 2\}$ **18.** $\{x \mid -5x \leqq -20\}$

(B)

Solve:

19. $x - 3 \leqq 10$ **20.** $5 = 4 + x$ **21.** $5 > 4 + x$

22. $3x + 2 > 8$ **23.** $5y - 1.8 = 2y$ **24.** $8x - 1 \leqq -17$

25. $2 + 4x \geqq 6x - 8$ **26.** $12x - 40 < 10 - 13x$

27. $3 - (8x - 13) - (2x + 16) \leqq 0$

28. $6(x - 2) < 4(x - 4) - 9$

29. $5x - 2 + 10 - 7x - 6 > 5x + 7 - 3(x - 2)$

Find the solution set defined by each of the following and draw its graph:

30. $4x + x \geqq 18 - 4x,\ x \in I$

31. $6x + 9 \leqq 2x - 3,\ x \in I$

32. $3(2x - 3) - 7x > 10$

33. $8x + 10 \geqq 6(x + 1)$

34. $4(x - 3) - 2(x - 3) + 1 \leqq 0$

35. $6(-x + 3) - (1 - 2x) - 9 < 4 - x$

36. $5(x - 3) > 3(x - 4) - 3,\ x \in I$

37. $2(3x - 5) - 5(2x - 3) \geqq 22$

38. $10x - 5(x - 4) - 7x + 3(5 - x) \leqq 0,\ x \in N$

39. $(x - 6)(x + 6) - x(x - 1) < -x - 8$

40. $(x - 5)(x + 4) \geqq x(x - 5)$

Find:

41. $\{x \mid 3x - 8 \geqq 4x - 15\}$

42. $\{y \mid 4(y + 2) \geqq 3 - 2(3y - 5)\}$

43. $\{b \mid b(4b - 2) - 2b(b + 2) \leqq (2b - 1)(b - 2)\}$

2·8 **Inequations involving products and fractional coefficients.** The questions in Exercise 2-6 are more difficult than those of the previous exercise because the expressions involved require more simplification.

Write solutions to the following problems and compare them with those on page 470.

1. Solve $2x^2 \geqq 2(2x - 3)^2 - 3(2x - 5)(x + 1)$.
2. Find $\left\{x \mid 1 - \frac{1 - 2x}{4} > \frac{x - 2}{6}, x \in I\right\}$ and draw the graph of the solution set.

Exercise 2-6

(B)

Solve each of the following:

1. $6 - \frac{8x - 1}{3} > \frac{2x + 1}{5}$
2. $\frac{3x - 5}{3} + \frac{5x + 3}{6} < \frac{13}{3}$
3. $(3x + 1)^2 - 3(x - 2)^2 > 6x^2 + 1$
4. $4(2y - 1)(y + 2) - 2(3y + 4)^2 - 5y(3 - 2y) + 11y \leqq 0$

Find the solution set defined by each of the following and draw the graph:

5. $5(x + 2) - 3(x - 1) - 2[2(x + 3) - 3(x + 1) - 2] > 0, \ x \in R$
6. $(x - 3)(x + 4) - (x + 2)(x + 1) < -16, \ x \in I$
7. $\frac{x - 5}{2} - \frac{x - 3}{4} < \frac{x - 7}{10}, \ x \in N$
8. $3(x - 2)(x + 2) - (x - 2)^2 \geqq 2x(x + 4), \ x \in I$
9. $4(x - 1)(x + 2) + 2x(x - 1) - 3(2x - 1)^2 \leqq 2x(5 - 3x)$
10. $\frac{x - 3}{6} \neq \frac{x + 1}{7} - \frac{2}{3}, \ x \in I$

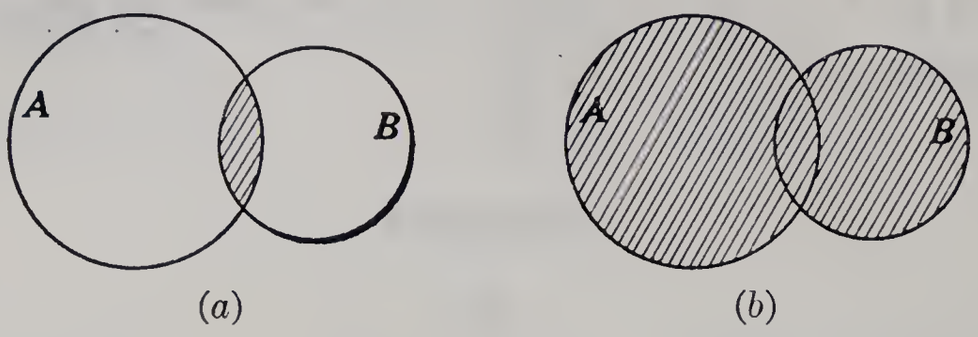
Find:

11. $\{x \mid 3x(4x - 1) - (4x - 3)(3x + 2) \geqq 3\}$
12. $\left\{x \mid \frac{x + 6}{2} - \frac{x + 3}{2} < \frac{x}{2} - \frac{2}{3}\right\}$
13. $\left\{x \mid \frac{x + 1}{2} - 6\frac{2}{3} \leqq \frac{x + 6}{4} - \frac{x + 2}{3}\right\}$
14. $\{x \mid 3x(x + 2) - 2(x - 3)(x + 1) > 20 + x^2\}$

2·9 Graphs of sets defined by compound sentences. In Review *A* the intersection and union of sets were discussed. If A and B are two sets of objects, then

(a) A and B, $(A \cap B)$, represents the intersection set of A and B. It contains those elements which are in both set A and set B.

(b) A or B, $(A \cup B)$, represents the union set of A and B. It contains the elements in at least one of set A and set B.



(a) (b)

Fig. 2-5

In the following examples, sets are defined by algebraic sentences or conditions connected by *and* and *or*, implying the intersection and union, respectively, of the sets defined by the individual sentences or conditions.

Example 1. Draw the graph of $\{x \mid 4x - 5 < 7 \text{ and } 2x + 1 > -3\}$.

Solution. $\{x \mid 4x - 5 < 7 \text{ and } 2x + 1 > -3\}$ is the intersection set of the two sets $A = \{x \mid 4x - 5 < 7\}$, $B = \{x \mid 2x + 1 > -3\}$.

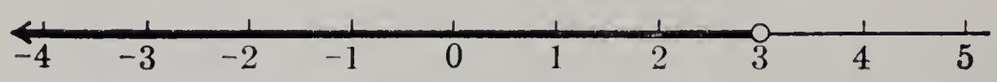
$$A \cap B = \{x \mid 4x - 5 < 7\} \cap \{x \mid 2x + 1 > -3\}$$

Solving the two defining sentences:

$4x - 5 < 7$	$2x + 1 > -3$
$\therefore\ 4x < 12$	$\therefore\ 2x > -4$
$\therefore\ x < 3$	$\therefore\ x > -2$
$A = \{x \mid x < 3\}$	$B = \{x \mid x > -2\}$

The graphs of (i) A (ii) B and (iii) $A \cap B$ are shown in *Fig. 2-6.*

(i)

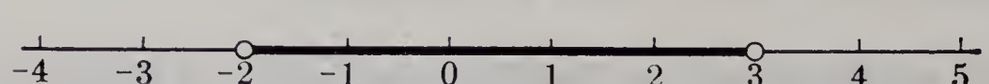


Graph of A or $\{x \mid x < 3\}$

(ii)

Graph of B or $\{x \mid x > -2\}$

(iii)



Graph of $A \cap B$

Fig. 2-6

Example 2. Draw the graph of $\{x \mid 3x + 5 \leqq 2 \text{ or } 5x + 2 \geqq 17\}$.

Solution. $\{x \mid 3x + 5 \leqq 2 \text{ or } 5x + 2 \geqq 17\}$ is the union set of the two sets $A = \{x \mid 3x + 5 \leqq 2\}$, $B = \{x \mid 5x + 2 \geqq 17\}$.

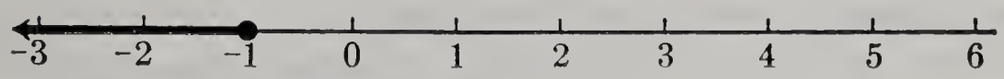
$$A \cup B = \{x \mid 3x + 5 \leqq 2\} \cup \{x \mid 5x + 2 \geqq 17\}$$

Solving the two defining sentences:

$3x + 5 \leqq 2$	$5x + 2 \geqq 17$
$\therefore\ 3x \leqq -3$	$\therefore\ 5x \geqq 15$
$\therefore\ x \leqq -1$	$\therefore\ x \geqq 3$
$A = \{x \mid x \leqq -1\}$	$B = \{x \mid x \geqq 3\}$

The graphs of (i) A (ii) B (iii) $A \cup B$ are shown in *Fig. 2-7.*

(i)

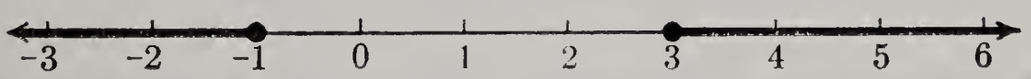


Graph of A, $\{x \mid x \leqq -1\}$

(ii)

Graph of B, $\{x \mid x \geqq 3\}$

(iii)



Graph of $A \cup B$

Fig. 2-7

Exercise 2-7

(A)

List the members of the following sets where $x \in I$:

1. $\{x \mid x > -1 \text{ and } x < 3\}$
2. $\{x \mid x \leqq -2 \text{ and } x \geqq -5\}$
3. $\{x \mid x > -1 \text{ or } x < 3\}$
4. $\{x \mid x = 3 \text{ or } x < 3\}$
5. $\{x \mid x < -4\} \cap \{x \mid x > -6\}$
6. $\{x \mid x \leqq -4\} \cup \{x \mid x \leqq -6\}$
7. $\{x \mid x < -4\} \cap \{x \mid x < -6\}$
8. $\{x \mid x \neq -2\} \cup \{x \mid x < 0\}$

(B)

Draw the graph of each of the following sets:

9. $\{x \mid 5x - 2 \leqq 8\}$ and $\{x \mid 4x + 7 \geqq 3\}$
10. $\{x \mid 6(x - 1) \geqq x - 6\}$ and $\{x \mid 12x - 17 \leqq 2x - 7\}$
11. $\{x \mid 3x + 7 \geqq 6x + 10\}$ or $\{x \mid 3x - 8 > -2\}$

Draw the graph of each of the following sets:

12. $\{x \mid 2(4x+3)-(3x+11)<0\}$ or $\{x \mid 7x-3(x+4)=0\}$

13. $\{x \mid 5(x-2)<0\} \cap \{x \mid 2(2-x) \leqq 10\}$

14. $\{x \mid 2(4x-1)-3(x+2)<3x\} \cup \{x \mid 5(3x+2) \leqq 7x+2\}$

15. $\{x \mid 6-(4-x) \neq 4,\ x \in I\} \cup \{x \mid 8x-5 \geqq x+16, x \in I\}$

16. $\{x \mid 2(3x+4)-(10+7x)>0,\ x \in I\} \cap \{x \mid 0 \neq 20(x+2)+20,\ x \in I\}$

17. $\{x \mid x>-6 \text{ and } x<-1\}$ or $\{x \mid x>1 \text{ and } x<3\}$

18. $\{x \mid x \leqq 5 \text{ and } x \not> 2,\ x \in I\}$ and $\{x \neq 0 \text{ and } x \geqq -2, x \in I\}$

(C)

Draw the graph of each of the following sets:

19. $\{x \mid (2x-3)^2-4(x-2)^2 \leqq 1 \text{ or } x(2x-1)-(2x-1)(x-3) \geqq 21\}$

20. $\{x \mid (x-6)(x-2)-(x-4)(x-5)<0 \text{ and } x^2-(x-2)^2+8>0\}$

21. $\left\{x \,\middle|\, \frac{x+1}{2}-\frac{x+4}{6} \leqq \frac{x+2}{4},\ x \in N\right\} \cap \left\{x \,\middle|\, \frac{x+7}{2}-\frac{x+3}{4} \geqq 4,\ x \in N\right\}$

22. $\{x \mid 5-3x(x-3)+(3x-7)(x+1)>8,\ x \in I\} \cup$
$\{x \mid 3(x-3)^2-(3x-5)(x-1)<32,\ x \in I\}$

***2·10* Absolute value.** Real numbers indicate both magnitude and sense or direction. $+3$ indicates a magnitude of 3 in one sense, and -3 indicates a magnitude of 3 in the opposite sense. The magnitude of a real number is referred to as its *absolute value*. It is indicated by placing the numeral between vertical bars:

$|+3|$ represents the absolute value of $+3$;

$|-3|$ represents the absolute value of -3.

The natural number 3 indicates the magnitude involved in the real numbers $+3$ and -3, and we might be inclined to define $|+3|$ and $|-3|$ as the natural number 3. But we are interested in the absolute values of real numbers, such as $|-2.42|$. Most of these do not correspond to natural numbers. Thus, it is customary to define the absolute value of a real number as the corresponding positive real number. Thus

$$|+3| = +3 \qquad |-3| = +3$$
$$|+2.42| = +2.42 \qquad |-2.42| = +2.42$$

The above examples suggest the following definition of absolute value.

If $x \in R$, and (i) if $x>0$, then $|x| = x$;

(ii) if $x=0$, then $|x| = 0$;

(iii) if $x<0$, then $|x| = -x$.

The absolute value of a real number may be thought of as the length of the line segment joining the origin to the point on the real number line of which the number is the coordinate. The length of a line segment is always positive. This idea is illustrated in *Fig. 2-8.*

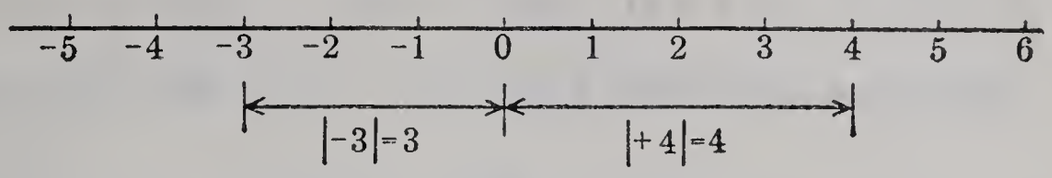


Fig. 2-8

Example 1. Write the absolute value of (i) + 15 (ii) − 15 .

Solution. (i) $|+15| = +15$ (ii) $|-15| = -(-15) = +15$

Example 2. Find the length of the line segment on a number line whose end points have coordinates (i) (+ 6) and (+ 2) (ii) (+ 2) and (− 2) (iii) (− 6) and (− 2). Illustrate graphically.

Solution: The lengths are:

(i) $|(+6) - (+2)| = |6 - 2| = |+4| = 4$
(ii) $|(+2) - (-2)| = |2 + 2| = |+4| = 4$
(iii) $|(-6) - (-2)| = |-6 + 2| = |-4| = 4$

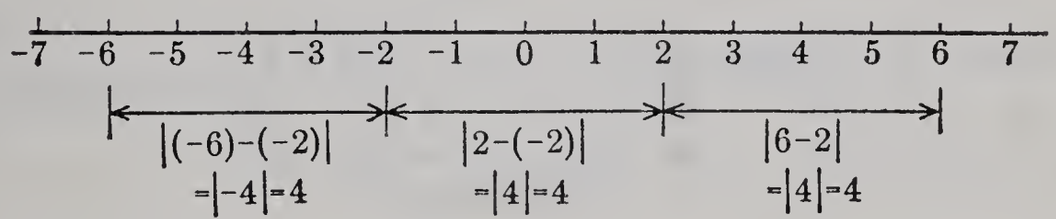


Fig. 2-9

Example 3. Find: (i) $|12 - (-4)|$ (ii) $|-3 + (-4)|$

Solution. (i) $|12 - (-4)| = |12 + 4| = |+16| = 16$
(ii) $|-3 + (-4)| = |-3 - 4| = |-7| = 7$

Exercise 2-8

(A)

Find:

1. $\left|\frac{3}{4}\right|$ **2.** $|-4|$ **3.** $|6 - 4|$

Find:

4. $|7 + 2 - 6|$ **5.** $|7 - 2 - 6|$ **6.** $|6| + |-3|$

7. $|-3|(-2) + 6$ **8.** $(-4)(|-3| + 3)$ **9.** $(|-3|)^2$

10. $\left|\frac{x-4}{4-x}\right| \cdot \frac{3}{4}$, $x < 4$ **11.** $|4x|$, $x < 0$ **12.** $|x - 3|$, $x > 4$

13. $|2x + 6|$, $x < -3$ **14.** $|0|$ **15.** $|x^2|$, $x < 0$

(B)

Evaluate or simplify:

16. $|7| + |-6|$ **17.** $|-10| + |-7| - |-6|$

18. $|5| - |-5|$ **19.** $|-6| + |-3|$

20. $|-7| - |-4|$ **21.** $|25 - 7| - |17 - 19|$

22. $|(-3) + (-4)|$ **23.** $|17 - (-5)|$

24. $|4x + 3|$, $x > 1$ **25.** $|x^2 + 2x + 1|$, $x \in R$

Using the symbols $>$, $=$, $<$, *complete the following to make true sentences:*

26. $|9 - 2|$? $|9| - |2|$ **27.** $|3 - 10|$? $|3| - |10|$

28. $|10 - (-2)|$? $|10| - |-2|$ **29.** $|(-3) - 12|$? $|-3| - |12|$

30. $|(-9) - (-4)|$? $|-9| - |-4|$

31. $|(-4) - (-9)|$? $|-4| - |-9|$

Evaluate the following, if $x = 2$, $y = -3$, $z = 1$:

32. $|x| - |y| + |z|$ **33.** $|x + y| - |y - z|$ **34.** $|x^2| + |y^2| - |z^2|$

35. $|x^2 + y^2 + z^2|$ **36.** $|(x + y)^2| - |(y - z)^2|$

2·11 **Equations involving absolute value (supplementary).** The definition of absolute value states that if $x \in R$, and

(i) if $x > 0$, then $|x| = x$;

(ii) if $x = 0$, then $|x| = 0$;

(iii) if $x < 0$, then $|x| = -x$.

Thus, to solve the equation $|x| = 3$, $x \in R$, we consider the following cases.

(i) If $x > 0$, then $|x| = x$
$\therefore$ the equation $|x| = 3$ is equivalent to $x = 3$.

(ii) If $x = 0$, then $|x| = 0$
$\therefore$ the equation $|x| = 3$ becomes $0 = 3$ which is not true.

(iii) If $x < 0$, then $|x| = -x$
$\therefore$ the equation $|x| = 3$ is equivalent to $-x = 3$ or $x = -3$.

Thus the equation $|x| = 3$ is equivalent to

$$x = 3 \quad \text{or} \quad x = -3 .$$

$\therefore$ the solution set is $\{3, -3\}$.

By a similar argument, the equation $|x - 3| = 2$, $x \in R$ is equivalent to:

$$x - 3 = 2 \quad \text{or} \quad x - 3 = -2$$
$$\therefore \quad x = 5 \qquad \therefore \quad x = 1$$

The solution set is $\{1, 5\}$. The roots of the equation are 1 and 5.

Verification.

If $x = 5$, L.S. $= |5 - 3| = |2| = 2$ R.S. $= 2$

If $x = 1$, L.S. $= |1 - 3| = |-2| = 2$ R.S. $= 2$

The graph of the solution set is shown in *Fig. 2-10.*

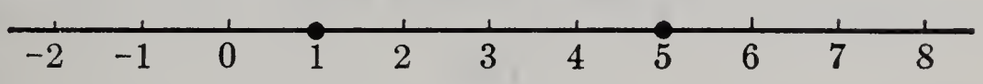


Fig. 2-10

Write solutions for the following problems and compare them with those on page 470.

1. Solve $|2x + 1| = 1$ and verify.

2. Find the solution set of $|3y - 2| = -3$ and verify.

Exercise 2-9

(A)

Solve:

1. $|x| = 3$ **2.** $|y| = -2$ **3.** $-|r| = -1$

4. $|x + 2| = 5$ **5.** $|2x| = 0$ **6.** $|-2x| = 4$

7. $|-3x| = -3$ **8.** $|2x + 1| = 3$ **9.** $-|-x| = 2$

(B)

Find the solution set and verify:

10. $|2x - 1| = 7$ **11.** $2|3x + 2| = 8$ **12.** $|3x - 1| = -1$

13. $-2|3x-2| = -8$ **14.** $-2|3x-2| = 2$ **15.** $|4x + 3| = 5$

16. $3 = \left|\frac{2x + 1}{2}\right|$ **17.** $|2(x-1)+3| = 4$ **18.** $|x + 4| = |-2|$

Find the solution set and verify:

19. $4\left|\frac{2x}{3}\right| = 14$ 20. $\left|\frac{3x+1}{2}\right| = 2$ 21. $\left|\frac{3x+1}{2}\right| = -2$

Solve and draw the graph of:

22. $|2x| = \frac{3}{2}$ 23. $|-3x| = 3$ 24. $2|x-2| = 2$

25. $|x-(2x-1)| = 2$ 26. $\left|\frac{2x+3}{3}\right| = |-2|$ 27. $|+3| = |-2x+1|$

28. $|3x-4| = |-3| - |2|$ 29. $|3x-4| = \left|\frac{-3}{-2}\right|$

30. $|3x-4| = \frac{|-3|}{-2}$

Practice Exercise 2-10

(B)

Solve and verify:

1. $\frac{1}{4}(x+1) = 5 - \frac{1}{3}(2x-5)$ 2. $\frac{x+1}{4} - \frac{x-1}{3} = 1$

3. $\frac{x-1}{2} = 3 - \frac{1}{3}(x-2)$

4. $\frac{1}{2}(x-2) + \frac{1}{4}(x-4) = 10 - \frac{1}{3}(x-3)$

5. $\frac{3x+5}{8} - \frac{x+21}{2} = 5(x-3)$

Find the solution set of each of the following:

6. $\frac{x-2}{4} - \frac{2x-5}{6} = 1 - \frac{3x}{20}$ 7. $.3x - 1.8 + .2x - .7 = 0$

8. $\frac{2}{3}(x-1) - \frac{2}{5}(4x-5) = 3 - \frac{7x+5}{10}$

Find:

9. $\left\{x \mid \frac{1}{5}(5x+1) - \frac{2x-5}{3} = \frac{3x}{5} - 4\right\}$

10. $\left\{x \mid \frac{2x+9}{4} - \frac{6}{5}(x-1) = 3 - \frac{4x}{7}\right\}$

Practice Exercise 2-11

(B)

Solve and verify:

1. $3(x - 8) - 7(x - 3) + 16 = 0$
2. $11x - 5(2x - 1) = 3(6 - x) + 1$
3. $(y + 3)(y - 2) = (y - 5)(y - 6)$
4. $(x + 5)(x + 2) = x^2 + 6x + 12$
5. $2(x + 1)(x + 3) + 8 = (2x + 1)(x + 5)$
6. $3(x + 2)(x - 1) - 2(x - 4)^2 = x^2 + 19$

Find the solution set of each of the following:

7. $(x - 6)(x - 3) + (x - 3)(x - 2) = 2(x - 3)^2$
8. $2(x + 2)^2 - (x - 1)(x - 2) = (x - 3)(x + 1) - 4$
9. $5(x + 1)^2 - 3x^2 + (2x - 5) = 2(x - 2)(x + 3) + 30$

Find:

10. $\{a|(a - 2)(a - 1) - (a - 3)(a + 2) = 9 - 4a\}$
11. $\{x|x(x - 3) - 2(x - 3)(x - 2) = 6 - (x - 4)(x + 3)\}$

Practice Exercise 2-12

(B)

Find the solution set of each of the following:

1. $5x - 3(2x - 11) > x - 1$
2. $2x^2 - (x + 1) \leqq 2x(x - 3)$
3. $4(x - 3) - 2(x + 1) < 5(2x - 1) - 3(3x - 3)$
4. $6(x - 2) + 4(x - 3) + 3(x - 4) \geqq 120$
5. $\frac{a}{4} - \frac{5}{6} < \frac{3a + 2}{3} - \frac{2a + 5}{2}$
6. $2(x - 1)^2 - 3(x - 2)(x + 3) \leqq 32 - (x - 3)(x - 4)$

Graph the solution set of each of the following:

7. $2(x - 3) \geqq 5(x + 1) - (1 - 2x),\ x \in I$
8. $15(x - 2) - 10(2x - 5) < 60 - 9x,\ x \in N$
9. $2(x - 2) + 3(x - 3) + 4(x - 4) = 20$
10. $\frac{7x + 11}{15} \neq \frac{4 + 5x}{9} - \frac{3 - 4x}{5}$

11. $2(x-2)^2 - 3(x-1)(x+3) \leqq 4 - (x^2 - x + 2)$

12. $(2x-1)(3x+1) - (3x-1)(2x-1) + 10 > 0$

Practice Exercise 2-13

(B)

Draw the graph of each of the following:

1. $\{x \mid 4x + 3 \geqq -7,\ x \neq 0\}$ **2.** $\{y \mid 3 > y \geqq -3,\ y \in I\}$

3. $\{x \mid 2x + 4 \geqq 0 \text{ and } 3x - 5 \leqq 4\}$

4. $\{x \mid 2x - 4 \geqq 0 \text{ or } 3x + 5 \leqq -1\}$

5. $\{x \mid 2x - 1 \geqq 5 - x,\ x \in I\}$ or $\{x \mid x - 2 > 5x + 6,\ x \in I\}$

6. $\{x \mid -9 - 7x < 2(3 - x)\}$ and $\{x \mid x - 2 \leqq 5x + 6\}$

7. $\{x \mid (x+5)^2 - (x+3)(x+6) > 0\} \cap \{x \mid (x+5)(2x-1) < x(2x+4)\}$

8. $\left\{x \mid \frac{2x-3}{5} \leqq \frac{x+1}{2} - \frac{1-2x}{10}\right\} \cup \left\{x \mid \frac{5x+3}{8} \leqq 5 - \frac{3x+5}{2}\right\}$

Practice Exercise 2-14

(B)

1. What is meant by the absolute value of a real number? Is $|x - 1| = -1$ a true statement? What is the solution set of this sentence?

2. If $a \in R$, state $|a|$ in the following cases:
(i) $a > 0$ (ii) $a = 0$ (iii) $a < 0$

Calculate the following:

3. $\left|\frac{14}{17}\right|$ **4.** $|-6|$ **5.** $|17 - 2 + 16 - 29|$

6. $|10| - |-3|$ **7.** $|x - 4|,\ x < 0$ **8.** $|-6|(-4) + |-2|$

Simplify the following:

9. $|10| - (-3)(+7) - |-36|$ **10.** $|4a| + (-4)| - 3b|$ $(a, b > 0)$

11. $|4 - 7| - 3|16 - 32| + (-6)$ **12.** $|3b - 4c|$ $(b > 0,\ c < 0)$

Find the solution sets for the following and draw the graphs:

13. $|x + 2| = 3$ **14.** $|2x - 3| \geqq 4,\ x \in I$

Review Exercise 2-15

(B)

1. If $a = 2$, $b = -4$, $c = -3$, find the value of:
$3(a - 2b) + 5(b - 2c) - (c - 2a) - 4a$

2. If $a = 2$, $b = -3$, $c = 0$, find the value of:
$(a^2 - b^2) + (b^2 - c^2) + (c^2 - a^2)$

3. Divide $15a^3b^2c - 12a^2bc^3$ by $-3abc$.

4. Find the product of $y^2 + 2y + 1$ and $y + 1$.

5. Simplify $a - b + c - (c - a + b) + (a + b + c) - (b - c + a)$.

6. Simplify $(x + 2y)^2 - (x - 2y)^2 - (2y)^2$.

7. Find the sum of $(2x - 3)(x - 4)$, $(x + 3)(2 - x)$, $(x + 3)(x - 1)$, and $6x - 3(x + 4)(x - 2)$.

8. From the sum of $3a - 2b + 4c$ and $2a - 5b - 6c$ subtract the sum of $5a - 7b + 6c$ and $4a - 3b - 2c$.

9. Simplify $\dfrac{2a^3 - 4a^2 - 6a}{2a} - \dfrac{6a^3 - 3a^4 + 3a^2}{-3a^2}$.

10. Divide $6x^4 + 4x^2 + 5x - x^3 - 6$ by $x + 3x^2 - 2$.

11. Divide $3(2x^2 + 3x - 1) - 2(16 - x)$ by $3x - 5$.

12. Solve $\dfrac{x + 1}{3} + 2 = \dfrac{4x - 2}{4} - \dfrac{x - 3}{2}$ and verify.

13. Find the solution set defined by:
$(a - 5)(a + 7) - (a + 1)(3a - 2) + 2(a - 1)^2 = -7$

14. Draw the graph of the solution set defined by:
$4(x - 3) - 3(3 - x) \leqq 5(x + 2) - 9(8 - x) + 20$

15. Find $\{x \mid 6(x - 2) - \frac{1}{2}(5 - x) > 26 - 7x, x \in I\}$ and draw the graph of the solution set.

16. Draw the graph of the solution set defined by:
$\frac{1}{2}(x - 1) - \frac{1}{3}(x + 2) \geqq \frac{1}{3}(x - 1)$

17. Draw the graph of each of the following:
(i) $\{x \mid 3x - 2 > 4\}$ or $\{x \mid 3(x - 1) - (11x + 8) < 0\}$
(ii) $\{x \mid (x - 1)^2 - (x - 2)^2 \leqq -5, x \in I\} \cap$
$\left\{x \mid \dfrac{3x}{5} - \dfrac{(x - 1)}{2} > -2, x \in I\right\}$

18. Solve the equations given below and draw the graph of each solution set: (i) $|y - 2| > 4$ (ii) $|2x + 1| \leqq 3$

Chapter III

PROBLEM SOLVING

***3·1* Introduction.** Problem solving is of vital importance in every phase of modern society. Engineers and scientists are called upon daily to solve problems in construction and industry; businessmen solve problems in economics; mathematicians work with statistical problems. Many people enjoy solving problems for the sake of finding a solution. The popularity of cross-word puzzles and problem columns in newspapers and magazines is evidence of this.

The great French mathematician Descartes (1596-1650) once wrote: "As a young man, when I heard of ingenious inventions, I tried to invent them myself . . . ." Descartes enjoyed finding solutions to problems which it had been demonstrated could be solved. In the process, he discovered that problem solving seemed to follow patterns and that a *systematic approach* was more likely to produce the desired result than an unorganized attempt. In this chapter we study the systematic solution of problems by algebraic methods.

***3·2* The algebraic equation and inequation.** Algebraic equations and inequations, such as $x + 3 = 6$, $x \in R$, and $x + 3 < 6$, $x \in R$, may be thought of as expressing a relationship between some real number, represented by x, and some given or known numbers, in this case represented by 3 and 6. The letter symbol x might represent the number or numbers to be determined in a problem.

To find an algebraic solution to a problem, it is necessary to express the relationship between the facts in the form of an equation or inequation. The method used to obtain this sentence is called the *analysis of the*

problem. Essentially the plan is to formulate, or establish, a numeral for the left side and another for the right side of the required equation or inequation. Once the equation or inequation is obtained, it can be solved and the results interpreted in terms of the problem.

***3·3* The analysis of a problem.** Usually the facts of a problem are presented in a few sentences.

Analysis step 1.

Read the question carefully two or three times:

(i) to get the general idea of the problem;

(ii) to discover specifically what is to be determined.

Do not try to absorb the problem as a whole in these readings. Generally this is too difficult; full comprehension is achieved as each fact is dealt with by itself.

Analysis step 2.

Represent the required number (or numbers) by a letter symbol (or symbols); specify the replacement set.

Example 1 (a). (For the following problem carry out the first two steps of the analysis.)

If the greater of two real numbers is twice the lesser, find the numbers if their sum is 52.

Solution. By reading the question, it is observed that two real numbers are to be determined and that one is twice the other. Therefore, a statement may be made representing these numbers by letter symbols.

Represent the lesser real number by x;
then the greater is $2x$.

Example 2 (a). (Carry out the same two steps for the following problem.)

Find three consecutive integers such that when the least is divided by 2, the next by 3, and the greatest by 4, the sum of the quotients is equal to the greatest of these integers.

Solution. By reading the question, it is observed that three consecutive integers are to be determined. It might seem that three letter symbols should be used to represent them, but the meaning of the word consecutive suggests that only one letter symbol need be used.

Represent the least integer by x,
the second by $x + 1$,
the third by $x + 2$.

NOTE: The three consecutive integers might have been represented as

(i) $(x - 1)$, x, $(x + 1)$,

or (ii) $(x - 2)$, $(x - 1)$, x .

Analysis step 3.

Study the facts carefully and represent them, one by one, in terms of the letter symbol (or symbols) introduced, so as to build up the two parts of the equation or inequation.

Analysis step 4.

Write the equation or inequation relating the given facts.

Example 1 (b). Complete steps 3 and 4 of the analysis of Example 1 (a).

Solution. The unused fact in the statement of the problem is:

their sum is 52.

The following statements represent this fact in terms of the symbol x.

Their sum is $x + 2x$ or $3x$. (L.S. of equation)

But this is 52. (R.S. of equation)

$\therefore\ 3x = 52$

Example 2 (b). Complete steps 3 and 4 of the analysis of Example 2 (a).

Solution. The unused facts in the statement of the problem are:

When the least is divided by 2, *the next by* 3, *and the greatest by* 4, *then the sum of the quotients is equal to the greatest.*

The following statements represent these in terms of the letter symbol x.

The least divided by 2 is $\frac{x}{2}$;

the next divided by 3 is $\frac{x+1}{3}$;

the greatest divided by 4 is $\frac{x+2}{4}$.

Their sum is $\frac{x}{2} + \frac{x+1}{3} + \frac{x+2}{4}$. (L.S. of equation)

But this sum is equal to $x + 2$. (R.S. of equation)

$\therefore\ \frac{x}{2} + \frac{x+1}{3} + \frac{x+2}{4} = x + 2$

With the analysis complete, it is only necessary to solve the equation and make the proper interpretation.

Write an analysis of the following problem; compare your analysis with that on page 471.

1. Determine all the integers such that four times the integer decreased by 5 is less than 7.

Exercise 3-1

(A)

Complete orally the analysis of each of the following:

1. When a certain real number is doubled and then increased by 19, the result is 41. Find the number.
Represent the real number by _____.
Double the number increased by 19 is _____.
But this is equal to _____.
∴ _____.

2. When four times a certain real number is increased by 9, the result is the same as 25 increased by twice the number. Find the number.
Represent the _____ _____ _____ _____.
Four times _____ _____ increased by _____ is _____.
25 increased by _____ _____ _____ is _____.
But these two results are _____.
∴ _____ .

3. Find all the real numbers such that five times the number increased by 3 is equal to or less than twice the number diminished by 2.
Represent any real number by _____ .
Five times the number increased by 3 is _____ .
Twice the number diminished by 2 is _____ .
But the first of these is equal to or less than the _____ .
∴ _____ $\leqq$ _____ .

4. When a certain real number is multiplied by 5 and 35 subtracted from the product, the result is 5. Find the number.
Represent any real number by _____ .
5 times the number less _____ is _____ .
But this number equals _____ .
∴ _____ = _____ .

5. Find a real number which when trebled is equal to 13 increased by double the number.
Represent any real number by _____ .
The number trebled is _____ .
13 increased by double the number is _____ .
But these two numbers are _____ .
∴ _____ = _____ .

6. Two real numbers differ by 7 and have a sum of 43. Find the numbers.

Complete orally the analysis of the following:

7. Find the real numbers which when divided by 3 and increased by 4 are equal to, or greater than, 7.

(B)

Write the analysis for the following problems:

8. If 21 is added to a real number, the sum is 8 times the number diminished by 21.

9. If 7 is added to 8 times a real number, the result is the same as that obtained when 5 is subtracted from 12 times the number. Find the number.

10. Find two real numbers whose sum is 83 such that one exceeds the other by 25.

11. Find three consccutive integers whose sum is 36.

12. Find the set of all real numbers such that the number decreased by $\frac{1}{2}$ is less than $\frac{3}{4}$ of the number increased by $1\frac{1}{2}$.

***3·4* The algebraic solution of a problem.** The following procedure suggests a general pattern for the algebraic solution of a problem.

Analysis.

1. Read the question carefully two or three times:
   (i) to get the general idea of the problem;
   (ii) to discover exactly what is to be found.
2. Represent the required number (or numbers) by a letter symbol (or symbols).
3. Represent the given facts, one by one, in terms of the symbol (or symbols) to obtain:
   (i) the left side of an equation or inequation;
   (ii) the right side of an equation or inequation.
4. Write the equation or inequation relating the given facts.

Completion.

5. Solve the equation or inequation.
6. State the conclusion: that is, interpret the solution of the equation or inequation in the context of the problem.
7. Verify by showing that the conclusion reached satisfies the conditions as stated in the problem.

3·5 Problems involving relationships between numbers.

Example 1. Find two real numbers whose sum is 23, such that the sum of the lesser and 3 times the greater is 54.

Solution. Represent the lesser real number by x .
Then the greater is $23 - x$.
The lesser plus three times the greater is

$$x + 3(23 - x) = x + 69 - 3x$$
$$= 69 - 2x .$$

But this sum equals 54 .

$$\therefore\ 69 - 2x = 54$$
$$\therefore\ -2x = -15$$
$$\therefore\ x = 7\tfrac{1}{2}$$

The lesser number is $7\frac{1}{2}$ and the greater is $15\frac{1}{2}$.

Verification. The lesser plus 3 times the greater is $7\frac{1}{2} + 3(15\frac{1}{2})$
$= 7\frac{1}{2} + 46\frac{1}{2}$
$= 54$.

Example 2. Find all the real numbers such that 3 times the number decreased by 2 is less than 4.

Solution. Represent any real number by x .
Then 3 times the number less 2 is $3x - 2$.
But this number is less than 4 .

$$\therefore\ 3x - 2 < 4$$
$$\therefore\ 3x < 6$$
$$\therefore\ x < 2$$

The required numbers are any real numbers less than 2.

Verification. $\because\ x < 2 \quad \therefore\ x + d = 2 \quad d \in {}^{+}R$ (Definition)
$\therefore\ x = 2 - d$

$(2 - d)$ represents all real numbers less than 2.

3 times the numbers decreased by 2 is $3(2 - d) - 2$
$= 6 - 3d - 2$
$= 4 - 3d$
< 4 as required.

Write complete solutions to each of the following and compare them with those on page 471.

1. Find three consecutive integers whose sum is 27.
2. Find all the real numbers such that the number decreased by 2, then divided by 3, is greater than 5.

Exercise 3-2

(B)

Write complete solutions for the following problems:

1. The sum of two numbers is 71. If the greater exceeds twice the lesser by 2, find the numbers.
2. At an election the successful candidate had a majority of 1,256 votes over his opponent. If the total vote was 6,424, find the number of votes cast for each candidate.
3. The sum of three numbers is 72. The second is 6 more than the first, and the third is 4 times the first. Find the numbers.
4. The sum of $\frac{1}{3}$ of a number and $\frac{1}{5}$ of the same number is 16. Find the number.
5. The sum of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{5}$ of a certain number is 106. Find the number.
6. Find two numbers that differ by 4 and such that $\frac{1}{2}$ of the greater exceeds $\frac{1}{6}$ of the lesser by 8.
7. Divide 132 into three parts so that the first part exceeds the second by 11, and the third exceeds the first by 26.
8. Find a number which exceeds 42 by the same amount that $\frac{1}{6}$ of the number exceeds 2.
9. Find three consecutive even integers whose sum is 48.
10. Find three consecutive odd integers whose sum is 63.
11. Find two consecutive even integers such that the sum of $\frac{1}{2}$ of the lesser and $\frac{1}{4}$ of the greater is 62.
12. Find the set of all real numbers such that 4 times the number increased by 5 is greater than twice the number decreased by 3.

***3·6* The context and variables in a problem.** The problems in Exercise 3-1 and 3-2 are straight-forward and permit a direct interpretation in terms of the letter symbol introduced. In many problem situations a number of *variables* are involved, and these are related to each other in some particular way. In other words, the given problem is a particular case of a general context involving certain variable quantities related in some way. The general relation is often not given in the problem, but the problem solver must know it if he is to do the problem.

Suppose we are planning a trip by car from town A to town B. This is a specific instance of a general situation involving three variables:

(i) *distance*, represented by d units (miles), $(d \in R,\ d > 0)$;

(ii) *speed* of travel, represented by v units (m.p.h.), $(v \in R,\ v > 0)$;

(iii) *time* of travel, represented by t units (hours), $(t \in R,\ t > 0)$.

These variables are related by the equation

$$d = vt$$

which may also be written,

$$v = \frac{d}{t} \quad or \quad t = \frac{d}{v}$$

and the form used depends on the particular variable in which we are interested. These equations are sometimes referred to as *the law relating the variables.*

To plan the trip mentioned, these three variables must be considered. For example, since the distance from A to B may be assumed to be known, then:

(i) if we have only 3 hours travelling time, the rate of travel,

$$v \text{ m.p.h., is } \frac{d}{3} \text{ m.p.h.;}$$

(ii) if our average speed is 45 m.p.h., then the time of travel,

$$t \text{ hours, is } \frac{d}{45} \text{ hours.}$$

It is necessary to examine the context to get the general idea of the problem. The following examples recall familiar general contexts, the variables involved and the law relating them.

Copy and complete the following table; compare your solution with that on page 471.

1.

GIVEN VARIABLE	DEPENDS ON THE VARIABLE(S)
(i) Time for a trip	
(ii) Rate of travel	
(iii) Area of a rectangle	
(iv) Area of a circle	
(v) Volume of a box	
(vi) Volume of a sphere	
(vii) Cost of a number of articles	
(viii) Gain (or loss)	
(ix) Current strength	
(x) Capacity of a cylindrical drum	
(xi) Interest	
(xii) Price of wheat	

Copy and complete the following table; compare your solution with that on page 472.

2.

VARIABLE QUANTITIES	EQUATION RELATING THE VARIABLES
(i) Area, length, width of a rectangle	
(ii) Volume, length, width, depth	
(iii) Circumference, radius (circle)	
(iv) Radius, area (circle)	
(v) Radius, volume (sphere)	
(vi) Distance, speed, time	
(vii) Total cost, cost per article, number	
(viii) Cost price, selling price, gain (loss)	
(ix) Interest, principal, rate, time	

3. Set up a table with the following headings in your notebook; complete the columns for each problem:

ASSOCIATED VARIABLE NOT STATED	LAW	EXPRESSION FOR ASSOCIATED VARIABLE

(i) The area of a rectangular desk top is 300 square inches; its length is 20 inches.

(ii) The area of a rectangle is A square inches; the width is 8 inches.

(iii) The radius of a circle is 10 inches.

(iv) The sum of \$30 is spent on shirts at \$$x$ per shirt.

(v) A boy wants a bicycle which sells for \$$y$. He can earn \$1$\frac{1}{2}$ every afternoon working as a delivery boy.

(vi) It is $(x - 4)$ miles to Montreal, and we have y hours to get there.

(vii) $(3x + 4)$ dollars is invested for 3 years at 5 per cent per annum.

Exercise 3-3

(A)

In each of the following, suggest at least one other variable associated with the data and express its value in terms of the given data:

1. The area of a rectangle is 200 sq. in. and its width is 10 in.
2. The area of a rectangle is A sq. in. and its length is 10 in.
3. The length of a rectangle is y ft. and its width is x ft.
4. At an average speed of 40 m.p.h. a man reaches Montreal in 8 hrs.
5. It is 50 miles to Hamilton and I take x hr. to get there.
6. I travel a distance of x miles at an average speed of y m.p.h.
7. I pay \$3.50 for 10 gallons of gasoline.
8. At x ¢ per gallon my weekly gasoline bill is \$5 .
9. A trip takes x gallons of gasoline at y ¢ per gallon.
10. Ten tons of coal cost \$280 .
11. x tons of coal cost $\$(y + 5)$.
12. I spend \$90 for coal at the rate of $\$x$ per ton.
13. The area of the floor of a rectangular coal bin is 270 sq. ft. and the depth of the bin is 6 ft.
14. The area of the base of a rectangular box is x sq. in. Its volume is 300 cu. in.
15. The diameter of a circle is 7 in.
16. The diameter of a circle is x in.
17. The circumference of a circle is y in.
18. The semi-perimeter of a rectangle is 20 in.; the length is 12 in.
19. The perimeter of a rectangle is 80 in. and the width is 15 in.
20. The perimeter of a rectangle is x in. and the width is y in.
21. I buy a house for \$18,000 and sell it for \$19,000 .
22. I buy a house for $\$x$ and sell it for $\$y$, which is less than I paid.
23. A boy wants a bicycle costing \$45 . He can earn \$1.50 each day after school.

3·7 **Problems involving measurements.** The following examples involve measurements in various contexts. It is necessary to examine carefully the variables mentioned in the problem, to recall any associated variables and the law which relates them.

Example. The rectangular top of a table has sides which are respectively 4 feet and 5 feet longer than the side of a table with a square top. If the

area of the rectangular table top is 47 square feet greater than the area of the square top, find the dimensions of the top of each table.

Solution.

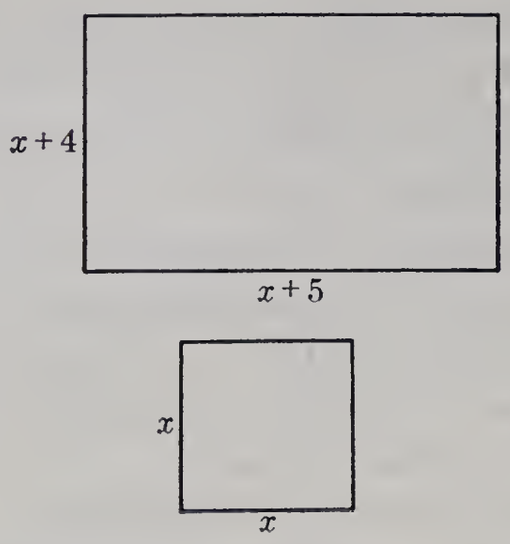


Fig. 3-1

This problem is a particular example of the general problem concerned with the variables: length, width, and area of a rectangle. The law relating these is expressed by

$$a = lw.$$

Represent the length of a side of the square by x ft. $(x \in R,\ x > 0)$.
Then the area of the square is x^2 sq. ft.
The lengths of the sides of the rectangle are $(x + 4)$ ft. and $(x + 5)$ ft.
The area of the rectangle is $(x + 4)(x + 5)$ sq. ft.
The difference in area is $[(x + 4)(x + 5) - x^2]$ sq. ft.

$$= (x^2 + 9x + 20 - x^2) \text{ sq. ft.}$$
$$= (9x + 20) \text{ sq. ft.}$$

But the difference in area is 47 sq. ft.

$$\therefore\ 9x + 20 = 47$$
$$\therefore\ 9x = 27$$
$$\therefore\ x = 3$$

The dimensions of the square table top are 3 ft. and 3 ft.
The dimensions of the rectangular table top are 7 ft. and 8 ft.

Verification.

Area of rectangular table top	$= 8 \times 7$ sq. ft.	$= 56$ sq. ft.
Area of square table top	$= 3 \times 3$ sq. ft.	$= 9$ sq. ft.
Difference in area		$= 47$ sq. ft.

Sometimes it is helpful in analyzing a problem to tabulate the data.
Represent the length of a side of the square by x ft. $(x \in R, x > 0)$.

VARIABLE	SQUARE TABLE TOP	RECTANGULAR TABLE TOP
Length in feet	x	$x + 5$
Width in feet	x	$x + 4$
Area in square feet	x^2	$x^2 + 9x + 20$

The difference in area is $(x^2 + 9x + 20 - x^2)$ sq. ft.
or $(9x + 20)$ sq. ft.

But the difference in area is 47 sq. ft.

$$\therefore\ 9x + 20 = 47$$

Exercise 3-4

(A)

State or write the analysis of each of the following problems:

1. Find the length of the side of an equilateral triangle with perimeter 72 inches.
2. The total weight of two iron castings is 130 lb. If one casting weighs 7 lb. more than twice the weight of the other, find the weight of each casting.
3. The length of a rectangular field is five times its width. If the perimeter is 72 rods, find its dimensions.
4. The sum of the interior angles of a polygon of n sides is $(2n - 4)$ right angles. Find the number of sides a polygon has if the sum of its interior angles is 10 right angles.
5. The difference between the length and width of a rectangular garden is 12 feet. If the perimeter of the garden is 60 feet, find its length and width.

(B)

Write a complete solution for each of the following problems:

6. A piece of lumber 20 feet long is to be divided into two parts. If the length of the longer part is to be $\frac{1}{2}$ foot longer than double the shorter part, find the length of each part.
7. In a stairway there are 55 steps of uniform height. However, if each step had been made one inch higher, only 50 would have been required. Find the height of each step.
8. Find the measurements of the angles of a triangle if the greatest is $50°$

larger than the smallest and 10° larger than the other and the sum of the three angles is 180°.

9. A rectangular room has its length 8 feet longer than its width. If each dimension is increased by 3 feet, the area is increased by 153 square feet. Find the original dimensions of the floor of the room.

10. One rectangular school garden has its length 10 feet longer than its width. A second garden, of equal area, is 5 feet longer and 2 feet narrower than the first. Find the dimensions of each garden.

11. A rectangular floor has a margin two feet wide around a rectangular carpet. If the length of the carpet is 3 feet more than its width and the area of the margin is 112 square feet, find the dimensions of the floor.

3·8 Problems involving rate per cent. If a person invests \$200 at an interest rate of 4% per annum for a period of 3 years, his income (amount of interest) from the investment is

$$\$\left(200 \times \frac{4}{100} \times 3\right) \text{ or } \$24$$

if the interest is simple interest.

In general, if an amount of money (*Principal*, P dollars) is invested at a rate of interest (r%) per annum for a period of time in years (t years), the income (I dollars) is expressed by the formula

$$I = P \times \frac{r}{100} \times t$$

if the interest is simple interest.

Example 1. The annual interest on a sum of money invested at 4% together with the interest on a sum twice as large invested at 5% is \$112. Find the amount of each sum of money invested.

Solution. Represent the sum of money invested at 4% by x dollars ($x \in R$, $x > 0$).

Then the sum of money invested at 5% is $2x$ dollars.

The total interest is $[.04(x) + .05(2x)]$ dollars

or $(.04x + .1x)$ dollars

or $.14x$ dollars.

But it is given that the total interest is \$112.

$$\therefore .14x = 112$$
$$\therefore 14x = 11{,}200$$
$$\therefore x = 800$$

The sum invested at 4% is \$800 and at 5% is \$1,600.

Verification. The total interest $= \$(.04 \times 800 + .05 \times 1{,}600)$
$= \$(32 + 80) = \$112$

Example 2. How many gallons of water must be added to 2.5 gallons of a 60% solution, by volume, of permanent antifreeze and water to make a 40% solution?

Solution. Represent the number of gallons of water added by x ($x \in R$, $x > 0$).

Then the number of gallons of antifreeze in a 40% solution is

$$.4(2.5 + x) \text{ or } (10 + .4x).$$

But the number of gallons of antifreeze is $.6 \times 2.5 = 1.5$.

$$\therefore\ .4(2.5 + x) = 1.5$$
$$\therefore\ 4(2.5 + x) = 15$$
$$\therefore\ 10 + 4x = 15$$
$$\therefore\ 4x = 5$$
$$\therefore\ x = 1.25$$

1.25 gallons of water should be added.

Verification. If 1.25 gallons of water are added, the strength of the solution is $\dfrac{1.5}{3.75} = \dfrac{150}{375} = \dfrac{2}{5} = 40\%$.

Write solutions to the following problems and compare them with those on page 472.

1. The yearly interest on an investment is \$48. If the annual rate of interest is 5%, find the amount of the investment.
2. A chemist has 60 cc. of a 12% solution of an acid. If he wishes to dilute it to a 5% solution by adding water, how many cc. of water should he add?

Exercise 3-5

(B)

1. A man invested \$4,000, part at 4% per annum and part at 5% per annum. If his total annual income is \$170, find the amount he invested at each rate.
2. A baseball sold for \$6.00 and provided a gain of 20% calculated on the cost. Find the cost of the baseball.
3. What sum of money invested at 5% will provide the same interest as a sum \$600 larger invested at 3%?
4. A salesman receives 15% commission on his total sales. If his total commission for a month was \$360, find his total sales for the month.
5. A man invested $\frac{1}{3}$ of his money at 3%, $\frac{1}{4}$ at 4%, $\frac{1}{5}$ at 5%, and the remainder at 6%. If his total yearly income is \$860, find the amount invested.

6. Divide \$6,200 into two parts so that the income from one part at 5% is equal to the income from the other part at 3%.
7. How many gallons of water must be added to 8 gallons of an 80% solution of permanent antifreeze to make a 50% solution?
8. How many gallons of water must be added to 100 gallons of milk, which tests 6% butterfat, to dilute it to test 4% butterfat?

3·9 **Problems involving distance, speed, and time.** The distance, speed, and time context discussed in section 3·6 is reviewed below.

If a jet aircraft travels 3,000 miles in 5 hours, its average speed is $\frac{3000}{5}$ m.p.h. or 600 m.p.h.

If the average rate and distance are known to be 600 m.p.h. and 3,000 miles, respectively, the time is $\frac{3000}{600}$ hours or 5 hours.

If the time and average speed are known to be 5 hours and 600 m.p.h., respectively, the distance is (600×5) miles or 3,000 miles.

In general, if d represents the distance in miles, v the average speed in miles per hour, and t the time in hours, then

$$\text{(i)}\quad v = \frac{d}{t} \qquad \text{(ii)}\quad t = \frac{d}{v} \qquad \text{(iii)}\quad d = vt\,.$$

It should be noted that these relations hold for any units of distance and time, provided the speed is expressed in consistent units.

Example 1. A boy travelled from home to a hill-climb by motorcycle at an average speed of 55 m.p.h. During the contest his machine was damaged and had to be left for repairs. He returned by bus at an average speed of 40 m.p.h. If his total travelling time to and from the meet was 3 hours and 48 minutes, how far from his home was the meet held?

Solution. Represent the distance by x miles $(x \in R,\ x > 0)$.

The given data may be tabulated as follows:

	DISTANCE IN MILES	SPEED IN m.p.h.	TIME IN HOURS
Motorcycle	x	55	$\frac{x}{55}$
Bus	x	40	$\frac{x}{40}$

Total travelling time is $\left(\frac{x}{55} + \frac{x}{40}\right)$ hours.

But this is given to be $3\frac{4}{5}$ hours.

$$\therefore\ \frac{x}{55} + \frac{x}{40} = \frac{19}{5}$$

$$\therefore\ 8x + 11x = 1{,}672$$
$$\therefore\ 19x = 1{,}672$$
$$\therefore\ x = 88$$

The distance is 88 miles.

Verification. Time to travel 88 miles at 55 m.p.h. is $\frac{88}{55}$ hr. or $1\frac{3}{5}$ hr.

Time to travel 88 miles at 40 m.p.h. is $\frac{88}{40}$ hr. or $2\frac{1}{5}$ hr.

Total time for the round trip is $3\frac{4}{5}$ hr. or 3 hr. and 48 min.

Example 2. If two persons, A and B, who are 12 miles apart, walk towards each other, they will meet after 2 hours. If A walks twice as fast as B, find their speeds of walking.

Solution. Represent B's speed of walking by x m.p.h. ($x \in R,\ x > 0$). Then the given data in the problem may be tabulated as follows:

	DISTANCE IN MILES	SPEED IN m.p.h.	TIME IN HOURS
A	$4x$	$2x$	2
B	$2x$	x	2

The total distance walked by A and B is $(4x + 2x)$ miles or $6x$ miles. But it is given that the total distance walked is 12 miles.

$$\therefore\ 6x = 12$$
$$\therefore\ x = 2$$

A's speed of walking is 4 m.p.h. and B's speed is 2 m.p.h.

Verification. Distance A walks in 2 hrs. at 4 m.p.h. is 8 miles.
Distance B walks in 2 hrs. at 2 m.p.h. is 4 miles.
Total distance is 12 miles.

Exercise 3-6

(B)

1. An athlete runs a certain distance at 8 m.p.h. and returns by car at 45 m.p.h. If his total time of travel is 53 minutes, how far did he run?
2. A member of a hiking party blows a whistle and hears an echo from a cliff in $5\frac{1}{2}$ seconds. If sound travels 1,130 feet per second, how far is the cliff from the party?

3. A freight train travelling at 50 m.p.h. takes 1 hour longer for a trip between two stations than an express train travelling at 60 m.p.h. Find the distance between the two stations.

4. A motorist leaves home at 2 p.m. travelling at an average speed of 50 m.p.h. A second motorist leaves the same point at 2:50 p.m. and travels at an average speed of 60 m.p.h. How many miles will he travel before overtaking the first motorist?

5. A bullet fired with a muzzle velocity of 1,000 feet per second strikes a target at a distance of 1,200 feet. The sound of the impact is heard 2.25 seconds after the bullet was fired. Calculate the approximate speed of sound in air.

6. A man flies by jet airliner at 600 m.p.h. from Toronto to Vancouver, a distance of 2,120 air miles, to take delivery of his own plane in which he returns to Toronto. If his total flying time is 21 hours 12 minutes, what is the average speed on the return trip?

7. A river flows at 5 m.p.h. A motor boat is driven upstream and back to the starting point in 12 minutes. If the boat moves at an average speed of 25 m.p.h. in still water, find the distance travelled upstream.

8. The road up one side of a mountain and the road down the other side are each a mile long. If a car is driven up the mountain at an average speed of 30 m.p.h., how fast must it be driven down the other side so that the average speed is 60 m.p.h.?

9. A light plane flies a certain distance into a headwind of 35 m.p.h. for 4 hours and returns the same distance in 2 hours 15 minutes. Find the speed at which the plane would have travelled if there had been no wind.

10. Find the length of time required for a radar beam from earth to be reflected back to earth from the moon if the speed of radar is 186,000 miles per second and the distance to the moon from the earth is 2.38×10^5 miles.

11. A man drove at an average speed of 50 m.p.h. outside a city and 30 m.p.h. within the city. If the complete trip of 30 miles took 40 minutes, how much of the trip was outside the city?

12. Two cars leave the same place at the same time and travel in opposite directions, one of the cars travelling 10 m.p.h. faster than the other. After 3 hours, they are 240 miles apart. What is the speed of each car in m.p.h.?

13. A turbo-prop aircraft leaves an airport at 6 p.m. At 7 p.m. a jet aircraft leaves the airport, flying the same course, and overtakes the first plane at 12 o'clock midnight. If the cruising speed of the jet is 73 m.p.h. more than that of the turbo-prop, find the speed of each.

3·10 **Problems in various contexts.** The importance of pattern and method in obtaining solutions to problems in particular contexts should now be evident. The problems in the exercise which follows are taken from various contexts. The skill and knowledge you have gained from previous experience should help you solve these problems.

Example 1. A has \$8 more than twice the amount of money B has. If A gives \$7 to B, he would now have only \$3 more than B. Find the amount each had at first.

Solution. In this problem there are two amounts of money to be found.

The first decision, therefore, will be whether to represent by $\$x$ the initial amount possessed by A or that possessed by B.

Exploration of the two possibilities reveals that if B's amount is represented by $\$x$, then A's amount is represented by $\$(2x + 8)$. Using this representation eliminates fractions. If A gives \$7 to B, A will have \$7 less than he had at the beginning, and B will have \$7 more.

Hence the new amounts which A and B have are $\$(2x + 8 - 7)$ and $\$(x + 7)$, respectively.

This information may be tabulated as follows:

	A'S AMOUNT (DOLLARS)	B'S AMOUNT (DOLLARS)
Initial	$(2x + 8)$	x
Final	$(2x + 8 - 7)$	$(x + 7)$

Since A still has \$3 more than B, it follows that

$$(2x + 8 - 7) = (x + 7) + 3.$$
$$\therefore 2x + 1 = x + 10$$
$$\therefore x = 9$$

A had \$26 and B had \$9.

Verification. Twice B's amount plus \$8 is \$26 or A's amount.
After exchange of \$7, A has \$19.
After exchange of \$7, B has \$16.
Difference is \$3 in favour of A.

Example 2. A man is 3 times as old as his nephew. 2 years ago the man was 5 times as old as his nephew was 4 years ago. Find their present ages.

Solution. In this problem two ages are to be found. Since the man is 3 times as old as his nephew, fractions will be avoided if the nephew's present age is represented by x years $(x \in R, x > 0)$.

Then the man's present age is $3x$ years.

2 years ago the man's age was $(3x - 2)$ years.

4 years ago the nephew's age was $(x - 4)$ years.

This information may be tabulated as follows:

	MAN'S AGE (YEARS)	NEPHEW'S AGE (YEARS)
Present	$3x$	x
2 years ago	$3x - 2$	
4 years ago		$x - 4$

But it is given that the man's age 2 years ago was 5 times the nephew's age 4 years ago.

$$\therefore \ 3x - 2 = 5(x - 4)$$
$$\therefore \ 3x - 2 = 5x - 20$$
$$\therefore \ 20 - 2 = 5x - 3x$$
$$\therefore \ 18 = 2x$$
$$\therefore \ x = 9$$

The nephew's present age is 9 years, and the man's is 27 years.

Verification. The man's age is 3 times the nephew's age ($27 = 3 \times 9$).
The man's age 2 years ago was 25 years.
The nephew's age 4 years ago was 5 years.
Hence the man's age 2 years ago was 5 times the nephew's age 4 years ago.

Example 3. The average mark in an algebra class was 62. Two students whose marks were 46 and 40 dropped out, raising the average to 63. Find the number of students originally in the class.

Solution. If the original number of students is represented by x ($x \in N$) the data of the problem may be tabulated as follows:

	BEFORE	AFTER
Number of students	x	$x + 2$
Average mark	62	63
Aggregate of marks earned	$62x$	$63(x - 2)$

Hence the difference in the aggregates $= (62x - 63(x - 2))$ marks
$= (62x - 63x + 126)$ marks
$= (126 - x)$ marks.

But the two students who dropped out received $(46 + 40)$ marks or 86 marks.

$$\therefore \ 126 - x = 86$$
$$\therefore \ -x = -40$$
$$\therefore \ x = 40$$

The original number of students was 40 .

Verification. First aggregate is 40×62 marks or 2,480 marks .
Second aggregate is 38×63 marks or 2,394 marks.
Difference is 86 marks.
This is the number of marks earned by the two students who dropped out.

Exercise 3-7

In this exercise there are some multiple choice questions. Each of these is followed by five answers marked (a), (b), (c), (d), (e). *For each question, decide upon the correct answer, then select the letter that precedes the correct answer.*

(B)

1. Divide \$1,236 among A, B, and C so that A receives twice B's share, and C receives \$150 more than A and B together.
2. A man is twice as old as his son. In 10 years the sum of their ages will be 83 years. Find their present ages.
3. Mr. X has \$10 . Mr. Y repays a debt to Mr. X of \$32 . Mr. X now has three-quarters as much money as Mr. Y has. Before repaying the debt, Mr. Y had (a) \$52 (b) \$63.50 (c) \$56 (d) \$88 (e) none of these.
4. A dealer buys 6 record players, 4 television sets, and 3 radios at a total cost of \$2,900 . If a television set costs \$50 less than twice the cost of a record player and a radio costs \$100 less than the cost of a record player, find the cost of each.
5. A boy is twice as old as his brother. Four years ago he was three times as old as his brother. Find their present ages.
6. A box contains two equal sums of money, one sum in dimes and the other in quarters. If there are 70 coins in all, how much money is there in the box?
7. A labourer in a factory is paid time and one-half for the hours worked in excess of 30 in any week. If he earns \$115.50 for a 38-hour week, his regular hourly rate is (a) \$2.62$\frac{1}{2}$ (b) \$2.75 (c) \$3.39 (d) \$1.80 (e) \$2.50 .
8. If \$15 more than $\frac{1}{3}$ of a certain sum of money is equal to \$25 less than $\frac{1}{2}$ of it, what is the sum of money?
9. In seven years a man will be twice as old as his brother. Find their present ages.
10. A man paid off a debt of \$6,200 in 5 years by paying in each successive year after the first, twice as much as the year before. Calculate his payment for the last year.

11. From a tank of gasoline two-thirds full, 16 gallons are drained leaving the tank filled to 40% of capacity. When it is filled to 60% of capacity, the number of gallons of gasoline it contains is (a) 40 (b) 32 (c) 60 (d) 24 (e) 36 .

12. A man 42 years of age has a daughter 8 years old. In how many years will the father be twice as old as his daughter?

13. A baseball glove costs three times as much as a baseball. If 2 gloves and 3 baseballs cost \$68.85, find the cost of each.

14. A man has \$125 in \$2 bills and \$5 bills. If he has 40 bills in all, how many has he of each?

15. If an article sells for \$11.25 after two successive price reductions of 10% and $16\frac{2}{3}\%$, what was the original price?

16. In a class the average mark was 58. Three students whose marks were 56, 62, and 84 were transferred into the class, raising the average mark to 59. Find the original number of students in the class.

17. John has $\frac{4}{5}$ as much money as Jim. If each spends 60¢, then John has $\frac{3}{4}$ as much as Jim. How much did John have to begin with?

18. If A has $(3x + 2)$ dimes and B has $(x + 5)$ dimes, the difference in the amount of money in nickels is (a) $20x - 30$ (b) 30 (c) $4x - 3$ (d) $2(2x - 3)$ (e) none of these.

19. In 3 years a boy will be $\frac{1}{3}$ as old as his father was 3 years ago. How old will the father be in 5 years, if the difference in their ages is now 28 years?

20. How much money has each of A, B, and C, if A and C together have \$105, B and C together have \$115, and B and A together have \$130?

21. The ages of five persons range from 20 to 50 years. Their average age in years could not be (a) 25 (b) 32 (c) 28 (d) 40 (e) 38.

22. The net price of an article after two successive discounts of 10% and 25% is \$2 more than it would have been, had there been a single discount of 35%. Find the original price of the article.

23. A pair of shoes costing \$$p$ was sold at a profit of $p\%$. If the selling price was $(p + 4)$ dollars, find the cost price of the shoes.

24. A car travelling 60 miles per hour overtakes a truck in 1 hour and 20 minutes. If the truck had a 20-mile head start, its average speed in miles per hour is (a) 50 (b) 42 (c) 45 (d) 40 (e) none of these.

25. How many grams of salt must be added to g grams of a $g\%$ solution of salt and water to make it a $2g\%$ solution?

26. How far is a man's home from downtown, if he walks downtown at 4 miles per hour and returns at once by subway train at 12 miles per hour and is away from home for 2 hours?

27. A bill of \$13.75 was paid in half-dollars and quarters. The number of half-dollars exceeded the number of quarters by 5. How many coins were there of each?

28. Of two sums of money, the second is \$500 more than the first. The first is invested at 2% and the second at 3%. If the total yearly interest is \$255, find the two sums.

29. The athletic society of a high school voted a total of \$145.38 to pay the travelling expenses of its hockey and basketball teams. If the expenses of the latter were \$17.43 more than $\frac{2}{3}$ of the expenses of the former, find the amount of money voted to each team.

30. A train travelling at 40 miles per hour takes 2 hours less in going a certain distance than a train travelling at 24 miles per hour. Find the distance the two trains travelled.

31. A bill was paid in quarters. If it had been paid in dimes, 90 more coins would have been necessary. Find the amount of the bill and the number of quarters required to pay it.

32. A 60% solution of sulphuric acid is obtained by adding 1 pint of water to an 80% solution. Find the total number of pints in the original 80% solution.

Chapter IV

FACTORING POLYNOMIALS

4·1 **The degree of a polynomial.** Expressions similar to $2x^3 - 4x^2 + x - 10$ are referred to as *polynomials*, being composed of a number of terms. The terms of this expression are $+ 2x^3$, $- 4x^2$, $+ x$, $- 10$.

For the purpose of working with polynomials, the $+$ and $-$ signs are considered to be signs of quality and, as such, a part of the term. It is understood that these terms are added and that the polynomial represents a sum. The given polynomial, as written, implies:

$$(+ 2x^3) + (- 4x^2) + (+ x) + (- 10) .$$

The polynomial is meaningless unless a *replacement set* for the variable, represented by x in this expression, is specified. The replacement set in this book, unless otherwise stated, is the set of real numbers.

The terms in polynomials are of two types:

(i) *Variable terms:* those composed of a numerical coefficient associated with a variable by multiplication;

(ii) *Constant terms:* those not containing a variable.

The variable terms in the given polynomial are $2x^3$, $- 4x^2$, x. Their numerical coefficients are 2, $- 4$, and 1, respectively.

The constant term in the given polynomial is $- 10$.

Examples of terms in one variable are:

$$16x^4, \quad 17y^3, \quad -\tfrac{3}{2}z^2, \quad m$$

$16x^4$ is composed of the product of 16, x, x, x, and x, and is said to be a term of the *4th degree* in one variable x, since x appears as a factor four times.

$17y^3$ is a term of the third degree in y;
$-\frac{3}{2}z^2$ is a term of the second degree in z;
m is a term of the first degree in m.

Examples of constant terms are: 15, 1, $\frac{3}{2}$, 256

It is customary to refer to the constant term as a term of degree zero, since any variable to exponent zero is unity. Thus,

$$15x^0 = 15 \times 1 = 15; \qquad \tfrac{3}{2}z^0 = \tfrac{3}{2} \times 1 = \tfrac{3}{2}.$$

The degree of a polynomial is that of the term of highest degree in the polynomial. Thus,

(i) $3x + 2$ is a polynomial of the first degree in x;

(ii) $4y^2$, $4y^2 + 2y$, and $3y^2 - y + 2$ are polynomials of the second degree in y;

(iii) $5y^3 - 3y + 2$ is a polynomial of the third degree in y.

Terms may contain more than one variable. Since the factorization of $3x^2y^3$ includes x as a factor twice and y three times, the degree of $3x^2y^3$ in x and y is 5, that is, the *sum of the exponents of the variables.*

$15x^3y + 3xy^2 + 2xy + 3$ is a polynomial of the *fourth degree* in x and y, since the term of highest degree, $15x^3y$, is a fourth degree term.

4·2 General polynomials in one variable.

a. *General polynomial of the first degree in one variable.*

$$ax + b, \; a \neq 0$$

is called the *general polynomial of the first degree in x.*

(i) x represents the variable;

(ii) a represents the numerical coefficient of x and cannot be zero if there is to be a first degree term;

(iii) b represents the constant term.

For example:

$3x + 2$ is a polynomial of the first degree in x, in which $a = 3$, $b = 2$;
$\frac{5}{2}x$ is a polynomial of the first degree in x, in which $a = \frac{5}{2}$, $b = 0$.

b. *General polynomial of the second degree in one variable.*

$$ax^2 + bx + c, \; a \neq 0$$

is called the *general polynomial of the second degree in x* or the *general quadratic expression.*

(i) x represents the variable;

(ii) a represents the coefficient of x^2, the second degree term, and cannot be zero if there is to be a second degree term;

(iii) b represents the coefficient of x, the first degree term;

(iv) c represents the constant or zero degree term.

For example:

$4x^2 - x + 2$ is a polynomial of the second degree in x, in which $a = 4$, $b = -1$, $c = 2$;

$2x^2 + 2$ is a polynomial of the second degree in x, in which $a = 2$, $b = 0$, $c = 2$;

$\frac{3}{5}x^2$ is a polynomial of the second degree in x, in which $a = \frac{3}{5}$, $b = 0$, $c = 0$.

c. *Polynomials of degree higher than the second in one variable.* In a similar manner, the following represent the general polynomials of the third and fourth degree respectively:

(i) $ax^3 + bx^2 + cx + d$, $a \neq 0$;

(ii) $ax^4 + bx^3 + cx^2 + dx + e$, $a \neq 0$.

Exercise 4-1

(A)

State the degree of each of the following terms:

1. x^2	**2.** $3y^3$	**3.** $4m$	**4.** 5
5. a^4	**6.** $7p^5$	**7.** $84q^7$	**8.** $95d^{15}$
9. $2xy$	**10.** $4x^2y$	**11.** $\frac{5}{2}x^4y^2$	**12.** $-12a^4b^2$
13. $10a^3b^4c^2$	**14.** $5x^0$	**15.** -3	**16.** $14m^2n^3p^4q^6$

State the degree of each of the following polynomials and the number of variables involved:

17. $2x - 3$	**18.** $4x^2 - 5x + 2$	**19.** $3x^2 - 1$
20. $3x^3 - x + 1$	**21.** $4x^3 - 2x^2 + x$	**22.** $5x^4$
23. 3	**24.** $5y - 4x$	**25.** $3x^2 - 4y + 3$
26. $2x^4 - 3xy + 2y^2$	**27.** $3xy^2 + 2xy$	**28.** $15x^5 - 2y^4 + 3z^3$
29. $5x^2y^4 + 3xy^2 + 2$	**30.** $9x^2 - 2x + 1$	**31.** $16z^4 - 6$

Compare the following expressions with the general expression $ax + b$, stating the values of a and b in each case:

32. $3x + 2$	**33.** $2x - 4$	**34.** $5x$
35. 2	**36.** $\frac{1}{5}x - 7$	**37.** $-7x + 2$
38. $-16x$	**39.** $\frac{4}{3}x + \frac{3}{4}$	**40.** $-\frac{1}{6}x - \frac{1}{3}$

Compare the following expressions with the general quadratic expression $ax^2 + bx + c$, *stating the values of* a, b, *and* c *in each case:*

41. $3x^2 - 4x + 2$ **42.** $5x^2 + 3x$ **43.** $17x^2$

44. $-4x^2 + 3$ **45.** $-\frac{1}{5}x^2$ **46.** $-\frac{3}{4}x^2 + \frac{1}{3}$

47. $15x^2 + 2x - 3$ **48.** $18x^2 + 2x - 5$ **49.** $-\frac{1}{3}x^2 - \frac{1}{5}x - 2$

50. State three examples of expressions of the first degree in one variable and state the values of a and b in each case.

51. State three examples of expressions of the second degree in one variable and state the values of a, b, and c in each case.

***4·3* A number and its factors.** Factoring a number (or expression) is the process of finding two or more expressions whose product equals the number.

In general, if a, b, c *represent elements in any set of numbers, and*

$$a = b \times c\,,$$

then b *and* c *are factors of* a, *and* $b \times c$ *is said to be a factorization of* a.

Among the factors of 32 are the numbers

$$1,\quad 2,\quad 4,\quad 8,\quad 16,\quad 32,$$

since the product of each with at least one other number is 32. Since each of the following indicated products,

$$1 \times 32,\ 2 \times 2 \times 2 \times 2 \times 2,\ 2 \times 16,\ 4 \times 8,\ 1 \times 16 \times 2\,,$$

is 32, each is a factorization of 32.

The factorization 1×32 is a special case, since the number 1 is a factor of every number and 32 is the number under consideration. In a still more special sense, $\frac{1}{8}$ is also a factor of 32, since $\frac{1}{8} \times 256 = 32$. Indeed, if we allow factors of this latter sort, *the set of factors* of a given number is *unlimited*. If we restrict ourselves to positive integers only, then the set of all positive integral factors of 32 is

$$\{1,\ 2,\ 4,\ 8,\ 16,\ 32\}\,.$$

Similarly, it may be seen that any number has an unlimited number of factors. Usually the context in which a factorization is to be made suggests which factorization will be the most useful.

In this chapter our interest lies in the factorization of polynomials in which the variables are real numbers. The following are examples:

(i) $5x^2y$ (ii) $3x^2 + 2x$ (iii) $\frac{1}{4}a - \frac{1}{4}b$

(iv) $a^2 - b^2$ (v) $-5x^2 - 10y$

It will be agreed that in the factorization of such expressions:

(i) the factors must have only positive integral exponents;

(ii) no fractions are to be introduced as coefficients.

For example:

(i) $5x^2y = 5 \cdot x^2 \cdot y$ or $5x^2y = 5 \cdot x \cdot xy$ or $5x^2y = 5 \cdot x \cdot x \cdot y$

(ii) $3x^2 + 2x = x(3x + 2)$; check by multiplying the factors.

(iii) $\frac{1}{4}a - \frac{1}{4}b = \frac{1}{4}(a - b)$; check by multiplying the factors. Here one factor is the fraction $\frac{1}{4}$, but it was not introduced in the process of finding the factors; it is the given numerical coefficient of each term in the polynomial expression.

(iv) $a^2 - b^2 = (a + b)(a - b)$; check by multiplying the factors.

(v) $-5x^2 - 10y = -5(x^2 + 2y)$; check by multiplying the factors.

4·4 Factoring a polynomial whose terms have a common factor.

Since $5 \cdot a \cdot b = 5ab$, $\therefore$ the factors of $5ab$ are 5, a, and b.

Since $3 \cdot a \cdot c = 3ac$, $\therefore$ the factors of $3ac$ are 3, a, and c.

Since a is a factor of both $5ab$ and $3ac$, a is said to be a factor common to both numbers and hence a *common factor* of $5ab$ and $3ac$.

The expression $5ab + 3ac$ is a polynomial whose terms have the common factor a.

By the distributive property of multiplication over addition

$$a(x + y) = ax + ay\,,$$

and

$$ax + ay = a(x + y)\,.$$

Thus, an expression whose two terms have a common factor may be factored by means of the distributive property.

Therefore, by the distributive property,

$$5ab + 3ac = a(5b + 3c)\,.$$

Similarly, the extended distributive property provides a means of factoring expressions with more than two terms, if the terms have a common factor.

$$a(x + y + z + \ldots) = ax + ay + az + \ldots\,,$$

and $ax + ay + az + \ldots = a(x + y + z + \ldots)\,.$

Thus, $3x + 6y + 9z = 3(x + 2y + 3z)\,.$ (D)

Example. Factor $7ax - 7ay$.

Solution. Since 7 and a are both common factors of $7ax$ and $7ay$,

$$\therefore\ 7ax - 7ay = 7a(x - y)\,. \qquad (D)$$

Write solutions for the following problems and compare them with those on page 473.

1. Factor $2x^3 - 4x^2 - 2x$.
2. Factor $a(x - y) + b(x - y)$.

Exercise 4-2

(A)

State the common factor in each of the following:

1. $3p - 3q$ **2.** $6a - 6$ **3.** $ax - xy - x$

4. $2a - 6ab$ **5.** $2x^2 - 6x - 2$ **6.** $4x^3 - 6x^2 - 8x$

7. For the expression $a^3 - 2a^2 + a$, state:

(i) the common factor;

(ii) how the second factor is obtained;

(iii) the number of terms in the second factor;

(iv) the second factor.

State the factors of each of the following:

8. $6a - 12$ **9.** $x^3 - x^2 - x$ **10.** $2\pi r^2 + 2\pi rh$

11. $a(x + y) + b(x + y)$ **12.** $x(x - 1) - 2(x - 1)$

(B)

Factor the following:

13. $3x - 6$ **14.** $6a^2 - 2a$ **15.** $x^2 - xy$

16. $ab - a^2$ **17.** $a^2 - ab - a$ **18.** $x - x^2 - xy$

19. $6x^2 - 3xy$ **20.** $ax^2 + a^2x - ax$ **21.** $\frac{1}{2}x - \frac{1}{2}y - \frac{1}{2}z$

22. $10x^2 - 15x^3$ **23.** $2a - 4a^2 + 8a^3$ **24.** $\frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$

25. $6r^3 + 12r^4 + 24r^5$ **26.** $abc - abd + abe$ **27.** $a^4 - 3a^3 - 6a^2$

28. $a(x + y) + a$ **29.** $a(x + y) - b(x + y)$

30. $a(x + y) + (x + y)$ **31.** $x(a - b) + 2(a - b)$

32. $2(p - q) - x(p - q)$ **33.** $a(x - 2) - (x - 2)$

34. $6x^2 + 2x(x - y)$ **35.** $2(x - y) - 4z(x - y)$

36. $(x - y) + (x - y)(x + y)$ **37.** $2(a - b)(a + b) - 4(a + b)$

38. $a(x + y) - b(x + y) + (x + y)$ **39.** $8(a - 2) + (a - 2)(a - 8)$

40. $(a - 2)(a + 2) + (a - 2)(a - 2)$

4·5 Factoring after grouping to obtain a common factor. The product

$$(a + b)(x + y) = a(x + y) + b(x + y) \qquad (D)$$
$$= ax + ay + bx + by. \qquad (D)$$

In order to factor the expression

$$ax + ay + bx + by$$

we should examine the above steps in reverse. This expression can be

factored if the terms are grouped in pairs, each pair in turn having a common factor. Thus,

$$\begin{aligned} ax + ay + bx + by &= (ax + ay) + (bx + by) && (AA) \\ &= a(x + y) + b(x + y) && (D) \\ &= (x + y)(a + b). && (D) \end{aligned}$$

The expression $ax + ay + bx + by$ may be described as an expression with a *common factor after grouping.*

Example. Factor $ax - dy - dx + ay$.

Solution.

		or	
$ax - dy - dx + ay$			$ax - dy - dx + ay$
$= ax - dx + ay - dy$	(CA)		$= ax + ay - dx - dy$ (CA)
$= x(a-d)+y(a-d)$	(D)		$= a(x+y)-d(x+y)$ (D)
$= (a - d)(x + y)$	(D)		$= (x + y)(a - d)$ (D)

It should be observed that in the second solution it was necessary to remove a negative common factor from the second group in order to obtain a repetition of the first bracketed expression.

In the worked examples of the text the authorities which are most significant in the development are given in brackets. When writing a solution you should think the authorities, but it is not necessary to include them in your solution.

Write solutions for the following problems and compare your solutions with those on page 473.

1. Factor $ax + a + x + 1$.
2. Factor $x^2 + y - xy - x$.
3. Factor $(a - b)^2 - ax + bx$.

Exercise 4-3

(B)

Factor:

1. $hx + hy + kx + ky$
2. $hx + my + hy + mx$
3. $ax - by - bx + ay$
4. $ax + by - bx - ay$
5. $a^2 + xy - ax - ay$
6. $(a + b)^2 + 2a + 2b$
7. $x^2 - xy + xz - yz$
8. $(x - y)^2 + x - y$
9. $(x - y)^2 - x + y$
10. $yz - xy + y^2 - xz$
11. $1 + bx + x + b$
12. $x^3 + x - x^2 - 1$
13. $x - a + (x - a)^2$
14. $a(x - 1) - x + 1$

Factor:

15. $6ab - d^2 - 3bd + 2ad$ **16.** $2ax - 3by - 3ay + 2bx$

17. $(ab - 1)^2 - ab + 1$ **18.** $(1 - x)^2 - y + xy$

19. $a(x - y) + b(y - x)$ **20.** $2 - x - 2y + xy$

(C)

Factor:

21. $p(x - y) - px^2 + pxy$ **22.** $x(3a + 3) + y(a + 1)$

23. $x^2 - y^2 - xy^2 + x^3$ **24.** $c(1 - r) - cr + cr^2$

4·6 Factoring trinomials of the form $x^2 + bx + c$. Trinomials of the general form $x^2 + bx + c$ may result from the product of two binomials. For example:

(i) $(x + 3)(x + 2) = x^2 + 5x + 6$
(ii) $(x - 3)(x - 2) = x^2 - 5x + 6$
(iii) $(x - 3)(x + 2) = x^2 - x - 6$
(iv) $(x + 3)(x - 2) = x^2 + x - 6$

In the general example $x^2 + bx + c$, b represents the coefficient of the term containing x, and c represents the constant term. The coefficient of x^2 in each case is 1 .

Comparing $x^2 + bx + c$ with the four examples above:

in (i) $b = 5, c = 6$; in (ii) $b = -5, c = 6$;
in (iii) $b = -1, c = -6$; in (iv) $b = +1, c = -6$.

The expression $x^2 + bx + c$ represents all trinomial expressions in one variable, x, arising from two factors of the form $(x + m)$ and $(x + n)$.

Compare the two products:

$(x + m)(x + n)$		$(x + 2)(x + 3)$	
$= x(x + n) + m(x + n)$	(D)	$= x(x + 3) + 2(x + 3)$	(D)
$= x^2 + nx + mx + mn$	(D)	$= x^2 + 3x + 2x + 6$	(D)
$= x^2 + (m + n)x + mn$	(D)	$= x^2 + (3 + 2)x + 6$	(D)
		$= x^2 + 5x + 6$	

It is seen that the sum of the two numbers m and n (or 2 and 3) is the coefficient of x in the product, and the product of the two numbers m and n (or 2 and 3) is the constant term.

Thus, to factor the polynomial $x^2 + 5x + 6$, the first step is to find two numbers such that:

(i) their product is the constant term 6 ;
(ii) their sum is the coefficient of x, that is, 5 .

Example 1. Factor $x^2 + 5x + 6$.

Solution. Since $3 \times 2 = 6$ and $3 + 2 = 5$

then
$$\begin{aligned} x^2 + 5x + 6 &= x^2 + (3 + 2)x + 6 \\ &= x^2 + 3x + 2x + 6 \qquad (D) \\ &= (x^2 + 3x) + (2x + 6) \\ &= x(x + 3) + 2(x + 3) \qquad (D) \\ &= (x + 3)(x + 2) \qquad (D) \end{aligned}$$

Once the principle is understood, the factors may be written down immediately.

Example 2. Factor $x^2 + 7x + 12$.

Solution. Since $4 \times 3 = 12$ and $4 + 3 = 7$,

$$\therefore\ x^2 + 7x + 12 = (x + 4)(x + 3) .$$

Example 3. Factor $x^2 - 11x + 24$.

Solution. Since $(-8)(-3) = 24$ and $(-8) + (-3) = -11$,

$$\therefore\ x^2 - 11x + 24 = (x - 8)(x - 3),$$

or carrying out the mathematical procedure in full,

$$\begin{aligned} x^2 - 11x + 24 &= x^2 - 8x - 3x + 24 \\ &= x(x - 8) - 3(x - 8) \\ &= (x - 8)(x - 3) . \end{aligned}$$

The required numbers (-8) and (-3) can usually be found by inspection, but the following procedure may be helpful.

Write down all the factors of 24 in order in one line and reverse the order in the line below. Since the sign of the product is positive and the sign of the sum is negative, a negative sign is prefixed to each row. It is understood that this sign is the sign of quality of each number in the row.

−	1	2	3	4	6	8	12	24
−	24	12	8	6	4	3	2	1

The product of each pair is 24. Choose the pair whose sum is (-11), that is, (-3) and (-8), or (-8) and (-3) .

Write solutions for each of the following problems and compare them with those on page 473.

1. Factor $a^2 - 5a - 36$.
2. Factor $x^2 + 8xy - 20y^2$.
3. Factor $2x^3 - 16x^2 - 18x$.

Exercise 4-4

(A)

State the factors of each of the following:

1. $x^2 + 3x + 2$ **2.** $a^2 - 4a + 3$ **3.** $b^2 - b - 2$

4. $x^2 + 2x - 3$ **5.** $a^2 + 6ab + 5b^2$ **6.** $x^2 - 2xy - 3y^2$

(B)

Factor each of the following expressions:

7. $x^2 - 5x + 6$ **8.** $x^2 - x - 6$ **9.** $x^2 + 7x + 6$

10. $x^2 - 5x - 6$ **11.** $x^2 - 6xy + 8y^2$ **12.** $a^2 - 2a - 8$

13. $b^2 + b - 20$ **14.** $c^2 - bc - 2b^2$ **15.** $x^2 + 13xy + 12y^2$

16. $x^2 + 2x - 15$ **17.** $x^2 - 8x + 15$ **18.** $x^2 - 14xy - 15y^2$

19. $a^2 + 11a + 18$ **20.** $a^2 + 6a - 16$ **21.** $a^2 - 8a + 16$

22. $b^2 + 10b - 24$ **23.** $x^2 + 10xy + 24y^2$ **24.** $a^2 - 10a - 24$

25. $y^2 + 5y - 24$ **26.** $y^2 - 3y - 18$ **27.** $a^2 - a - 30$

28. $x^2 + 12x + 32$ **29.** $x^2 - 10xy + 25y^2$ **30.** $y^2 - 6yz - 7z^2$

31. $a^3 - 18a^2 + 17a$ **32.** $x^3 - 3x^2 - 10x$ **33.** $3a^2 + 21a - 24$

34. $2x^3 - 22x^2 - 52x$ **35.** $2a^2b - 8ab - 24b$ **36.** $p^2q^2 - 7pq^2 - 18q^2$

(C)

Factor:

37. $x^2 - 15x - 100$ **38.** $a^2b^2 - 11ab - 60$

39. $x^4 - 29x^2 - 96$ **40.** $(a + b)^2 + 7(a + b) + 12$

41. $(x - y)^2 - (x - y) - 12$ **42.** $(a - b)^2 - 4(a - b) - 12$

***4·7* Factoring trinomials of the form $ax^2 + bx + c$.** Trinomials of the form $ax^2 + bx + c$ result from the product of two binomials such as the following:

(i) $(2x + 3)(3x + 4) = 6x^2 + 17x + 12$

(ii) $(4x - 1)(2x - 5) = 8x^2 - 22x + 5$

(iii) $(2x + 3)(5x - 2) = 10x^2 + 11x - 6$

In (i) $a = 6$, $b = 17$, $c = 12$; in (ii) $a = 8$, $b = -22$, $c = 5$; in (iii) $a = 10$, $b = 11$, $c = -6$.

The expression $ax^2 + bx + c$ represents all trinomials in one variable x arising from the product of two binomials of the form

$$(mx + n) \text{ and } (px + q).$$

Compare the two products:

(i) $(mx + n)(px + q)$		(ii) $(2x + 3)(3x + 4)$	
$= mx(px + q) + n(px + q)$	(D)	$= 2x(3x + 4) + 3(3x + 4)$	(D)
$= mpx^2 + mqx + npx + nq$	(D)	$= 6x^2 + 8x + 9x + 12$	(D)
$= mpx^2 + (mq + np)x + nq$	(D)	$= 6x^2 + (8 + 9)x + 12$	(D)
		$= 6x^2 + 17x + 12$	

In (i) the *product* of the coefficient of x^2 and the constant term is $mnpq$ or $(mq)(np)$; the *coefficient* of x is $(mq) + (np)$. Thus mq and np are two numbers whose product is $(mq)(np)$ and whose sum is the coefficient of x, $mq + np$. In (ii) the product of the coefficient of x^2 and the constant term is $6 \times 12 = 72 = 8 \times 9$; the coefficient of x is $8 + 9 = 17$. Thus 8 and 9 are two numbers whose product is $6 \times 12 = 72$ and whose sum is 17, the coefficient of x.

Thus, if the factors of $6x^2 + 17x + 12$ are of the form $(mx + n)$ and $(px + q)$, then to factor $6x^2 + 17x + 12$, it is first necessary to find two numbers such that:

(i) their product is 72; (ii) their sum is 17.

To obtain the two numbers, express 72 as the product of its prime factors; then associate these factors to obtain two numbers whose sum is 17.

$$\begin{aligned} 72 &= (2 \times 2 \times 2) \times (3 \times 3) \\ &= 8 \times 9 \\ 17 &= 8 + 9 \end{aligned}$$

To complete the factorization of the expression, write:

$$\begin{aligned} 6x^2 + 17x + 12 &= 6x^2 + 8x + 9x + 12 \\ &= (6x^2 + 8x) + (9x + 12) && (AA) \\ &= 2x(3x + 4) + 3(3x + 4) && (D) \\ &= (2x + 3)(3x + 4) && (D) \end{aligned}$$

Example 1. Factor $2x^2 + 7x + 3$.

Solution. By inspection,

$$2 \times 3 = 6 = 6 \times 1 \quad | \quad 7 = 6 + 1$$

Then,

$$\begin{aligned} 2x^2 + 7x + 3 &= 2x^2 + 6x + x + 3 \\ &= (2x^2 + 6x) + (x + 3) && (AA) \\ &= 2x(x + 3) + (x + 3) && (D) \\ &= (2x + 1)(x + 3) && (D) \end{aligned}$$

Example 2. Factor $10x^2 - 9x + 2$.

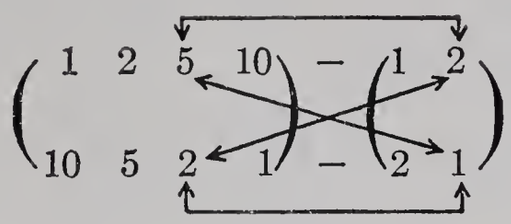
Solution.

$$\begin{aligned} 10 \times 2 &= 2 \times 2 \times 5 \\ &= 4 \times 5 \\ &= (-4)(-5) \end{aligned} \quad \Big| \quad -9 = (-4) + (-5)$$

$$\therefore\ 10x^2 - 9x + 2 = 10x^2 - 5x - 4x + 2$$
$$= 5x(2x - 1) - 2(2x - 1) \qquad (D)$$
$$= (2x - 1)(5x - 2) \qquad (D)$$

The following method may be preferred:

$$10x^2 - 9x + 2$$



Each of the first set of factor pairs has a product which is 10 .
Each of the second set of factor pairs has a product which is 2 .

The − signs are placed between the two groups to ensure the correct sign for the terms + 2 and − 9x. It is now necessary to select the correct cross-products whose sum is − 9 . By inspection, the marked cross-products have the sum (− 5) + (− 4) or − 9 .

Thus $10x^2 - 9x + 2 = (5x - 2)(2x - 1)$.

Write a solution for the following problem and compare it with that on page 473.

1. Factor $6a^2 - 11ab - 10b^2$.

Exercise 4-5

(B)

Factor each of the following:

1. $2x^2 + 5x + 2$	**2.** $3a^2 - 5a + 2$	**3.** $2y^2 - y - 3$
4. $2x^2 - 15x + 7$	**5.** $5m^2 + 3m - 2$	**6.** $7p^2 - 2p - 5$
7. $3a^2 + 16a + 5$	**8.** $5x^2 + 14x - 3$	**9.** $3x^2 - 22xy + 7y^2$
10. $6a^2 - 7a + 2$	**11.** $4x^2 - x - 3$	**12.** $9a^2 + 12ab - 5b^2$
13. $7x^2 - 12xy - 4y^2$	**14.** $5a^2 - 7a - 6$	**15.** $3x^2 - 10xy + 8y^2$
16. $8a^2 - 2a - 3$	**17.** $6p^2 + 13pq + 6q^2$	**18.** $7 + 11x - 6x^2$
19. $5 - 12y - 9y^2$	**20.** $25a^2 - 10a + 1$	**21.** $4a^2 + 12ab + 9b^2$
22. $6x^2 - 5xy - 4y^2$	**23.** $9 + 25b - 6b^2$	**24.** $12y^2 - 5y - 3$
25. $10 - 19a + 6a^2$	**26.** $8x^2 - 31xy - 4y^2$	**27.** $15 + m - 6m^2$
28. $15x^4 + 11x^2 - 12$	**29.** $16y^4 + 34y^2 + 15$	**30.** $16x^4 - 24x^2y^2 + 9y^4$
31. $18 - b - 4b^2$	**32.** $24x^2 + 13x - 2$	**33.** $6x^2 + 31xy - 30y^2$
34. $12 - 31x + 20x^2$	**35.** $18a^2 + 19a - 12$	**36.** $16h^2 - 9hk - 25k^2$

(C)

Factor:

37. $4\left(\frac{x}{y}\right)^2 - 8\left(\frac{x}{y}\right) + 3$

38. $6x^2 - 13 - \frac{15}{x^2}$

39. $3(a-b)^2 - 7(a-b) - 6$

40. $x^2 - (m+n)x + mn$

41. $abx^2 + (3a+2b)x + 6$

42. $abx^2 + (bc+ad)x + cd$

43. $abx^2 + (an - bm)x - mn$

***4·8* Factoring a trinomial square.** The trinomial $x^2 + 12x + 36$ can be factored as follows:

$$36 = 6 \times 6, \text{ and } 12 = 6 + 6$$

$$\begin{aligned} x^2 + 12x + 36 &= x^2 + 6x + 6x + 36 \\ &= x(x+6) + 6(x+6) && (D) \\ &= (x+6)(x+6) && (D) \\ &= (x+6)^2 \end{aligned}$$

Because the expression $x^2 + 12x + 36$ can be expressed as the square of $(x + 6)$, it is often called a *trinomial square.*

In general,

$$\begin{aligned} a^2 + 2ab + b^2 &= a^2 + ab + ab + b^2 \\ &= a(a+b) + b(a+b) && (D) \\ &= (a+b)(a+b) && (D) \\ &= (a+b)^2. \end{aligned}$$

Similarly, $a^2 - 2ab + b^2 = (a-b)^2$.

Thus, any trinomial of the form $a^2 + 2ab + b^2$ or $a^2 - 2ab + b^2$ is called a trinomial square and may be factored by inspection.

Example. Factor: (i) $16x^2 + 40xy + 25y^2$ (ii) $9a^2 - 24a + 16$

Solution.

	EXPLANATION
(i) $16x^2 + 40xy + 25y^2 = (4x + 5y)^2$	The first and last terms are the squares of $4x$ and $5y$, respectively. The middle term is $+ 2(4x)(5y)$.
(ii) $9a^2 - 24a + 16 = (3a - 4)^2$	The first and last terms are the squares of $3a$ and 4, respectively. The middle term is $-2(3a)(4)$.

Write solutions for the following problems and compare them with those on page 474.

1. Show that $36x^2 - 30x + 25$ is not a trinomial square.

2. Find the square roots of $36x^2 - 60xy + 25y^2$.

3. Simplify $\sqrt{9x^2 + 60x + 100}$.

Exercise 4-6

(A)

State the missing term in each of the following, so that the expression may be a trinomial square:

1. $x^2 + ____ + 9$ **2.** $16 - ____ + x^2$ **3.** $4x^2 - ____ + 9y^2$

4. $100x^2 + ____ + y^2$ **5.** $9x^2 - ____ + 16y^2$ **6.** $25 + ____ + 36a^2$

7. $x^2 + 8x + ____$ **8.** $y^2 - 18y + ____$ **9.** $x^2 + 3x + ____$

10. $4x^2 - 12x + ____$ **11.** $9a^2 + 30a + ____$ **12.** $____ - 20y + 25y^2$

Factor:

13. $25a^2 - 10a + 1$ **14.** $x^2 + 24x + 144$ **15.** $4x^2 - 12xy + 9y^2$

State the square roots of:

16. $9a^2 - 6ab + b^2$ **17.** $1 - 2x + x^2$ **18.** $4x^2 - 36x + 81$

Simplify:

19. $\sqrt{1 - 4x + 4x^2}$ **20.** $\sqrt{a^2 - 14a + 49}$ **21.** $\sqrt{36x^2 - 12x + 1}$

(B)

Rewrite each of the following as a trinomial square by supplying the missing term:

22. $16x^2 - ____ + 81$ **23.** $49y^2 + ____ + 4$ **24.** $400 - ____ + 9y^2$

25. $y^4 + 20y^2 + ____$ **26.** $16a^2 + 56a + ____$ **27.** $36y^2 - 60y + ____$

Write the square roots of each of the following:

28. $144x^2 - 120x + 25$ **29.** $64y^4 + 144y^2 + 81$ **30.** $4a^2b^2 - 44ab + 121$

Simplify:

31. $\sqrt{9a^2 - 120a + 400}$ **32.** $\sqrt{49 - 140a + 100a^2}$

33. $\sqrt{169x^2 - 2x + 1}$ **34.** $\sqrt{x^2 + 2x + 1} - \sqrt{x^2 - 2x + 1}$

35. $2\sqrt{a^2 - 6a + 9} - 3\sqrt{4 - 4a + a^2}$

Factor:

36. $81a^4 - 72a^2 + 16$ **37.** $1 - 16ab + 64a^2b^2$

38. $x^2 + x + \frac{1}{4}$ **39.** $x^2 - 2 + \frac{1}{x^2}$

40. $\frac{x^2}{9} + \frac{2x}{3} + 1$ **41.** $\frac{x^2}{16} - \frac{xy}{6} + \frac{y^2}{9}$

4·9 Factoring the difference of the squares of two numbers. For any two real numbers represented by a and b,

$$\begin{aligned}(a+b)(a-b) &= a(a-b)+b(a-b) && (D)\\ &= a^2-ab+ba-b^2 && (D)\\ &= a^2-b^2.\end{aligned}$$

$(a+b)$ is the *sum* of the two numbers a and b;

$(a-b)$ is the *difference* of the two numbers a and b;

a^2-b^2 is the *difference of the squares* of a and b.

Thus, the product of the sum and difference of the same two real numbers is equal to the difference of the squares of the two numbers.

Also, the factors of the difference of the squares of two real numbers are the sum and difference of the two numbers.

Thus, $a^2-b^2=(a-b)(a+b)$.

Example. Factor: (i) x^2-16 (ii) $25-4y^2$

Solution. (i) $x^2-16=(x-4)(x+4)$

(ii) $25-4y^2=(5-2y)(5+2y)$

Write solutions for the following problems and compare them with those on page 474.

1. Factor: (i) x^4-16y^4 (ii) $(a-b)^2-c^2$

2. Factor: (i) $9-(x-y)^2$ (ii) $(a+b)^2-(c-d)^2$

Exercise 4-7

(A)

State the factors of:

1. a^2-4	**2.** a^2-9	**3.** p^2-q^2
4. m^2-16n^2	**5.** $100-81x^2$	**6.** 10^2-a^2
7. 10^2-9^2	**8.** 16^2-b^2	**9.** 98^2-2^2
10. $64a^2-b^2$	**11.** $25x^2-36y^2$	**12.** $1-a^2b^2$
13. $(a+b)^2-c^2$	**14.** $(x-y)^2-4$	**15.** $(a-b)^2-25c^2$

(B)

Factor:

16. b^2-1	**17.** $1-x^2$	**18.** $1-16y^2$
19. $36a^2-25$	**20.** $144-49y^2$	**21.** $m^2n^2-p^2$
22. a^3-4a	**23.** x^3y-xy^3	**24.** $2a^2-32b^2$

Factor:

25. $10x^3 - 90x$ **26.** $3 - 48x^2$ **27.** $2a^3 - 50ab^2$

28. $x^4 - 1$ **29.** $1 - 16a^4$ **30.** $162a^4 - 2b^4$

31. $(x + y)^2 - z^2$ **32.** $(a - b)^2 - 1$ **33.** $(2x - 3)^2 - y^2$

34. $(x^2 + 1)^2 - x^2$ **35.** $(x^2 + y^2)^2 - x^2y^2$

36. $(a^2 + 2b^2)^2 - 4a^2b^2$

37. $x^2 - \dfrac{1}{x^2}$ **38.** $9(a - 2b)^2 - 1$ **39.** $(a-b)^2-(c+d)^2$

40. $(x-y)^2-(p-q)^2$ **41.** $a^2 - (b + c)^2$ **42.** $a^2 - (b - c)^2$

(C)

43. Find four factors of $x^4 - 10x^2 + 9$.

44. What factor is common to all of the following?

(i) $a^2 - 3a$ (ii) $a^2 - 9$

(iii) $a^2 - 6a + 9$ (iv) $a^2 - 2a - 3$

45. Factor: (i) $(a^2 + 4)^2 - 25a^2$ (ii) $a^8 - b^8$

***4·10* Factoring after grouping to obtain the difference of the squares of two numbers.** In section 4·6 an expression was factored by first grouping terms having a common factor. Grouping of terms in an expression may be suggested by relationships other than that of a common factor. For example, in the expression

$$a^2 - 2ab + b^2 - c^2,$$

the term $-2ab$ suggests an association with the terms a^2 and b^2. It is observed that $a^2 - 2ab + b^2$ is a trinomial square which may be replaced by $(a - b)^2$. This leads to the observation that

$$a^2 - 2ab + b^2 - c^2 \text{ which is equivalent to } (a - b)^2 - c^2$$

is a difference of two squares, an expression which can be factored by known methods.

Thus,
$$\begin{aligned} a^2 - 2ab + b^2 - c^2 &= (a^2 - 2ab + b^2) - c^2 \\ &= (a - b)^2 - c^2 \\ &= (a - b - c)(a - b + c). \end{aligned}$$

Terms such as $-2ab$ in an expression may be thought of as *clue* terms, since their presence is suggestive of combinations in groupings which may be used to change the form of an expression. Some expressions, not directly factorable, may be rewritten and, by proper grouping, expressed as the difference of the squares of two numbers.

Example 1. Factor $x^2 - z^2 + 8xy + 16y^2$.

Solution. The term $8xy$ suggests grouping $x^2 + 8xy + 16y^2$ which is $(x + 4y)^2$.

$$\begin{aligned} x^2 - z^2 + 8xy + 16y^2 &= (x^2 + 8xy + 16y^2) - z^2 \\ &= (x + 4y)^2 - z^2 \\ &= (x + 4y - z)(x + 4y + z) \end{aligned}$$

Example 2. Factor $m^2 - p^2 + 4m + 4$.

Solution.
$$\begin{aligned} m^2 - p^2 + 4m + 4 &= (m^2 + 4m + 4) - p^2 \\ &= (m + 2)^2 - p^2 \\ &= (m + 2 - p)(m + 2 + p) \end{aligned}$$

Write solutions for the following problems and compare them with those on page 474.

1. Factor $x^2 + y^2 - z^2 - p^2 - 2xy + 2zp$.
2. Factor $4y^2 + 6x - 9x^2 - 1$.

Exercise 4-8

(A)

Factor:

1. $(a + b)^2 - c^2$
2. $(x - y)^2 - z^2$
3. $(p - q)^2 - 1$
4. $a^2 + 2ab + b^2 - 1$
5. $x^2 - 2xy + y^2 - 4$
6. $x^2 + 2x + 1 - y^2$

(B)

Factor:

7. $x^2 - 6x + 9 - y^2$
8. $a^2 - 10a + 25 - m^2$
9. $a^4 - 2a^2 + 1 - b^2$
10. $x^2 - y^2 + 4x + 4$
11. $9x^2 - 4y^2 + 6x + 1$
12. $25 - 10a + a^2 - b^2$
13. $a^4 + 2a^2 + 1 - a^2$
14. $25x^2 + 1 - 10x - y^2$
15. $16a^4 - 8a^2 - b^4 + 1$
16. $9a^2 - 25b^2 + 24ac + 16c^2$
17. $a^2 + 8ab + 16b^2 - c^2$
18. $4m^2 - p^2 - 16pq - 64q^2$
19. $a^2 - b^2 + c^2 - d^2 - 2ac - 2bd$

***4·11* The quadratic equation.** If $a, b \in R$ and $a = 0$ or $b = 0$, then $ab = 0$.

Thus, if $x + 2 = 0$ or $x - 3 = 0$, then

$$(x + 2)(x - 3) = 0 \text{ and } x^2 - x - 6 = 0.$$

The equation $(x + 2)(x - 3) = 0$ (or $x^2 - x - 6 = 0$) is called the *joint equation* of the equations $x + 2 = 0$ and $x - 3 = 0$.

If $a, b \in R$ and $ab = 0$, then $a = 0$ or $b = 0$.

Thus, if $x^2 - x - 6 = 0,$

that is, $(x + 2)(x - 3) = 0$

then, $x + 2 = 0 \quad \text{or} \quad x - 3 = 0.$

Thus the statements:

(i) $x + 2 = 0$ or $x - 3 = 0$

(ii) $(x + 2)(x - 3) = 0$

(iii) $x^2 - x + 6 = 0$

are equivalent.

Since $x + 2 = 0 \quad \text{or} \quad x - 3 = 0$ is equivalent to

$x = -2 \quad \text{or} \quad x = 3,$

therefore the solution set of the equation $x^2 - x + 6 = 0$ is $\{-2, 3\}$.

The equations $x^2 - x + 6 = 0$ and $(x + 2)(x - 3) = 0$ are called *quadratic equations* in one variable, since the highest power in which the variable appears is the second power.

Other examples of quadratic equations are:

$3b^2 - 4 = 0, \quad 7a^2 - 2a = 0, \quad \sqrt{3}y^2 - \sqrt{2}y + 3.14 = 0$

Example 1. By factoring, solve $4x^2 - 25 = 0$.

Solution.

$$4x^2 - 25 = 0$$

$$\therefore (2x - 5)(2x + 5) = 0$$

$$\therefore 2x - 5 = 0 \quad \text{or} \quad 2x + 5 = 0$$

$$\therefore x = \frac{5}{2} \quad \Big| \quad \therefore x = -\frac{5}{2}$$

The solution set is $\left\{\frac{5}{2}, -\frac{5}{2}\right\}$.

Verification.

$$\text{L.S.} = 4\left(\frac{5}{2}\right)^2 - 25 \qquad \text{R.S.} = 0$$

$$= 25 - 25$$

$$= 0$$

$$\text{L.S.} = 4\left(-\frac{5}{2}\right)^2 - 25 \qquad \text{R.S.} = 0$$

$$= 25 - 25$$

$$= 0$$

Example 2. A number of footballs was purchased for \$150. If each football had cost \$5.00 more, 5 fewer would have been received. Find the original price of each football.

Solution. Represent the price of one football by $\$x$ ($x \in R$, $x > 0$).

	PRICE PER FOOTBALL IN DOLLARS	TOTAL COST IN DOLLARS	NUMBER PURCHASED
At original price	x	150	$\frac{150}{x}$
At increased price	$x + 5$	150	$\frac{150}{x+5}$

The difference in number purchased is $\frac{150}{x} - \frac{150}{x+5}$.

$$\therefore \frac{150}{x} - \frac{150}{x+5} = 5$$

$$\therefore x^2 + 5x - 150 = 0$$

$$\therefore (x + 15)(x - 10) = 0$$

$$\therefore x = -15 \text{ or } 10$$

The original price of each football is \$10.00 .

Verification. At \$10 each the number of footballs purchased is $\frac{150}{10}$ or 15 .
At \$15 each the number of footballs purchased is $\frac{150}{15}$ or 10 .
Difference is 5 .

Write solutions for the following problems and compare them with those on page 474.

1. Solve $2x^2 + 3x - 2 = 0$ and verify .

2. Solve $x^2 - 6 = x$.

Exercise 4-9

(A)

By factoring, state an equivalent quadratic equation for each of the following:

1. $x^2 + 3x + 2 = 0$ **2.** $a^2 - 4 = 0$ **3.** $x^2 + 2x + 1 = 0$

4. $n^2 - n - 12 = 0$ **5.** $4x^2 - 9 = 0$ **6.** $3x^2 + 2x - 1 = 0$

(B)

Solve and verify:

7. $(m-1)(n-7)=0$ **8.** $(a-2)(a+3)=0$ **9.** $x^2 - x - 2 = 0$

10. $p^2 - p = 12$ **11.** $b^2 - b - 20 = 0$ **12.** $y^2 + 15 = 8y$

Solve and verify:

13. $x^2 + 2x - 63 = 0$ **14.** $p^2 + 13p + 12 = 0$ **15.** $2a^2 + 3a - 2 = 0$

16. $10x^2+29x+10=0$ **17.** $a^2 - 2a + 1 = 0$ **18.** $b^2 + 6b + 9 = 0$

19. $c^2 - c - 42 = 0$ **20.** $m^2 = 3m + 10$ **21.** $x^2 - 1 = 0$

22. $p^2 - 100 = 0$ **23.** $x^2 - \frac{1}{4} = 0$ **24.** $a^2 - .25 = 0$

25. $4x^2 - 100 = 0$ **26.** $7a^2 - 343 = 0$ **27.** $y^2 - 7y + 6 = 0$

28. $x^2 - x - 90 = 0$ **29.** $a^2 - 12 = 11a$ **30.** $p^2 - 15p = -50$

31. $a^2 - 169 = 0$ **32.** $b^2 - \frac{4}{25} = 0$ **33.** $c^2 = 36$

34. $a^2 = \frac{25}{49}$ **35.** $a^2 - a - 20 = 0$ **36.** $b^2 + 2b - 35 = 0$

37. $x^2 + 5x + 4 = 0$ **38.** $y^2 - 7y + 12 = 0$ **39.** $b^2 - 16b = 17$

40. $c^2 + 21c + 68 = 0$ **41.** $m^2-16m+64=0$ **42.** $3x + 88 = x^2$

43. $2a^2 - 18a = 0$ **44.** $5b^2 - 5b = 280$ **45.** $2b^2 + 3b = 5$

46. $3p^2 - 14 = p$ **47.** $10x^2 - 5x = 30$ **48.** $7y^2 = -36y - 5$

49. Twice the square of a certain number decreased by 6 times the number is 80. Find the number.

50. Find two consecutive whole numbers whose product is 72 .

51. Find two consecutive even integers whose product is 80 .

52. Two trains make the same trip of 210 miles. The "express" goes 16 m.p.h. faster on the average than the "local" and covers the distance in $1\frac{1}{2}$ hours less time. Find the average speed of the two trains.

***4·12* Factoring an incomplete trinomial square (supplementary).** Consider the expressions

(i) $a^4 + a^2b^2 + b^4$, and

(ii) $x^4 + 2x^2 + 9$.

$a^4 + a^2b^2 + b^4$ would be a trinomial square if its middle term were $+ 2a^2b^2$ or $- 2a^2b^2$.

$x^4 + 2x^2 + 9$ would be a trinomial square if its middle term were $6x^2$ or $- 6x^2$.

Expressions such as (i) and (ii) are sometimes called *incomplete squares.*

Equivalent expressions for these may be written with alterations as indicated in the following:

$$
\begin{aligned}
& a^4 + a^2b^2 + b^4 \\
&= a^4 + 2a^2b^2 + b^4 - a^2b^2 \\
&= (a^2 + b^2)^2 - (ab)^2 \\
&= (a^2 + b^2 - ab)(a^2 + b^2 + ab) \\
&= (a^2 - ab + b^2)(a^2 + ab + b^2)
\end{aligned}
$$

$$
\begin{aligned}
& x^4 + 2x^2 + 9 \\
&= x^4 + 6x^2 + 9 - 4x^2 \\
&= (x^2 + 3)^2 - (2x)^2 \\
&= (x^2 + 3 - 2x)(x^2 + 3 + 2x) \\
&= (x^2 - 2x + 3)(x^2 + 2x + 3)
\end{aligned}
$$

The two expressions have been rewritten in the form of the difference of the squares of two numbers by adding the number required to complete the square and then subtracting it to maintain equality. They have then been factored as indicated.

If an incomplete square can be expressed in the form of the difference of the squares of two numbers, it can be factored.

Example. Factor $4x^4 + 8x^2 + 9$.

Solution.
$$\begin{aligned} 4x^4 + 8x^2 + 9 &= (4x^4 + 12x^2 + 9) - 4x^2 \\ &= (2x^2 + 3)^2 - (2x)^2 \\ &= (2x^2 + 3 - 2x)(2x^2 + 3 + 2x) \\ &= (2x^2 - 2x + 3)(2x^2 + 2x + 3) \end{aligned}$$

Write solutions to the following problems and compare them with those on page 475.

1. Factor $a^4 + 4b^4$.

2. Factor $9y^4 - 51y^2 + 49$.

Exercise 4-10

(B)

Factor:

1. $x^4 + x^2 + 1$ **2.** $x^4 - 3x^2 + 1$ **3.** $4x^4 + 1$

4. $a^4 + 5a^2b^2 + 9b^4$ **5.** $b^4 - 9b^2c^2 + 16c^4$ **6.** $4y^4 + 8y^2 + 9$

7. $64 + x^4$ **8.** $64a^4 + 39a^2 + 9$ **9.** $25a^4 - 89a^2b^2 + 16b^4$

10. $81x^4 + 144x^2 + 100$ **11.** $121 - 166y^2 + y^4$ **12.** $4x^4 + 81$

(C)

13. Find three factors of $b^8 + b^4 + 1$.

14. Find four factors of $16y^4 - 65y^2 + 49$.

15. Factor $16 + 8x^4 + x^8$. **16.** Factor $x^4 - 14x^2 + 1$.

4·13 The Factor Theorem.

a. *Multiplication by zero.* By definition the product of any real number and zero is zero.

$$\text{If } a \in R, \text{ then } a \times 0 = 0 \times a = 0 .$$

Also, if the product of two real numbers is zero, then one, or the other, or both real numbers are zero.

In general,

if $a, b \in R$ and $a \cdot b = 0$, then $a = 0$ or $b = 0$.

b. *Identical equations.* The product $(x - 1)(x - 3)$ when expanded is $x^2 - 4x + 3$. Thus, for all real values of x, the two expressions are equivalent and

$$\therefore\ (x - 1)(x - 3) = x^2 - 4x + 3 \text{ for all } x \in R\,.$$

This means that $(x - 1)(x - 3) = x^2 - 4x + 3$ is an equation with the solution set $\{x \mid x \in R\}$. Equations whose solution sets are the set of numbers under consideration are called *identical equations* or *identities* in that set.

c. *The Factor Theorem.* Since $(x - 1)(x - 3) = x^2 - 4x + 3$ is an identity in the set of real numbers, then if $x = 1$

$$\begin{aligned} \text{L.S.} &= (1 - 1)(1 - 3) \\ &= 0(-2) \\ &= 0 \end{aligned} \qquad \begin{aligned} \text{R.S.} &= 1^2 - 4 \times 1 + 3 \\ &= 1 - 4 + 3 \\ &= 0 \end{aligned}$$

By definition $(x - 1)$ is a factor of $x^2 - 4x + 3$, and it is seen that if x is replaced by $(+1)$ in $x^2 - 4x + 3$, the expression equals zero.

Also if $x = 3$, then

$$\begin{aligned} \text{L.S.} &= (3 - 1)(3 - 3) \\ &= 2 \times 0 \\ &= 0 \end{aligned} \qquad \begin{aligned} \text{R.S.} &= 3^2 - 4 \times 3 + 3 \\ &= 9 - 12 + 3 \\ &= 0 \end{aligned}$$

By definition $(x - 3)$ is a factor of $x^2 - 4x + 3$, and it is seen that if x is replaced by $(+3)$ in $x^2 - 4x + 3$, the expression equals zero.

No real replacements for x other than 1 or 3 make the expression $x^2 - 4x + 3$ equal to zero, since only the replacements 1 or 3 make either factor $(x - 1)$ or $(x - 3)$ equal to zero.

This suggests the conclusion that:

(i) if $x = 1$ makes $x^2 - 4x + 3$ equal to zero, then $(x - 1)$ is a factor of $x^2 - 4x + 3$;

(ii) if $x = 3$ makes $x^2 - 4x + 3$ equal to zero, then $(x - 3)$ is a factor of $x^2 - 4x + 3$.

In general,

if $x = a$ makes an expression equal to zero, then
$(x - a)$ is a factor of the expression.

This theorem, called the *Factor Theorem,* is not proved here, but it is accepted and used as a means of factoring some expressions.

Example 1. Find the factors of $x^3 + 2x^2 - 5x - 6$.

Solution. By the Factor Theorem, if a replacement a can be found for x which makes the expression equal to zero, then $x - a$ is a factor. Also, since a product like $(x - 1)(x - 2)(x - 3)$ produces a constant term

in its expansion which is the product of $(-1)(-2)(-3) = -6$, it is seen that the replacements for x may be limited to the integral factors of the constant term which in this case are $\pm 1, \pm 2, \pm 3, \pm 6$.

If $x = 1$, the expression $= 1 + 2 - 5 - 6 \neq 0$

$\therefore$ $x - 1$ is not a factor of the expression.

If $x = -1$, the expression $= (-1)^3 + 2(-1)^2 - 5(-1) - 6$

$= -1 + 2 + 5 - 6$

$= 0$.

$\therefore$ $x + 1$ is a factor of the expression.

Similarly, other factors may be found or, having found one factor, the second factor may be found by division.

$\therefore x^3 + 2x^2 - 5x - 6$

$= (x + 1)(x^2 + x - 6)$

$= (x + 1)(x - 2)(x + 3)$

$$
\begin{array}{r|l}
x + 1 & x^3 + 2x^2 - 5x - 6 \\
 & \underline{x^3 + x^2} \\
 & \quad x^2 - 5x \\
 & \quad \underline{x^2 + x} \\
 & \quad\quad -6x - 6 \\
 & \quad\quad \underline{-6x - 6}
\end{array}
\qquad x^2 + x - 6
$$

Example 2. Show by the Factor Theorem that $x - 1$ is a factor of $2x^3 - 4x^2 + 6x - 4$.

Solution. If $x = 1$, then the expression $= 2 - 4 + 6 - 4 = 0$.

$\therefore$ $x - 1$ is a factor of the expression.

Write solutions for the following problems and compare them with those on page 475.

1. Factor $2x^3 - 11x^2 + 5x + 4$.

2. Factor: (i) $x^3 - y^3$ (the difference of two cubes)

(ii) $x^3 + y^3$ (the sum of two cubes)

3. Show by the Factor Theorem that $(y + 3)$ is a factor of $y^3 + 2y^2 - 7y - 12$.

Exercise 4-11

(A)

1. Show that $x - 1$ is a factor of $x^3 - 3x^2 + 2$. State how you would obtain a second factor.

Show that $x - 1$ is a factor of:

2. $4x^3 - x - 3$ **3.** $8x^4 - 6x^3 - 2$ **4.** $x^5 - 1$

Show that $x + 1$ is a factor of:

5. $2x^3 + x^2 + 1$ **6.** $x^3 - 2x - 1$ **7.** $x^5 + 1$

Show that $x - a$ is a factor of:

8. $x^3 - a^3$ **9.** $x^3 - 7ax^2 + 6a^3$ **10.** $x^5 - a^5$

Show that $x + a$ is a factor of:

11. $x^3 + a^3$ **12.** $x^3 + 10a^2x + 11a^3$ **13.** $x^5 + a^5$

14. If the polynomial $y^3 + y^2 - 14y - 24$ equals zero when $y = -2$, 4, or -3, state the three factors of the polynomial.

(B)

Factor:

15. $x^3 - 3x + 2$ **16.** $x^3 - 3x - 2$ **17.** $x^3 - x^2 - x - 2$

18. $6x^3+7x^2-9x+2$ **19.** $2x^3+15x^2-16x-1$ **20.** $a^3 - 2a^2 - 9$

21. $y^3 - 7y^2 + 16y - 12$ **22.** $2a^3 - 3a^2 - 8a - 3$

23. $x^3 + 2x^2 - 4x - 3$ **24.** $y^3 - 6y^2 - 9y + 14$

25. $3x^3 - 22x^2 + 43x - 12$ **26.** $y^3 - 125$

27. $6x^3 + 13x^2 + x - 2$ **28.** $4x^3 + 8x^2 - 11x + 3$

(C)

29. Use the Factor Theorem to show that $x - y$, $x - 2y$, and $x + 3y$ are factors of $x^3 - 7xy^2 + 6y^3$.

Factor:

30. $x^3 - 2xy^2 + y^3$ **31.** $2x^3 + 5x^2y + 4xy^2 + y^3$

32. $x^3 - 8a^3$ **33.** $x^3 + 27y^3$

34. $10a^3 + 3a^2b - 6ab^2 + b^3$ **35.** $5a^3 - 6a^2x + x^3$

36. Show that $x + y$ is a factor of $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

Factor:

37. $x^4 - 10x^3 + 35x^2 - 50x + 24$

38. If $x - 1$ is a factor of $x^3 + 3x^2 + 2x - k$, find the value of k.

39. If $x + 2$ is a factor of $x^4 - x^2 + p$, find the value of p.

40. If $x - 3$ and $x + 2$ are factors of $2x^3 + hx^2 - 13x + k$, find the values of h and k and the remaining factor.

41. If $x^4 + x^3 + ax^2 + bx - 3$ is divisible by $x - 1$ and $x + 3$, find a and b and the remaining factor.

4·14 Summary.

EXPRESSIONS	METHODS OF FACTORING
Binomials	
$ax + bx$	Common factor
$a^2 - b^2$	Difference of squares
$a^3 - b^3$	Difference of cubes (Factor Theorem)
$a^3 + b^3$	Sum of cubes (Factor Theorem)
$a^4 + 4b^4$	Incomplete square
Trinomials	
$ax + ay + az$	Common factor
$x^2 + bx + c$ $ax^2 + bx + c$	Inspection
$a^2 + 2ab + b^2$ $a^2 - 2ab + b^2$	Trinomial square
$a^4 + a^2b^2 + b^4$	Incomplete square
Trinomials of the third degree in the variable	Factor Theorem
Polynomials of four or more terms	
$ax + ay + az + aw$	Common factor
$ax + ay + bx + by$	Grouping to obtain a common factor
$x^2 + 2xy + y^2 - z^2$	Grouping to obtain a difference of squares
Polynomials of the third degree in the variable	Factor Theorem

Practice Exercise 4-12

(B)

Factor:

1. $x^2 - x - 56$
2. $x^3 - x$
3. $a^4 - 1$
4. $2x^2 - 12x + 18$
5. $a^3 - a^2 - 20a$
6. $4x^2 - 3x - 1$
7. $27x^3 - 3x$
8. $27x^3 - 1$
9. $(2a - b)^2 - 4$
10. $(x - 2y)^2 - x^2y^2$
11. $x(y - 1) + 2y - 2$
12. $a^2 - c^2 - 2ab + b^2$

Factor:

13. $x^4 - 9y^2 - x^2 - 6xy$
14. $100b^4 - 20b^2 + 1$
15. $100b^4 - 15b^2 - 1$
16. $100b^4 - 24b^2 + 1$
17. $64y^3 + 1$
18. $1 - x^3y^3$
19. $30a^2 + 70ab - 30b^2$
20. $xy - x^3y^3$
21. $24a^4 - 2a^3 - a^2$
22. $x^3 - 4x^2 - x + 4$
23. $x - y + xy - 1$
24. $9a^2 - 60ab + 100b^2$
25. $x^2y - xy^2 + x^2y^2$
26. $mx + my - m^2 - xy$
27. $16x^4 + 7x^2 + 1$
28. $a^3 - a^2 - 42a$
29. $4x^4 - 17x^2 + 4$
30. $36b^4 - 16b^2c^2 + c^4$
31. $9x^4 + 35x^2 - 4$
32. $20x^2 - 26x - 6$
33. $a^2 + 1 + 4b^2 + 4a^2b^2$
34. $a^2 - 1 + 4b^2 - 4a^2b^2$
35. $a^2 - 1 + 4b^2 - 4ab$
36. $a^2 + b^2 - a^2b^2 - 1$
37. $9x^2 + 12x - 12$
38. $16y^4 - 12y^3 - 18y^2$
39. $4x^4 + 3x^2y^2 + 9y^4$
40. $4a^4 - 13a^2b^2 + 9b^4$
41. $10x^2y^2 + 7xy^2 - 12y^2$
42. $10x^4y^4 - 22x^2y^2 + 12y^4$
43. $(x^2 - 1)^2 - 3x^2 + 3$
44. $(x^2 + 1)^2 - 2x^3 - 2x$

Practice Exercise 4-13

(B)

Factor:

1. $2x^2 - x - 1$
2. $x^2 + x - x^3 - 1$
3. $16x^2 - 15x - 1$
4. $5 - 80a^2$
5. $1 - 6y - 9y^4 + 9y^2$
6. $4y^3 - 3y + 1$
7. $a^3 + 5a^2 - 4a - 20$
8. $a^2 - ab - a + b$
9. $64y^3 - 4y$
10. $64y^4 + y$
11. $a^3b^3 - 2a^2b^2 - 3ab$
12. $5x^2y^2 - 2xy^2 - 7y^2$
13. $y^3 - y^2 - 22y + 40$
14. $12y^2 - y - 20$
15. $5 - 40b^3$
16. $ax^2 - a - x^2 + 1$
17. $a^4 + 7a^2 + 64$
18. $a^4 - 18a^2 + 81$

Practice Exercise 4-14

(B)

Factor:

1. $ax^2 - a^2x + ax$
2. $ax^2 - 2ax + a$
3. $a^2x^2 - ax - 2$
4. $a^2x^2 - a^2 + 1 - 2ax$

Factor:

5. $a^2x^4 - 3a^4x^2 + a^6$
6. $a^4x^2 - 5a^2x^4 + 4x^6$
7. $10ax^2 + 38ax - 8a$
8. $6y^3 - 19y^2 + 11y + 6$
9. $ax^2 - 2 - x + 2ax$
10. $a^3x^3 + y^3$
11. $2a^2 - 5a - 18$
12. $64a^2 - 48a + 9$
13. $1 - 17x^2 + 16x^4$
14. $3x - 27x^3$
15. $(3a - b)^2 - 6a^2 + 2ab$
16. $x^3 - xy^2 - x^2y + y^3$
17. $25x^4 + 6a^2x^2 + a^4$
18. $25x^4 + 16a^2x^2 - 9a^4$
19. $2a^3 + 7a^2 + 2a - 3$
20. $3x^3 - 5x^2 - 16x + 12$

Practice Exercise 4-15

(A)

Factor:

1. $ax + bx + cx$
2. $cx - cy + c$
3. $x^2 + 7x + 12$
4. $x^2 - 2x + 1$
5. $x^2 - 9$
6. $x^2 - x - 6$
7. $x^2 - x$
8. $x(a - b) - y(a - b)$
9. $a(x + y) - x - y$
10. $25x^2 - 10x + 1$
11. $(a - b)^2 - c^2$
12. $(x + y)^2 - 1$
13. $a^3 - b^3$
14. $a^3 + 1$
15. $a(x - y) + b(x - y) + c(x - y)$

(B)

Factor:

16. $2x^2 + 3x - 2$
17. $3 - 5x - 2x^2$
18. $4y^3 - 7y^2 + 3y$
19. $2y^3 - 18y$
20. $30x^2 + 9x - 12$
21. $25x^2 - 60xy + 36y^2$
22. $(x - y - 1)^2 - 1$
23. $x^2 + 25y^2 - 1 - 10xy$
24. $(a - b)^2 - a + b$
25. $(a - 1)^2 + 2 - 2a$
26. $81a^4 + 3a$
27. $81a^4 - 9a^2$
28. $2ax + by - ay - 2bx$
29. $c - xd - cd + x$
30. $72x^2 - x - 1$
31. $6y^2 - 3y - 9$
32. $6y^3 + 23y^2 - 4y$
33. $a^2bc + bcd^2 - ab^2d - ac^2d$
34. $1 - x - x^2 + x^3$
35. $98x^2 - 28x + 2$
36. $a^5 - 81a$
37. $x^2y^2 - x^2 + 1 - 2xy$
38. $15x^2 - 4x - 3$
39. $11 - 6y - 5y^2$

Factor:

40. $40 + 65a - 30a^2$
41. $x - a + (ax - a^2)$
42. $ax + b(x - y) - ay$
43. $x^2 + a^2 - 1 - a^2x^2$
44. $b^4 + b^2c^2 + c^4$
45. $b^4 + b^2c^2 - 2c^4$
46. $x^4 - 3x^2 + 1$
47. $x^4 - 3x^2 - 4$
48. $4x + 4x^4$
49. $1 + 4x^4$
50. $64x^4 + y^4$
51. $64x^4 - y^4$
52. $y^3 + 4y^2 + y - 6$
53. $y^3 + y^2 - y - 1$
54. $4y^3 - 3y^2 - 1$
55. $4y^3 - 3y^2 - y$
56. $12x^3 + 16x^2 - 3x$
57. $8x^3 - 72x^2 + 162x$
58. $9y^3 - 31y + 10$
59. $9y^2 - 31y + 12$
60. $64x^4 + x$
61. $4x^3 - 16x^2 + 13x - 3$

Practice Exercise 4-16

(B)

Perform the operation indicated:

1. $(5x^3)(3x^5)$
2. $\left(-\frac{2}{3}x^2y\right)\left(\frac{3}{4}xy^2\right)$
3. $-2a(a^2 - a - 3)$
4. $(2x - 3y)(4x + y)$
5. $(2a - b)(3a^2 - ab - 2b^2)$
6. $\dfrac{-20a^4b}{-5a^2b}$
7. $\dfrac{\frac{1}{2}x^2y^3}{\frac{3}{4}xy^2}$
8. $\dfrac{3a^3 - 6a^2 - 3a}{-3a}$
9. $(4a^3 - 11ab^2 + 3b^3) \div (2a - 3b)$

If $a = -2$, $b = -3$, *evaluate:*

10. $\dfrac{10ab}{a + b}$
11. $(a - b)^2$
12. $b^2 - a^2$
13. $2a^2 - 3ab - 2b^2$
14. $a(a - b) - 2b(a + b)$
15. $b(a - b) - a(2a - b) - (a - b)(a + b)$

Simplify:

16. $(2x - 1)^2 - (2x - 1)(2x + 1)$
17. $a(a - 1)^2 - 2a(a + 1)(a - 2) - a(1 - 2a)$
18. $3(2x - 3y)(x + 2y) - 2(3x - y)^2$
19. $\dfrac{a^3 + a^2}{a^2} + \dfrac{a^2 - a}{a}$
20. $\dfrac{3x^2 + 6xy}{3x} - \dfrac{4y^2 - 2xy}{2y}$

Chapter V

SIMPLIFICATION OF RATIONAL EXPRESSIONS

5·1 Rational expressions. The following expressions

$$3x + 2, \quad \frac{1}{x-1}, \quad \frac{x^2 - 3x + 2}{x^3}, \quad \frac{x^2 - y^2}{(x-2)(y+3)}, \quad \frac{x^4y^3z^2}{x^2y^3}$$

are called *rational expressions* because of the ratio or fractional form in which they appear. The numerator and denominator of each is a polynomial. If the denominator is a polynomial without a variable, such as 1, 2, etc., the rational expression is called a *polynomial* or *integral rational expression;* if the denominator is a polynomial with a variable, the rational expression is called a *fractional rational expression.*

It is essential to understand that fractional rational expressions are not defined for values of the variable which make the denominator equal to zero, since division by zero is not defined.

5·2 Simplification of rational expressions. If the variables in a rational expression are real numbers, then the rational expression represents a real number.

Thus, rational expressions may be added, subtracted, multiplied, or divided in accordance with the definition of these operations for real numbers. In the following sections, applications of these operations are illustrated.

***5·3* Reducing rational expressions to lowest terms.** The principle of equivalent fractions (*PEF*) may be used to simplify rational expressions as follows:

(i) $\dfrac{21}{28} = \dfrac{3 \times 7}{4 \times 7}$

$= \dfrac{3}{4}$

(ii) $\dfrac{18a^2b}{24ab^3} = \dfrac{3a \cdot 6ab}{4b^2 \cdot 6ab}$

$= \dfrac{3a}{4b^2}, \; a, \; b \neq 0$

(iii) $\dfrac{x^2 - 9}{x^2 + 3x} = \dfrac{(x - 3)(x + 3)}{x(x + 3)}$

$= \dfrac{x - 3}{x}, \; x \neq -3$

(iv) $\dfrac{x^2 - 7x + 12}{x^2 - 8x + 15}$

$= \dfrac{(x - 4)(x - 3)}{(x - 5)(x - 3)}$

$= \dfrac{x - 4}{x - 5}, \; x \neq 3$

The principle of equivalent fractions does not permit multiplication and division of numerator and denominator by zero. Therefore, when numerator and denominator are multiplied or divided by an expression involving a variable, it is necessary to indicate any restriction that must be placed on the variable. For example, in (iii) since division by $x + 3$ is performed, x cannot be (-3) if the expressions are to be equivalent.

Write solutions to each of the following problems and compare them with those on page 476.

1. Reduce to lowest terms $\dfrac{2x^4 - 2y^4}{6x - 6y}$.

2. Reduce to lowest terms $\dfrac{ax - bx + ay - by}{ax - bx - ay + by}$.

3. Reduce to lowest terms $\dfrac{x^4 - 15x^2 + 9}{x^2 - 3x - 3}$.

Exercise 5-1

(A)

State the values of the variables for which the following rational expressions are not defined:

1. $\dfrac{1}{x}$ **2.** $\dfrac{1}{xy}$ **3.** $\dfrac{x - 1}{x - 2}$

4. $\dfrac{3xy}{(x - 3)(x - 4)}$ **5.** $\dfrac{2}{x^2 - 4}$ **6.** $\dfrac{3(z - 3)}{z(z + 6)}$

7. $\dfrac{(x-2)(x-3)}{3x^2-27}$ **8.** $\dfrac{a^3b^2}{(c-d)d}$ **9.** $\dfrac{1}{(x^2-1)(2+x)}$

Copy and complete the following:

10. $\dfrac{1}{2}=\dfrac{\quad}{4}=\dfrac{9}{\quad}=\dfrac{4\frac{1}{2}}{\quad}=\dfrac{\quad}{2a}=\dfrac{\quad}{8bc}$

11. $\dfrac{a}{b}=\dfrac{\quad}{b^2}=\dfrac{a^2}{\quad}=\dfrac{\quad}{bc}=\dfrac{abc}{\quad}=\dfrac{3a^3}{\quad}$

12. $\dfrac{a-b}{a+b}=\dfrac{a^2-b^2}{\quad}=\dfrac{(a-b)^2}{\quad}=\dfrac{\quad}{3(a^2-b^2)}$

(B)

Reduce to lowest terms:

13. $\dfrac{9abc}{3c}$ **14.** $\dfrac{2pq}{6pqr}$ **15.** $\dfrac{(x+4)(x-2)}{(x-2)(x+1)}$

16. $\dfrac{7x+7y}{7z}$ **17.** $\dfrac{x^2-7x+12}{x^2-4x}$ **18.** $\dfrac{x^3-8}{x^2-4}$

19. $\dfrac{x^4-y^4}{x^2+y^2}$ **20.** $\dfrac{x^3-5x^2+4x}{x^3-x}$

21. $\dfrac{27a^3-b^3}{9a^3-ab^2}$ **22.** $\dfrac{(x-1)(x-2)(x^2-6x+5)}{(x^2-7x+10)(x^2-2x+1)}$

23. $\dfrac{(x-5y)^2}{2x^2-12xy+10y^2}$ **24.** $\dfrac{x^4-y^4}{x^3-y^3}$

25. $\dfrac{(m^2+m-6)(m^2-3m-4)}{(m^2-m-2)(m^2-m-12)}$ **26.** $\dfrac{x^4+12x^2+64}{x^3+2x^2+8x}$

27. $\dfrac{ax^2+ay^2}{bx^4-by^4}$ **28.** $\dfrac{a^2-x^2+4x-4}{a-2+x}$

29. $\dfrac{ax+bx+ay+by}{ax+bx-ay-by}$ **30.** $\dfrac{x^3-x^2+x-1}{x^4-1}$

31. $\dfrac{a^2-2ab+b^2-c^2}{a^2-ab+ac}$ **32.** $\dfrac{x^2-8x+7}{x^3-5x+4}$

33. $\dfrac{x^2-ax-bx+ab}{x^2-ax+bx-ab}$ **34.** $\dfrac{z^2-x^2-y^2+2xy}{y^2-z^2-x^2+2xz}$

(C)

Reduce to lowest terms:

35. $\dfrac{m^4 - 7m^2 - 18}{m^3 - 3m^2 + 2m - 6}$

36. $\dfrac{a^3 - b^3 - 2a^2b + 2ab^2}{a^3 + b^3}$

37. $\dfrac{(a - 2b)^2 - 4(a - 2b)}{a^2 - 4ab + 4b^2 - 16}$

38. $\dfrac{x^5 - x^4 - x + 1}{x^4 - x^3 - x^2 + x}$

39. $\dfrac{a^4 - a^2b^2 + b^4}{a^6 + b^6}$

40. $\dfrac{x^6 - 1}{x^4 + x^2 + 1}$

***5·4* Equivalent rational expressions formed by multiplying the numerator and denominator by (− 1).** The negative of a real number is obtained by multiplying the number by (− 1) . Thus, the negative

(i) of 36 is (− 1)(36) or − 36 ;
(ii) of a is (− 1)(a) or $-a$;
(iii) of $(a - b)$ is (− 1)$(a - b)$ or $(-a + b)$ or $(b - a)$.

It is sometimes convenient to use this fact, together with the principle of equivalent fractions, to change the signs of the terms of a rational expression. The following examples illustrate this.

Example 1. Express $\dfrac{+x}{-y}$ with denominator $(+y)$.

Solution. $\dfrac{+x}{-y} = \dfrac{(+x)(-1)}{(-y)(-1)} = \dfrac{-x}{+y}$

Example 2. Express $\dfrac{a - b}{q - p}$ with denominator $(p - q)$.

Solution. $\dfrac{a - b}{q - p} = \dfrac{(a - b)(-1)}{(q - p)(-1)} = \dfrac{b - a}{p - q}$

Example 3. Reduce $\dfrac{a - b}{b - a}$ to lowest terms.

Solution.
$$\frac{a - b}{b - a} = \frac{(a - b)(-1)}{(b - a)(-1)}$$
$$= \frac{(a - b)(-1)}{(a - b)}$$
$$= (-1), \; a \neq b$$

Write a solution to the following problem and compare it with that on page 476.

1. Reduce $\dfrac{2a^2 - ab - b^2}{(a + b)(b - a)}$ to lowest terms.

Exercise 5-2

(A)

Read each of the following fractions with the denominator positive:

1. $\frac{+5}{-2}$
2. $\frac{-5}{-6}$
3. $\frac{+x}{-y}$
4. $\frac{-x}{-y}$
5. $\frac{2}{-a-b}$
6. $\frac{-2}{-p-q}$
7. $\frac{a-b}{-5}$
8. $\frac{x+y}{-m}$
9. $\frac{x}{(-y)(-z)}$

Express with denominator $(a-b)$:

10. $\frac{5}{b-a}$
11. $\frac{x-y}{b-a}$
12. $\frac{-5}{b-a}$
13. $\frac{x+y}{b-a}$
14. $\frac{(+a)(-b)}{b-a}$
15. $\frac{3(x-y)}{b-a}$

Reduce to lowest terms:

16. $\frac{p-q}{q-p}$
17. $\frac{m-n}{n-m}$
18. $\frac{5x-3y}{3y-5x}$
19. $\frac{3d-3c}{c-d}$
20. $\frac{a^3-1}{1-a^2}$
21. $\frac{x^2-7x+12}{9-x^2}$
22. $\frac{x^4-16}{4-x^2}$
23. $\frac{x^3-9x}{-3-x}$
24. $\frac{1-x^2}{x^3-7x+6}$

5·5 Multiplication and division of rational expressions. Multiplication and division of fractions are defined as follows:

If $a, b, c, d \in R$, then

$$\text{(i)}\quad \frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd} \qquad\qquad \text{(ii)}\quad \frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}.$$

These definitions also apply to the multiplication and division of rational expressions in which the variables are real numbers.

Example 1. Simplify $\frac{5}{3}\times\frac{12}{28}\div\frac{15}{7}$.

Solution.
$$\frac{5}{3}\times\frac{12}{28}\div\frac{15}{7}=\frac{5}{3}\times\frac{12}{28}\times\frac{7}{15}$$
$$=\frac{5\times3\times4\times7}{3\times4\times7\times3\times5}=\frac{1}{3}$$

In line 2 of the solution, the numerators and denominators are expressed in factored form so that the fraction may be reduced to lowest terms at the same time that the product or quotient is determined.

Example 2. Simplify $\dfrac{x^2-1}{x^2-9} \times \dfrac{x^2+3x}{x^2+x}$.

Solution.
$$\frac{x^2-1}{x^2-9} \times \frac{x^2+3x}{x^2+x} = \frac{(x+1)(x-1)}{(x+3)(x-3)} \times \frac{x(x+3)}{x(x+1)}$$
$$= \frac{x-1}{x-3}, \quad x \neq 0 \text{ or } -1 \text{ or } -3$$

Write solutions for each of the following problems and compare them with those on page 476.

1. Simplify $\dfrac{ax+ay}{a-b} \times \dfrac{a^2-b^2}{a^2+ab} \div (x+y)$.

2. Simplify $\dfrac{2x^2-3x-2}{x^2+x-6} \times \dfrac{x^3-1}{4x^2-1} \div \dfrac{x^3+x^2+x}{2x^2+5x-3}$.

Exercise 5-3

(A)

Simplify:

1. $\dfrac{x}{y} \times \dfrac{y}{z}$ **2.** $\dfrac{a}{b} \times \dfrac{b}{c} \times \dfrac{c}{a}$ **3.** $\dfrac{a-b}{a+b} \times \dfrac{a+b}{a-b}$

4. $\dfrac{4a}{5b} \times \dfrac{25b}{8c}$ **5.** $\dfrac{x^2-y^2}{x+y} \times \dfrac{1}{x-y}$ **6.** $\dfrac{a^2+ab}{b} \times \dfrac{ab-b^2}{a}$

(B)

Simplify:

7. $\dfrac{2x}{3y^2} \times 3y$ **8.** $\dfrac{2x}{3y} \times \dfrac{6yz}{5x^2}$

9. $\dfrac{a^2}{bc} \times \dfrac{c^2}{ab} \div \dfrac{ac}{b^2}$ **10.** $\dfrac{2a}{5b} \times \dfrac{4b}{3c} \div \dfrac{6a}{5c}$

11. $\dfrac{a+1}{a-1} \times \dfrac{a+2}{a^2-1} \times \dfrac{a-1}{(a+2)^2}$ **12.** $\dfrac{x^2-xy}{x^2+xy} \div \dfrac{x^2-y^2}{x+y}$

13. $\dfrac{a^2-b^2}{x^2-xy} \div \dfrac{ab+a^2}{xy-y^2}$ **14.** $\dfrac{4a^2-6ab}{15xy} \times \dfrac{45xy}{8ab-12b^2}$

Simplify:

15. $\dfrac{x^2+3x+2}{x^2-5x+6}\times\dfrac{x^2-7x+12}{x^2+x}$ **16.** $\dfrac{x(y-x)}{y^2+2yx+x^2}\times\dfrac{y(y+x)}{y^2-2yx+x^2}$

17. $\dfrac{8(a^2-b^2)}{20(a-b)^2}\times\dfrac{5(a+b)}{12(a+b)^2}$ **18.** $\dfrac{x^3+1}{x^2+x+1}\times\dfrac{x^3-1}{x^2-1}$

19. $\dfrac{a-b}{c-d}\times\dfrac{d-c}{b-a}$ **20.** $\dfrac{x^2-3x-4}{x^2-x-12}\times\dfrac{x+3}{x+1}$

21. $\dfrac{x^2-2x+1}{x^2-5x+6}\times\dfrac{x^2-6x+9}{x^2-3x+2}\times\dfrac{x^2-4x+4}{x^2-4x+3}$

22. $\dfrac{a^2+a-2}{a^2+2a-15}\times\dfrac{a^2+12a+35}{10+3a-a^2}\div\dfrac{a^2+6a-7}{a^2-8a+15}$

23. $\dfrac{(a-b)^2-c^2}{(a-c)^2-b^2}\times\dfrac{(b-c)^2-a^2}{a^2-(b-c)^2}$

(C)

24. $\dfrac{x^2-xy+y^2}{x^2-2xy+y^2}\times\dfrac{x^3-y^3}{x^3+y^3}\times\dfrac{x^2-y^2}{x^4+x^2y^2+y^4}$

25. $\dfrac{a^6-b^6}{a^4+a^2b^2+b^4}\times\dfrac{a^2+b^2}{a^2-ab+b^2}\div\dfrac{a^3-b^3}{a+b}$

26. $\dfrac{a^3+a}{a^3+1}\times\dfrac{(a^2+a)^2}{a^4+a^2+1}\times\dfrac{a^6-1}{(a^3-a)^2}\times\dfrac{a^3-1-3a(a-1)}{a^3-a^2+a-1}$

***5·6* Addition and subtraction of rational expressions.** The addition and subtraction of fractions are defined as follows:

If $a,\ b,\ c,\ d \in R$, then

(i) $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad+bc}{bd}$ (ii) $\dfrac{a}{b}-\dfrac{c}{d}=\dfrac{ad-bc}{bd}$.

In practice, the definitions usually are not applied directly, but the fractions to be added or subtracted are written first as equivalent fractions with the same or a common denominator; the numerators are then added and the common denominator retained.

For example:

(i) $\dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad}{bd}+\dfrac{bc}{bd}=\dfrac{ad+bc}{bd}$

(ii) $\dfrac{a}{b}-\dfrac{c}{d}=\dfrac{ad}{bd}-\dfrac{bc}{bd}=\dfrac{ad-bc}{bd}$

Example. Simplify: (i) $\frac{3}{x} - \frac{2}{y}$ (ii) $\frac{5}{x+4} - \frac{3}{x+1}$

Solution.

(i) $\frac{3}{x} - \frac{2}{y}$

$= \frac{3y}{xy} - \frac{2x}{xy}$

$= \frac{3y - 2x}{xy}$

(ii) $\frac{5}{x+4} - \frac{3}{x+1}$

$= \frac{5(x+1)}{(x+4)(x+1)} - \frac{3(x+4)}{(x+1)(x+4)}$

$= \frac{5x + 5 - 3x - 12}{(x+4)(x+1)}$

$= \frac{2x - 7}{(x+4)(x+1)}$

Write a solution to the following problem and compare your solution with that on page 476.

1. Simplify $\frac{3x}{x-2} - \frac{2x}{x+2} + \frac{5x^2 - 1}{x^2 - 4}$.

Exercise 5-4

(B)

Simplify:

1. $\frac{a}{4} + \frac{b}{3} - \frac{c}{2}$ **2.** $\frac{x^2}{5} + \frac{x}{10} - \frac{1}{15}$ **3.** $\frac{3}{a} + \frac{4}{b} - \frac{2}{c}$

4. $\frac{3}{x^2} - \frac{4}{x} + \frac{5}{1}$ **5.** $\frac{pq}{r} + \frac{pr}{q}$ **6.** $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$

7. $\frac{1}{xy} + \frac{1}{yz} + \frac{1}{zx}$ **8.** $\frac{5}{ab} - \frac{3}{a^2b} + \frac{2}{ab^2}$

9. $\frac{x+3}{5} - \frac{x+4}{6}$ **10.** $\frac{a-b}{2} + \frac{a+b}{3} - \frac{2a-b}{4}$

11. $\frac{3}{x+1} - \frac{2}{x-1} + \frac{5x}{x^2-1}$ **12.** $\frac{12}{a-b} + \frac{5}{a+b} - \frac{a+2b}{a^2-b^2}$

13. $\frac{6}{4a^2 - 9b^2} - \frac{3}{2a+3b} - \frac{1}{2a-3b}$

14. $\frac{a-b}{a^2 - ab + b^2} - \frac{a+b}{a^2 + ab + b^2}$

Simplify:

15. $\dfrac{3}{(x-y)^2}+\dfrac{8}{x^2-y^2}-\dfrac{2}{x^2+2xy+y^2}$

16. $\dfrac{x^2-x+1}{x}+\dfrac{x}{x^2+x+1}$

17. $\dfrac{3}{a^2+3a+2}+\dfrac{2}{a^2+5a+6}-\dfrac{5}{a^2+4a+3}$

18. $\dfrac{3}{a-b}+\dfrac{2}{b-a}$

19. $\dfrac{5}{x-y}-\dfrac{2}{xy-y^2}+\dfrac{4}{x^2-xy}$

20. $\dfrac{a^2}{a-a^3}-\dfrac{a}{1+a^2}$

21. $\dfrac{y}{x(x^2-y^2)}+\dfrac{x}{y(x^2+y^2)}$

22. $\dfrac{3}{a^2-4}+\dfrac{1}{(a-2)^2}$

23. $\dfrac{1}{3-x}-\dfrac{1}{3+x}-\dfrac{2x}{9+x^2}$

24. $\dfrac{1}{m+1}-\dfrac{1}{(m+1)(m+2)}+\dfrac{1}{(m+1)(m+2)(m+3)}$

25. $\dfrac{1}{4(1+x)}+\dfrac{1}{4(1-x)}+\dfrac{1}{2(1+x^2)}$

26. $\dfrac{1}{2p^2+p-1}+\dfrac{1}{3p^2+4p+1}$

27. $\dfrac{a}{a^2+5a+6}+\dfrac{15}{a^2+9a+14}-\dfrac{12}{a^2+10a+21}$

5·7 Addition and subtraction of rational expressions involving simplification. In the addition and subtraction of rational expressions it is advisable to reduce the expressions to lowest terms before completing the addition and subtraction. For example:

(i) $\dfrac{35}{70}+\dfrac{1}{3}$

$=\dfrac{1}{2}+\dfrac{1}{3}$

$=\dfrac{3+2}{6}$

$=\dfrac{5}{6}$

(ii) $\dfrac{x^2-2x}{x^2-4}-\dfrac{3x}{x+2}$

$=\dfrac{x(x-2)}{(x-2)(x+2)}-\dfrac{3x}{x+2}$

$=\dfrac{x}{x+2}-\dfrac{3x}{x+2}$, $x\neq 2$

$=\dfrac{-2x}{x+2}$

Write a solution to the following problem and compare it with that on page 477.

1. Simplify $\dfrac{x^2 - 9x + 20}{x^2 - 7x + 12} - \dfrac{x^2 - 9x + 18}{x^2 - 9}$.

Exercise 5-5

(B)

Simplify:

1. $\dfrac{ax}{ay} + \dfrac{by}{bx}$

2. $\dfrac{3x^8}{x^9} - \dfrac{2}{x}$

3. $\dfrac{a^{99}}{a^{100}} + \dfrac{b^{77}}{b^{78}} + \dfrac{c^{44}}{c^{45}}$

4. $\dfrac{642}{1284a} - \dfrac{144}{432b}$

5. $\dfrac{3x^2 - x}{x} + \dfrac{5x^4 + 3x^3}{x^3}$

6. $\dfrac{ab + ac}{a} - \dfrac{mc - mb}{m}$

7. $\dfrac{m^2 - n^2}{m - n} + \dfrac{5m}{3} - \dfrac{2n}{5}$

8. $\dfrac{a - b}{b - a} + \dfrac{b - c}{c - b} + \dfrac{c - a}{a + c}$

9. $\dfrac{m + 5}{m^2 - 3m - 10} + \dfrac{3}{m + 2}$

10. $\dfrac{3x}{x^2 - 5x} + \dfrac{4x}{x^2 + 5x} - \dfrac{5x + 2}{x^2 - 25}$

11. $\dfrac{x^2 - 4a^2}{x^2 - 2ax} - \dfrac{x^2 + 2ax - 8a^2}{x^2 - 4a^2}$

12. $\dfrac{x^2 + 5xy + 4y^2}{x^2 - 16y^2} - \dfrac{2xy}{2x^2 + 8xy}$

13. $\dfrac{a^3 + b^3}{2a^2 - 2ab + 2b^2} - \dfrac{a^3 - b^3}{3a^2 + 3ab + 3b^2}$

14. $\dfrac{a^2 - 4}{a^4 - 16} + \dfrac{a}{a^3 - 4a} + \dfrac{a^2 + 5}{a^4 - 16}$

15. $\dfrac{x^2 - x - 6}{x^2 - 7x + 12} - \dfrac{x^2 + 3x - 10}{x^2 + 2x - 8} - \dfrac{x^2 + 3x + 3}{x^2 - 16}$

Practice Exercise 5-6

(B)

Reduce each of the following fractions to lowest terms:

1. $\dfrac{a^2 - b^2}{ab + b^2}$

2. $\dfrac{(a + b)^2 - c^2}{a^2 - (b + c)^2}$

Reduce each of the following fractions to lowest terms:

3. $\dfrac{x^3 - 9x}{x^2 - 7x + 12}$

4. $\dfrac{2a^2 - 16a + 24}{a^2 - 36}$

5. $\dfrac{x^3 + 2x^2 + x + 2}{x^2 - 4}$

6. $\dfrac{a^3 - 4a^2 - 2a + 8}{2a^3 - 128}$

7. $\dfrac{6x(x - 3)^2}{3x^3 - 27x}$

8. $\dfrac{a^3b - ab^3}{a^3b^2 + a^2b^3}$

9. $\dfrac{a^2 + 2ab + b^2 - c^2}{a^2 - b^2 - 2bc - c^2}$

10. $\dfrac{12x^2 + xy - 35y^2}{6x^2 + 14xy - 40y^2}$

11. $\dfrac{x^4 - y^4}{x^2 - xy}$

12. $\dfrac{8x^3 - y^3}{4x^2 - y^2}$

13. $\dfrac{x^3 - x^2 - x + 1}{(x - 1)^2}$

14. $\dfrac{x^4 + x^2y^2 + y^4}{x^2 - xy + y^2}$

Practice Exercise 5-7

(B)

Simplify:

1. $\dfrac{a}{b} \times \dfrac{b}{c} \times \dfrac{c}{d}$

2. $\dfrac{a^2b}{xy^2} \times \dfrac{x^2y}{ab^2}$

3. $\dfrac{x^2 - 4}{x^3 - 8} \times \dfrac{x^2 + 2x + 4}{x^2 + 2x}$

4. $\dfrac{x^2 - 7x + 12}{x^2 - 9} \times \dfrac{x^2 + 3x}{x^2 - 16}$

5. $\dfrac{x^2 - y^2}{x^2 + xy} \times \dfrac{(x + y)^2}{(x - y)^2}$

6. $\dfrac{m^2 + 3m}{m^2 - 9} \times \dfrac{m^2 - 3m}{m^2}$

7. $\dfrac{a^2 - 25}{a^2 - 6a + 5} \times \dfrac{a^2 - a - 6}{a^2 - 16} \times \dfrac{a^2 - 5a + 4}{a^2 + 7a + 10}$

8. $\dfrac{ac + 2bc - ad - 2bd}{mc - md - nc + nd} \div \dfrac{a + 2b}{m - n}$

9. $\dfrac{3a^2 - 6ab + 3b^2}{(a - b)^2} \div \dfrac{a^4 - b^4}{2ab - 2b^2}$

10. $\dfrac{64x^3 - y^3}{16x^2 - y^2} \div \dfrac{x^4 - 16y^4}{4x^2 - 7xy - 2y^2}$

11. $\dfrac{a^4 - b^4}{(a - b)^2} \times \dfrac{a^2 - b^2}{a^3 + ab^2} \div \dfrac{a - b}{a^2 + b^2}$

Practice Exercise 5-8

(B)

Simplify:

1. $\frac{p}{q}+\frac{r}{t}$
2. $\frac{5}{a}-\frac{3}{b}+\frac{4}{c}$
3. $\frac{5}{3x}-\frac{7}{2y}$
4. $\frac{3}{ab}-\frac{6}{bc}+\frac{1}{ac}$
5. $\frac{3x-2y}{5}+\frac{2x-3y}{3}$
6. $\frac{3x-2y}{5}-\frac{2x-3y}{3}$
7. $\frac{2x-1}{2}+\frac{4x-5}{3}-\frac{x}{6}$
8. $\frac{a-b}{3}+\frac{b-a}{2}-\frac{a+b}{5}$
9. $\frac{3}{x-1}+\frac{2}{x+1}$
10. $\frac{7}{a-b}-\frac{2}{a+b}$
11. $\frac{x-1}{x+1}-\frac{x+1}{x-1}$
12. $\frac{3a-2b}{a-b}+\frac{3a+2b}{a+b}$
13. $\frac{3x}{x^2-7x+12}-\frac{2x}{x^2-9}$
14. $\frac{x^2-x-2}{x+1}-\frac{x^2-5x+6}{x-2}$

Practice Exercise 5-9

(B)

Simplify:

1. $\frac{5}{x-1}+\frac{3}{x+1}-\frac{8x}{x^2-1}$
2. $\frac{4}{a-b}-\frac{2}{a+b}+\frac{3a-b}{a^2-b^2}$
3. $\frac{1}{x^2-7x+12}-\frac{2}{x^2-9x+20}+\frac{3}{x^2-8x+15}$
4. $\frac{5}{x^2+xy}-\frac{3}{x^2-xy}+\frac{3x-y}{x^3-xy^2}$
5. $\frac{a}{a+3}-\frac{2a}{a-3}+\frac{2b}{2a^2b-18b}$
6. $\frac{1}{x-1}-\frac{2}{x-2}+\frac{3}{x-3}$
7. $\frac{m^2-9}{m^2-7m+12}-\frac{m^2-16}{m^2-9m+20}$
8. $\frac{x-3}{3}+\frac{3}{x-3}-\frac{2}{3x-9}$
9. $\frac{4}{a^2-b^2}-\frac{3}{a^2+b^2}+\frac{2a^2-3b^2}{a^4-b^4}$

Practice Exercise 5-10

(B)

Simplify:

1. $\frac{a^2}{bc} \times \frac{b^2}{ac} \times \frac{c^2}{ab}$

2. $\frac{2a}{3b} \times \frac{6bc}{5a^2}$

3. $\frac{m+1}{m-1} \times \frac{m+2}{m^2-1} \times \frac{m-1}{(m+2)^2}$

4. $\frac{a^2-4x^2}{a^2+4ax} \div \frac{a^2-2ax}{ax+4x^2}$

5. $\frac{a^4-b^4}{a^2+b^2+2ab} \div \frac{a-b}{a^2+ab}$

6. $\frac{x^2-3x+2}{x^2-6x+9} \times \frac{x^2-2x+1}{x^2-5x+6}$

7. $\frac{ab-b^2}{(a+b)^2} \div \frac{b^2}{a^2-b^2}$

8. $\frac{3x^2-6xy+3y^2}{(x-y)^2} \times \frac{2xy-2y^2}{x^4-y^4}$

9. $\frac{2a^2-18a+40}{3a^2-75} \times \frac{a^2+5a}{(a+3)^2-9} \times \frac{a^2+a-12}{a^3-16a}$

10. $\frac{(a-4)^2-4}{a^2-2a} \times \frac{a^2-36}{(a-5)^2-1}$

11. $\frac{(a+b)^2-c^2}{(a+c)^2-b^2} \times \frac{(b+c)^2-a^2}{a+b-c}$

12. $\frac{x^4-y^4}{(x+y)^2} \times \frac{x^2-y^2}{(x^2-y^2)^2} \times \frac{x^2+2xy+y^2}{x^2+y^2}$

Review Exercise 5-11

(B)

1. Reduce $\frac{6a^2b^3}{2ab}$ to lowest terms.

2. Simplify $\frac{1}{6x} + \frac{1}{8y} + \frac{1}{2yz}$.

3. Simplify $\frac{x^3-9x}{x^2-7x+12} \times \frac{x^2-16}{x^2+4x}$.

4. Multiply $x^5+x^4+x^3+x^2+x+1$ by $x-1$.

5. Using the Factor Theorem, test whether $x-1$ is a factor of $x^5-6x^4+7x^3-8x+5$.

6. Simplify $\frac{3}{a-b} - \frac{1}{a+b} + \frac{2a-5b}{2a^2-2b^2}$.

7. Multiply $3x^2-4x+2$ by $2x^2+5x-6$.

8. If $m=4$, find the value of $\frac{3}{m-3} - \frac{2}{m+3} - \frac{2m-8}{m^2-9}$.

9. By how much does $\dfrac{x^2 - 4x + 3}{x^2 - 5x + 4}$ exceed $\dfrac{x^2 - 7x + 10}{x^2 - 8x + 12}$?

10. Simplify $\dfrac{2ab}{a^4 - b^4} \times 0 \times \dfrac{a^4 + b^4}{6ab}$.

11. Find the square of $3x^2 - 5x + 2$.

Factor:

12. $x^6 - 64$

13. $x^2 - 4y^2 + x - 2y$

14. $4(x - y)^3 - (x - y)$

15. $9x^3 - 4xy^2$

16. $a^2 - b^2 - 2bc - c^2$

17. $a^4 + 3a^2 - 4$

18. $6y^2 - 7y - 3$

19. $2m^4 - m^3 + 4m - 2$

20. Find the first four terms of the quotient $(1 - 3x) \div (1 + 2x)$.

21. Without multiplying, show that
$(a^3 - 1)(a^3 + 1) = [(a^2 + 1)^2 - a^2](a^2 - 1)$.

22. Find two numbers whose difference is 11, and one-fifth of whose sum is 9 .

23. If $a = 3,\ b = 2,\ c = 1$, find the value of $\dfrac{a}{b + c} + \dfrac{b}{c + a} + \dfrac{c}{a + b}$.

24. Divide $x^4 - 10x^2 + 9$ by $x^2 - 2x - 3$.

25. If $P = a - 2b + c,\ Q = 3a - 5b - 4c,\ R = 2a + 3b - 6c$, find $3P - 4Q + 5R$.

26. Solve: $(5x + 1)(x - 2) - (4x - 3)(3x - 1) = 10 - (7x + 2)(x + 1)$

27. Without actual division, show that $x^5 - 3x^2 - 20$ is divisible by $x - 2$.

28. Reduce $\dfrac{a^4 - a^2 - 2a + 2}{2a^3 - a - 1}$ to lowest terms.

29. Find the remainder when $x^5 - 4x^4 + 7x^3 - 2x^2 + x - 5$ is divided by $x^2 - x + 1$.

30. Simplify $\dfrac{a^2 + b^2}{a^2 - ab} \times \dfrac{ab - b^2}{a^4 - b^4} \times \dfrac{a}{b}$.

31. Simplify $\dfrac{2}{a^2 - 3a + 2} + \dfrac{2}{a^2 - a - 2} - \dfrac{1}{a^2 - 1}$.

32. The distance an object falls is given by the formula $D = \frac{1}{2}gt^2$. Find D, given $g = 32$ and $t = 10$.

33. Simplify $\dfrac{a^3 - 4a}{a^4 - 16} - \dfrac{5a}{a^3 + 4a}$.

Chapter VI

RELATIONS

***6·1* Binary relations.** The familiar words "relation" and "relationship" bring to mind a great many properties which relate one individual to another: for example,

Bill "is older than" Jim.
John "is taller than" George.
Jane "is the sister of" Mary.
Jack "likes" Jill.
George "is heavier than" John.
Helen "is the mother of" Mary.
Sally "is cleverer than" Ann.
Mr. Smith "is the employer of" Mr. Jones.

These relationships apply between two people. Other relations, such as, "is at war with", "trades with", "is opposed to", "is a supporter of", could apply between two countries. Still other relations, such as, "is greater than", "is equal to", "is less than", "is the square of", apply between two numbers.

The common characteristic of the above relations is that each relation applies between pairs of things of the same general type. For this reason, these relations are called *binary relations*.

If the relation "is older than" is examined as it applies to students in a particular classroom, it would not be difficult to list the names of all pairs of students, such that the student named second in each pair is older than the student named first. This would involve forming all possible pairs of the form (Bill, Jim), (Bill, Mary), (Bill, Jack), (Jim, Mary), (Jim, Jack),

(Mary, Jack), and so on, such that the second member of each pair is older than the first. Since the order in which the names are written in each of these pairs is important, each pair is called an *ordered pair*. The set of all ordered pairs formed as indicated above specifies completely the relation "is older than" as it applies to the students in the particular classroom under consideration.

The basic idea of a binary relation is an *association* from one element of a set to another. Such an association determines a set of ordered pairs. Thus, it is natural and convenient to study a binary relation by studying the set of ordered pairs of the relation.

It should be understood that there are relations which involve three members. These are called *ternary relations* and determine a set of *ordered triples*. Similarly, there are relations which involve more than three members. However, the only relations discussed in this book are binary relations.

Example 1. Given $A = \{2, -5, 7, -7, -2, 5\}$, list the ordered pairs of each of the following relations:

(i) R_1 "is greater than", on A;

(ii) R_2 "is the additive inverse of", on A;

(iii) R_3 "is the square of", on A.

In listing the ordered pairs of a relation, the second element in each pair is in the given relation to the first element.

Solution. (i) $R_1 = \{(2, 7), (2, 5), (-5, 2), (-5, 7), (-5, -2), (-5, 5), (-7, 2), (-7, -5), (-7, 7), (-7, -2), (-7, 5), (-2, 2), (-2, 7), (-2, 5), (5, 7)\}$

(ii) $R_2 = \{(2, -2), (-2, 2), (-5, 5), (5, -5), (7, -7), (-7, 7)\}$

(iii) $R_3 = \emptyset$

Example 2. Given $A = \{1, 2, 3, 4, 5\}$, list the ordered pairs of the relation R "is equal to", on A.

Solution. $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$

Exercise 6-1

(B)

1. The following is a list of the heights of the boys on a basketball team: Andy 6′ 1″, George 6′ 2″, Michael 5′ 11″, Ted 6′ 4″, Martin 5′ 7″. List the ordered pairs such that the second member in each pair "is shorter than" the first.

2. The following is a list of the densities of certain materials in grams per cubic centimetre: Aluminium 2.72, Copper 8.90, Diamond 3.5, Gold 19.34, Iron 7.85 . List the ordered pairs such that the density of the second member in each pair "is greater than" the density of the first.

3. Given the set $A = \{-3, -1, 0, 1, 3, 5\}$, list the set of ordered pairs of each of the following relations on A :

(i) R_1 "is three times" (ii) R_2 "is equal to"

(iii) R_3 "is the square of" (iv) R_4 "is two more than"

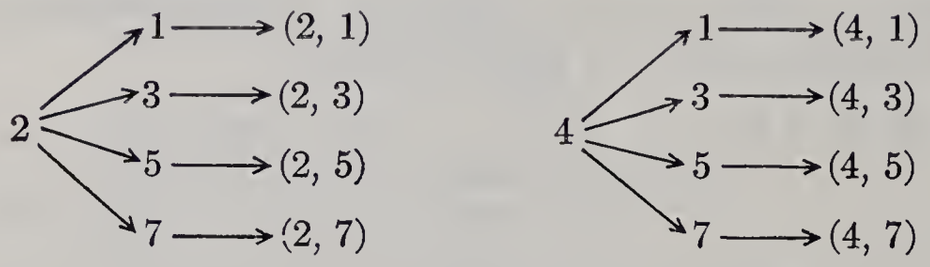
(v) R_5 "is greater than"

4. Given the set $B = \{0, \frac{1}{3}, \frac{1}{2}, 1, 2, 3\}$, list the set of ordered pairs of each of the following relations on B:

(i) P_1 "is less than" (ii) P_2 "is the reciprocal of"

(iii) P_3 "is one more than twice as great as"

6·2 Ordered pairs in cross-products. If $A = \{2, 4\}$ and $B = \{1, 3, 5, 7\}$, it is possible to form a set of eight ordered pairs whose first member is an element of A and whose second member is an element of B in the following manner:



The resulting set is $\{(2, 1), (2, 3), (2, 5), (2, 7), (4, 1), (4, 3), (4, 5), (4, 7)\}$ which is referred to as a *cross-product* of the two sets A and B. This cross-product is designated $A \times B$ (read, "A cross B") and, using set-builder notation, is defined as follows:

$$A \times B = \{(x, y) | x \in A, y \in B\}$$

DEFINITION: *The cross-product, $A \times B$, of two sets A and B is the set of all ordered pairs whose first member is an element of A and whose second member is an element of B.*

The sets A and B also give rise to a second cross-product, $B \times A$, in which the first member of each ordered pair is from the set B and the second from the set A.

$$B \times A = \{(x, y) \mid x \in B, y \in A\}$$

$$B \times A = \{(1, 2), (1, 4), (3, 2), (3, 4), (5, 2), (5, 4), (7, 2), (7, 4)\}$$

If the set B is identical to the set A, then the cross-product

$$A \times A = \{(x, y) | x \in A, y \in A\} .$$

From the manner in which the ordered pairs are formed, it is seen that the number of ordered pairs in a cross-product is the product of the number of elements in each of the two sets.

Thus, the number of ordered pairs in $A \times B$ is $2 \cdot 4$ or 8, and in general:

If a set A has m elements and a set B has n elements, then the cross-product $A \times B$ has $m \cdot n$ or mn ordered pairs, and the cross-product $A \times A$ has m^2 ordered pairs.

Example. Given $A = \{1, 3, 5\}$.

(i) Determine the number of elements in $A \times A$.

(ii) List the ordered pairs of $A \times A$.

Solution. (i) Number of elements in $A \times A = 3 \cdot 3 = 9$

(ii) $A \times A = \{(1, 1), (1, 3), (1, 5),$
$(3, 1), (3, 3), (3, 5),$
$(5, 1), (5, 3), (5, 5)\}$

Write solutions to the following problems and compare them with those on page 477.

1. Given $A = \{1, 3, 5\}$, select the subset B of ordered pairs from $A \times A$ such that the second member is less than the first member.

2. Find the set of all ordered pairs of $N \times N$ (N is the set of natural numbers) such that the second member of each pair exceeds the first member by 2 .

3. Given $A = \{8, 27, 64\}$, select the subset C of ordered pairs in $A \times A$ such that the second member of each pair is the cube of the first member.

Exercise 6-2

(A)

1. If A is the set of integers having only one digit and B is the set of odd integers having one digit, how many ordered pairs are there in $A \times B$?

2. If $A = \{1, 4, 7\}$, list the elements of $A \times A$.

3. Find $A \times B$ and $B \times A$ when $A = \{2, 5\}$ and $B = \{1, 3, 4\}$.

4. Under what condition is $A \times B = B \times A$?

5. Which ordered pair in $\{(1, 1), (3, 9), (2, 4), (4, 8), (6, 36), (5, 25)\}$ does not exhibit the same pattern as the other ordered pairs?

(B)

6. List $\{(x, y) | x, y \in A\}$ where $A = \{2, 3, 4\}$.

7. List the set of all ordered pairs in $\{5, 7\} \times \{4, 6, 8\}$.

8. If $A = \{1, 2, 3, \ldots, 9\}$, list the set of all ordered pairs whose second member exceeds its first member by 1.

9. List the elements of the set of all ordered pairs in $N \times N$ whose second member is double its first member.

10. If $A = \{1, 4, 7\}$, list the set of all ordered pairs in $A \times A$ in which the second member is the cube of the first member.

11. The Eastern Conference Football League consists of teams from Hamilton, Montreal, Ottawa, and Toronto. List ordered pairs to indicate all possible games, the second member of the pair being the visiting team.

***6·3* Graphs of ordered pairs.** It has been noted previously that there is a one-to-one correspondence between the real numbers and the points on the real number line. The points designated by the real numbers 2, -1.5, $\sqrt{2}$, $-\frac{1}{4}$ are indicated graphically in *Fig. 6-1.*

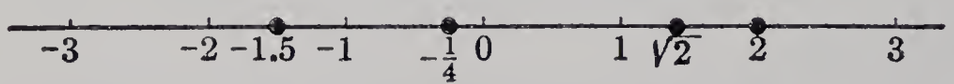


Fig. 6-1

To draw the graph of an ordered pair, two such real number lines are required. They are usually drawn at right angles, as indicated in *Fig. 6-2.*

The horizontal number line is known as the *x-axis* and the vertical number line as the *y-axis.*

If lines are drawn parallel to the x-axis through points on the y-axis whose coordinates are the integers $\ldots, -3, -2, -1, 1, 2, 3, \ldots$, and if lines are drawn parallel to the y-axis through points on the x-axis having integral coordinates, a *lattice* is formed (*Fig. 6-4*). The points of intersection of these lines are called *lattice points* which can be identified with ordered pairs of integers. The lines are called *lattice lines.*

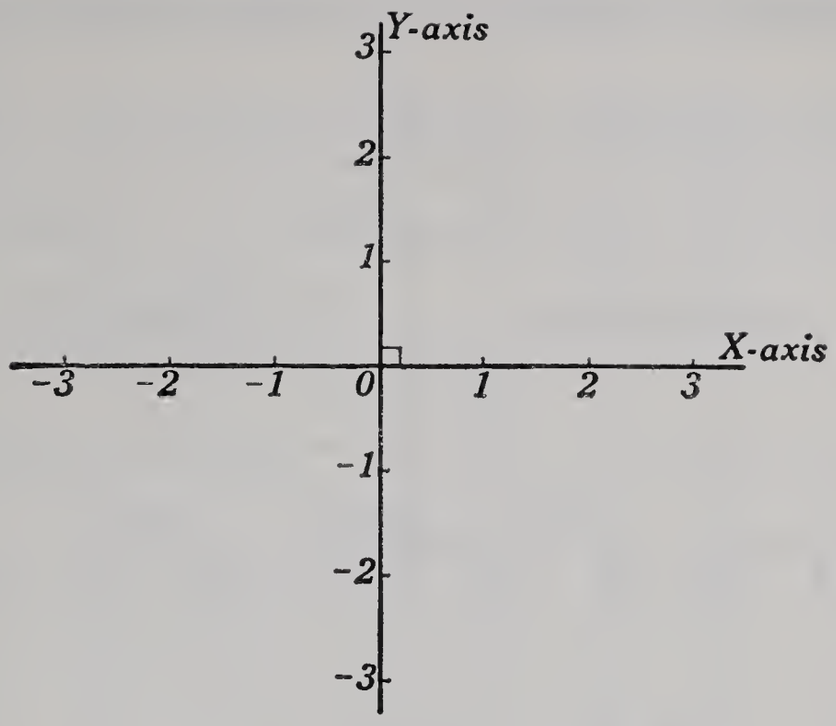


Fig. 6-2

To draw the graph of an ordered pair of integers (p, q), it is necessary to locate a lattice point which is p units from the y-axis, measured in a direction parallel to the x-axis, and q units from the x-axis, measured in a direction parallel to the y-axis (*Fig. 6-3*); p is called the *x-coordinate* or *abscissa*, and q the *y-coordinate* or *ordinate* of the point.

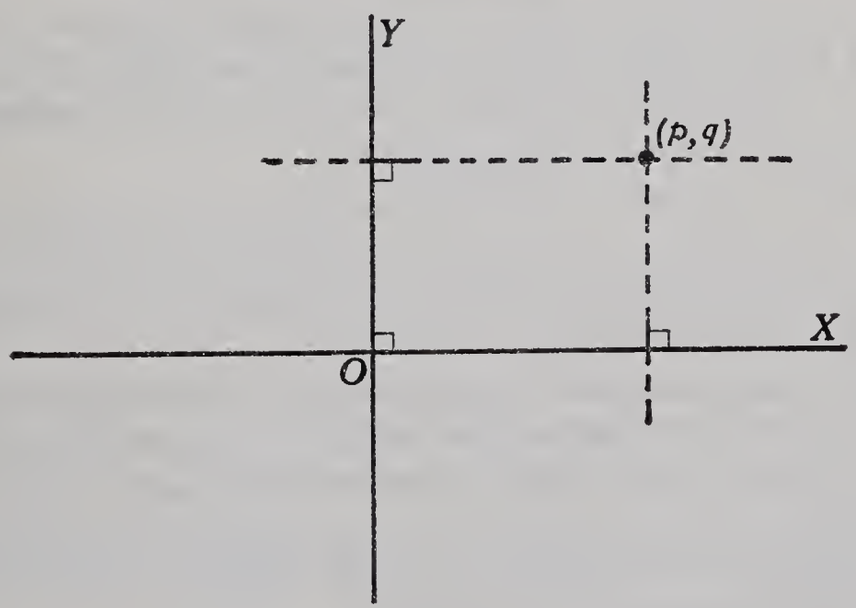


Fig. 6-3

Thus, the point identified by the ordered pair $(-3, 4)$ is 3 units to the left of the y-axis and 4 units above the x-axis, as shown in *Fig. 6-4*. Other points, with their coordinates, are also shown.

The number plane is divided into four *quadrants*, numbered as in *Fig. 6-4.*

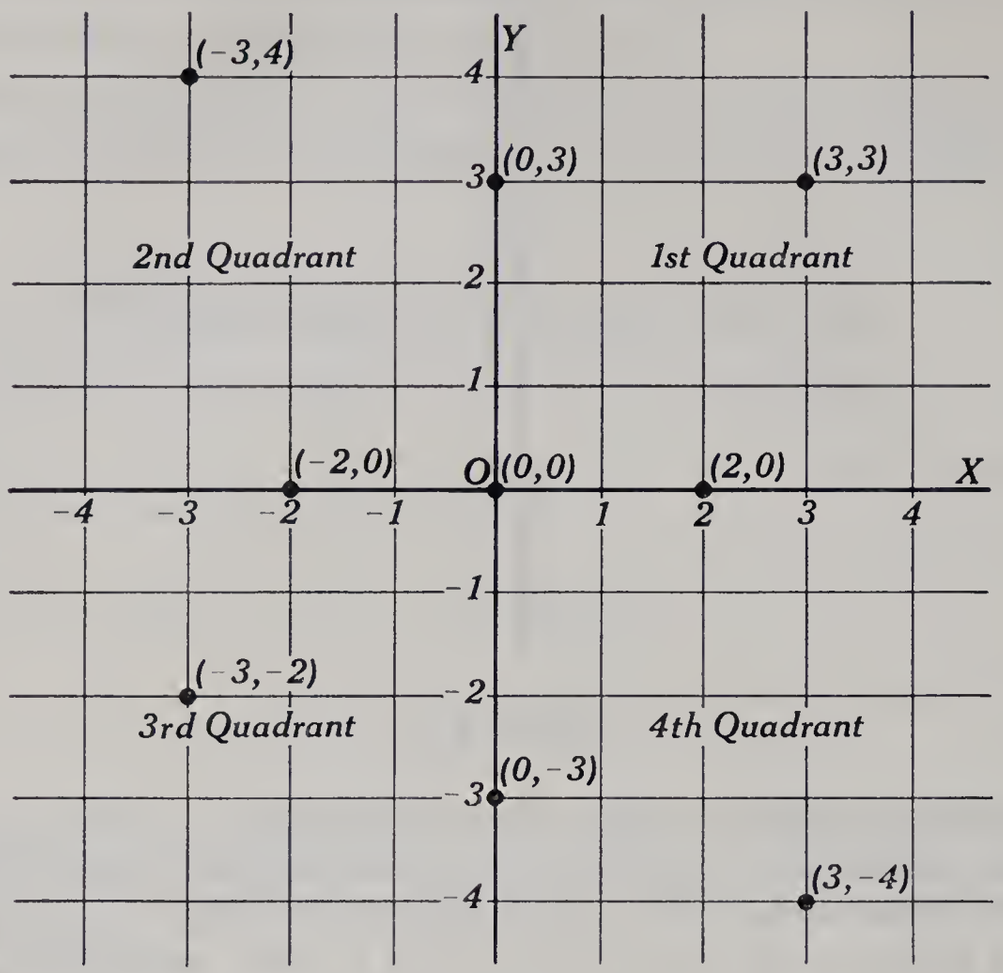


Fig. 6-4

Since the set of integers includes 0, lattice points appear on the axes, as illustrated by the points whose coordinates are (2, 0), (0, 3), (− 2, 0), and (0, − 3). The point of intersection of the axes has coordinates (0, 0) and is known as the *origin* (designated by O).

Lines parallel to the axes may be drawn through any points $(x, 0)$ and $(0, y)$, $x, y \in R$ on the axes to meet in a unique point (x, y) on the number plane. Thus, every point in the real number plane can be associated with an ordered pair of real numbers.

The French mathematician and philosopher, René Descartes (1596-1650), is credited with developing this method of representing the position of a point in a plane with respect to number-line axes of reference. In 1637 he published the first book which may properly be called a treatise on analytic geometry or Cartesian coordinate geometry. The fundamental assumption of this geometry is as follows:

> *There is a one-to-one correspondence between the set of all points in a plane and the set of all ordered pairs of real numbers.*

The set of all ordered pairs of real numbers is the cross-product

$$R \times R = \{(x, y) | x \in R, y \in R\} .$$

Cross-products are often referred to as *Cartesian products* in deference to Descartes.

Example 1. If $A = \{-3, 1, 4\}$, determine $A \times A$ and draw its graph.

Solution.

$$A \times A = \{(-3, 4), (1, 4), (4, 4), (-3, 1), (1, 1), (4, 1), (-3, -3), (1, -3), (4, -3)\}$$

The graph is shown in *Fig. 6-5.*

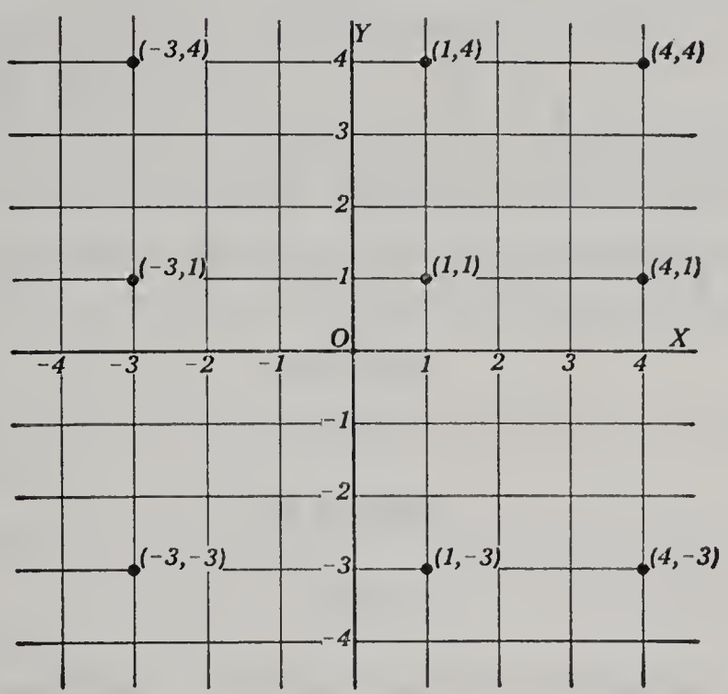


Fig. 6-5

The graph of the Cartesian set $A \times A$ consists of all the points of intersection in the number plane formed by drawing parallels to the axes through the points on the axes corresponding to the elements of A. In other words, there is a one-to-one correspondence between the set of ordered pairs in $A \times A$ and these points in the number plane.

Example 2. Draw the graph of the Cartesian set $A \times A$ where

$$A = \left\{-\frac{3}{5}, \frac{7}{10}\right\}.$$

Solution.

$$A \times A = \left\{\left(-\frac{3}{5}, \frac{7}{10}\right), \left(\frac{7}{10}, \frac{7}{10}\right), \left(-\frac{3}{5}, -\frac{3}{5}\right), \left(\frac{7}{10}, -\frac{3}{5}\right)\right\}$$

The graph is shown in *Fig. 6-6*.

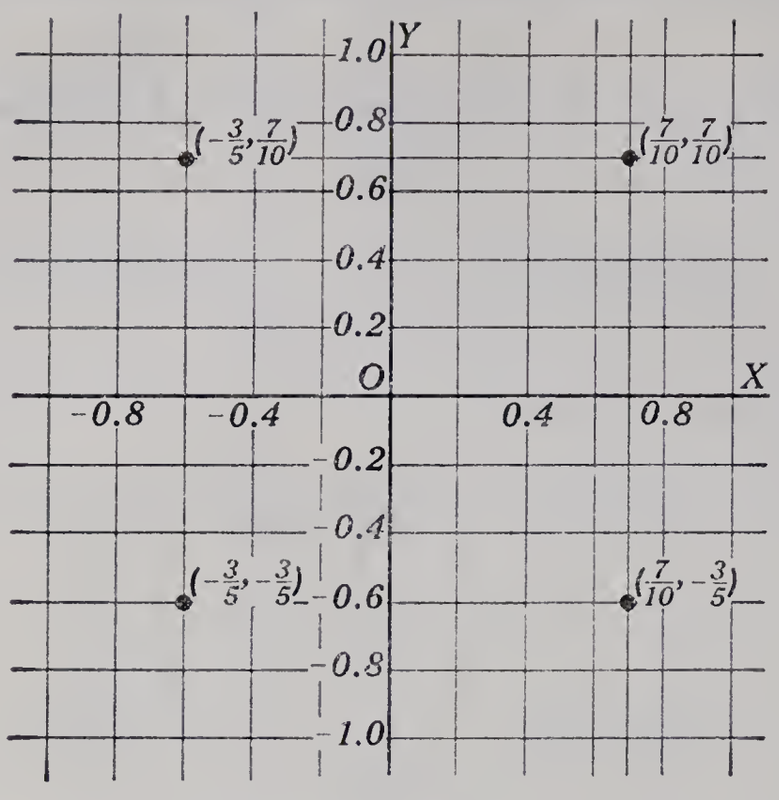
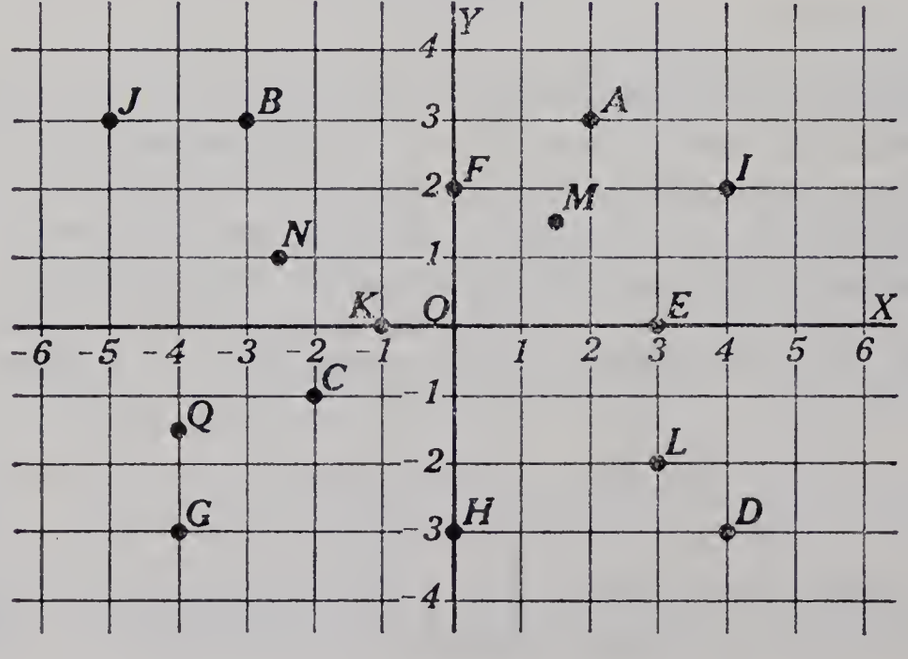


Fig. 6-6

Exercise 6-3

(A)

1. In the accompanying diagram, state the coordinates of the points indicated.



(B)

2. Write the coordinates of the points indicated in the diagram of question 1.

3. Using convenient axes and scale, plot the points $P(10, 3)$, $Q(-6, 4)$, $R(0, 10)$, $S(-2, -8)$, $T(6, 0)$, and $V(2, -9)$.

4. Plot the points $A(-3, 5)$, $B(7, 5)$, and $C(7, -3)$. Measure (i) the length of AB (ii) the length of BC.

5. In the diagram for question 4, find the coordinates of the fourth point, D, of a rectangle $ABCD$ and of the midpoints of AB and BC.

6. Plot the point $A(3, 4)$. Find the distance from A to the origin in terms of the unit employed.

7. Plot the points $A(-4, 0)$ and $B(0, 3)$. Find the length of AB by using a right-angled triangle. Check the result by measurement.

8. Plot the points $P(6, 2)$ and $Q(-2, -6)$. Check that these points are the same distance from the origin (i) by using right-angled triangles and (ii) by measurement.

9. If $A = \{1, 2, 3, 4\}$, list the ordered pairs in $A \times A$ and draw the graph of the set.

10. Draw the graph of $A \times A$ when $A = \{-3, -1, 1, 3\}$.

11. If $A = \{-2, 0, 2\}$, plot the graph of $A \times A$ and list the elements of the Cartesian set.

***6·4* Relations in $A \times B$ and $A \times A$.** If with each element of set A there is associated an element of set B, then this association is called a *relation* from A to B in $A \times B$. Since the association determines a set of ordered pairs, a relation in $A \times B$ determines a subset of $A \times B$. Also, a relation in $A \times A$ determines a subset of $A \times A$.

A relation P involves:

(i) a universal set U on which the relation is applied;
(ii) a defining sentence (rule of correspondence);
(iii) a set of ordered pairs determined by the defining sentence.

The relation P "is less than" applied on set U is conveniently expressed

$$P = \{(x, y) | y < x\,,\ x\,,\ y \in U\}$$

which indicates that the relation P:

(i) is applied on U;
(ii) has the defining sentence $y < x\,,\ x\,,\ y \in U$;
(iii) determines a set of ordered pairs (x, y) which is a subset of $U \times U$.

Example. Given $A = \{0, 1, 2, 3, 4\}$.

(i) Find all the ordered pairs (x, y) in $A \times A$ and draw the lattice lines determining the points which have these ordered pairs as coordinates. Draw the graph of $A \times A$.

(ii) Select and graph the ordered pairs of the following relations:

(a) $R_1 = \{(x, y) \mid y = x, x, y \in A\}$

(b) $R_2 = \{(x, y) \mid y > x, x, y \in A\}$

(c) $R_3 = \{(x, y) \mid y < x, x, y \in A\}$

Solution.

(i) $A \times A$

$= \{(0, 4), (1, 4), (2, 4), (3, 4), (4, 4),$
$(0, 3), (1, 3), (2, 3), (3, 3), (4, 3),$
$(0, 2), (1, 2), (2, 2), (3, 2), (4, 2),$
$(0, 1), (1, 1), (2, 1), (3, 1), (4, 1),$
$(0, 0), (1, 0), (2, 0), (3, 0), (4, 0)\}$

The graph is shown in *Fig. 6-7*.

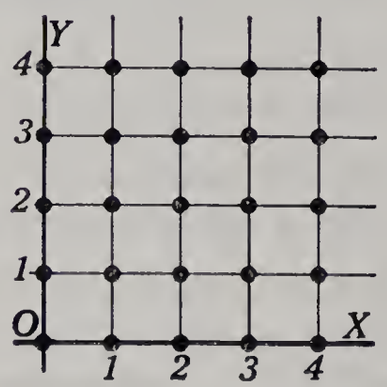


Fig. 6-7

(ii) (a)

$R_1 = \{(x, y) \mid y = x, x, y \in A\}$

$= \{(4, 4),$
$(3, 3),$
$(2, 2),$
$(1, 1),$
$(0, 0),\}$

The graph is shown in *Fig. 6-8*.

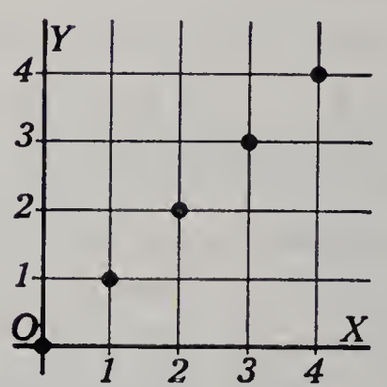


Fig. 6-8

(b)

$R_2 = \{(x, y) \mid y > x, x, y \in A\}$

$= \{(0, 4), (1, 4), (2, 4), (3, 4),$
$(0, 3), (1, 3), (2, 3),$
$(0, 2), (1, 2),$
$(0, 1)\}$

The graph is shown in *Fig. 6-9*.

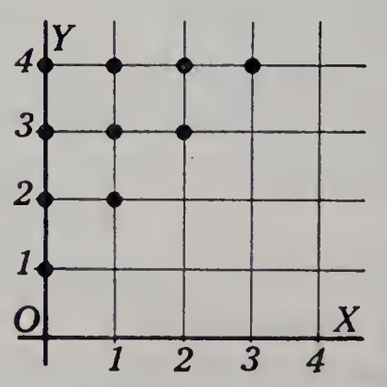


Fig. 6-9

(c)

$$R_3 = \{(x, y) | y < x,\ x, y \in A\}$$
$$= \{(4, 3),$$
$$(3, 2), (4, 2),$$
$$(2, 1), (3, 1), (4, 1),$$
$$(1, 0), (2, 0), (3, 0), (4, 0)\}$$

The graph is shown in *Fig. 6-10.*

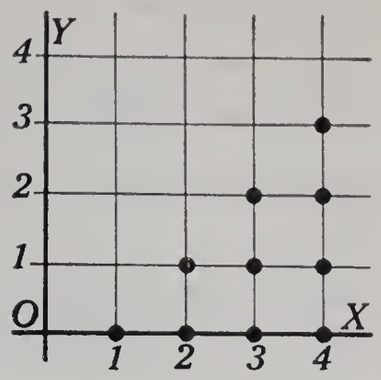


Fig. 6-10

6·5 Domain and range of a relation.

DEFINITION: *The domain of a relation is the set of all first members of the ordered pairs of the relation.*

DEFINITION: *The range of a relation is the set of all second members of the ordered pairs of the relation.*

Example. List the domain and range of the relation

$$F = \{(x, y) | y = 2x,\ x, y \in A\}, \text{ where } A = \{0, 1, 2, 4\}.$$

Solution. $F = \{(0, 0), (1, 2), (2, 4)\}$

$\therefore$ the domain is $\{0, 1, 2\}$, and the range is $\{0, 2, 4\}$.

Write solutions to the following problems and compare them with those on page 477.

1. If $A = \{0, 1, 2, 3, 4, 5\}$, find the domain and range of the relation $B = \{(x, y) | y = x^2 + 1,\ x, y \in A\}$.

2. Find the domain and range of the relation $C = \{(x, y) | y = 2x - 1,\ x, y \in N_0\}$.

3. Find the domain and range of the relation $D = \{(x, y) | y = x,\ x, y \in R\}$.

4. Which of the ordered pairs:
$(3, 4), (0, 0), (0, -5), (4, -3), (5, 0), (0, 25), (4, 3), (6, 1)$
do not satisfy the relation $E = \{(x, y) | x^2 + y^2 = 25,\ x, y \in I\}$?

Exercise 6-4

(A)

1. Find the domain and range of the relations:

(i) $R_1 = \{(x, y) | y = 2x + 1,\ x, y \in A\}$ where $A = \{1, 3, 5, 7\}$

(ii) $R_2 = \{(x, y) | y = 2x^2,\ x, y \in N_0\}$

Find the domain and range of the relations:

(iii) $R_3 = \{(x, y)|2x + y = 3, x, y \in N\}$

(iv) $R_4 = \{(x, y)|x + y = 0, x, y \in R\}$

2. List three ordered pairs of each of the following relations on the set I:

(i) $P_1 = \{(x, y)|y = x + 1\}$ (ii) $P_2 = \{(x, y)|x + y = 5\}$

(iii) $P_3 = \{(x, y)|y = x^2\}$ (iv) $P_4 = \{(x, y)|x = y^2\}$

(B)

3. Which of the ordered pairs (1, 2), (2, 6), (18, 3), (0, 0), (4, 32) do not satisfy the relation $Q = \{(x, y)|2x^2 = y, x, y \in N\}$?

4. If $U = \{0, 1, 2, 3\}$ and (x, y) is an element of $U \times U$, determine the ordered pairs, and state the domain and range of each of the following relations:

(i) $A = \{(x, y)|y = x + 1\}$ (ii) $B = \{(x, y)|x = 3\}$

(iii) $C = \{(x, y)|y > x\}$ (iv) $D = \{(x, y)|y = 4x\}$

(v) $E = \{(x, y)|y = 4\}$ (vi) $F = \{(x, y)|y = x^2\}$

5. If $A = \{-1, 0, 1\}$ and $B = \{-2, 0, 2\}$, find $A \times B$; select the ordered pairs for which the second member is double the first member, and state the domain and the range of this relation.

6. Find the domain and range of:

(i) $F_1 = \{(x, y)|y = x^2, x, y \in R\}$

(ii) $F_2 = \{(x, y)|x = -2, x, y \in R\}$

(iii) $F_3 = \{(x, y)|y = 3, x, y \in R\}$

(iv) $F_4 = \{(x, y)|x = 0, x, y \in R\}$

6·6 Graphs of relations. The graph of a binary relation is the set of points whose coordinates are the ordered pairs of the relation. Example 1 illustrates a method of obtaining the graph of a relation which is defined over a set of numbers containing only a limited number of elements.

Example 1. If $A = \{-2, -1, 0, 1, 2, 3\}$, draw the graph of the relation $B = \{(x, y)|y = x, x, y \in A\}$.

Solution. $B = \{(x, y)|y = x, x, y \in A\}$

The ordered pairs of B may be determined by constructing a table of values in which each x in the domain is associated with the corresponding value of y in the range.

Table of values for B:

x	-2	-1	0	1	2	3
y	-2	-1	0	1	2	3

The table implies the following about this relation:

(i) the domain, which is $\{-2, -1, 0, 1, 2, 3\}$;

(ii) the range, which is $\{-2, -1, 0, 1, 2, 3\}$;

(iii) the set of ordered pairs, which is $\{(-2, -2), (-1, -1), (0, 0), (1, 1), (2, 2), (3, 3)\}$.

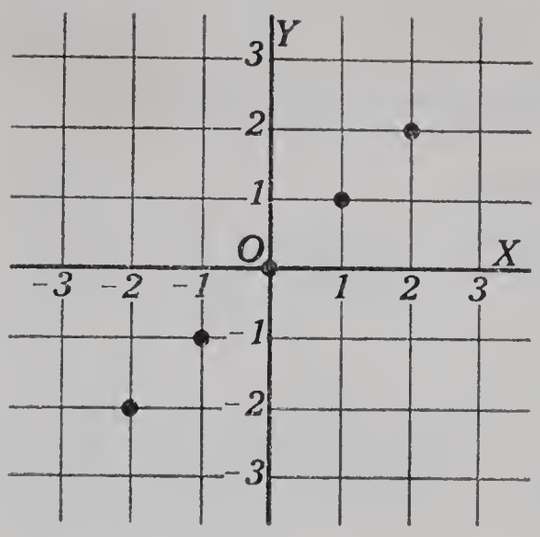


Fig. 6-11

The graph, *Fig. 6-11*, consists of the six points whose coordinates are the ordered pairs of the relation.

Examples 2 to 5 deal with relations which have the same defining equation, but which are defined on different number sets. The development is designed to suggest what is meant by the graph of a relation defined on the set of real numbers.

Example 2. Draw the graph of the relation

$$P_1 = \{(x, y) | y = x + 1, x, y \in C\}$$

where $C = \{-3, -2, -1, 0, 1, 2, 3\}$, and state the domain and range of the relation.

Solution.

$P_1 = \{(x, y) | y = x + 1, x, y \in C\}$

For P_1:

x	-3	-2	-1	0	1	2
y	-2	-1	0	1	2	3

The graph is shown in *Fig. 6-12*.

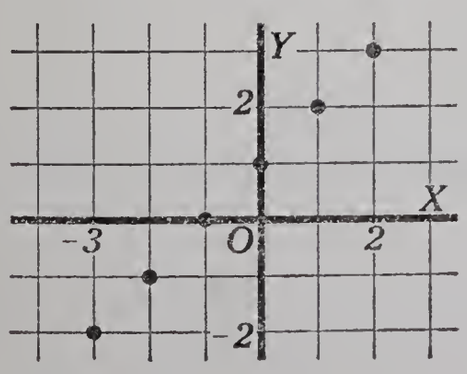


Fig. 6-12

DISCUSSION. This table gives the domain and range, and the ordered pairs of the relation.

However, in this example, neither the domain nor the range contains every element of C. For example, if $x = 3$, then $y = 4$, but $4 \notin C$; therefore the relation is not defined for all elements of C.

Domain is $\{-3, -2, -1, 0, 1, 2\}$.

Range is $\{-2, -1, 0, 1, 2, 3\}$.

Example 3. Draw the graph of the relation

$$P_2 = \{(x, y) | y = x + 1, x, y \in I\}$$

and state its domain and range.

Solution.

$P_2 = \{(x, y) | y = x + 1, x, y \in I\}$

For P_2:

x	-3	-2	-1	0	1	2
y	-2	-1	0	1	2	3

The partial graph of P_2 is shown in *Fig. 6-13.*

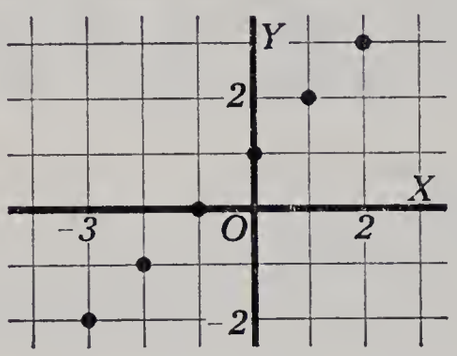


Fig. 6-13

Domain is $\{x | x \in I\}$.
Range is $\{y | y \in I\}$.

DISCUSSION. Since the set I is an infinite set, the relation determines an infinite set of ordered pairs. The graph consists of an infinite set of points. To obtain the *partial graph* of P_2, arbitrarily select a convenient number of replacements for x from the set I and construct the corresponding table of values. Plot the corresponding points; these points are a partial graph of P_2. The full graph cannot be drawn. The partial graph implies that the full graph is an extension of this graph in the directions indicated by the plotted points.

The table does not state the domain and range nor list all the ordered pairs. The domain and range are inferred from the nature of the graph or as follows:

$\because$ $y = x + 1$, y is an integer for all replacements of x from I,

$\therefore$ the domain is $\{x | x \in I\}$.

$\because$ $x = y - 1$, x is an integer for all replacements of y from I,

$\therefore$ the range is $\{y | y \in I\}$.

Example 4. Draw the graph of the relation

$$P_3 = \{(x, y) | y = x + 1, x, y \in C\}$$

where C is the set of all real numbers between and including -3 and $+3$.

Solution.

$P_3 = \{(x, y) | y = x + 1, x, y \in C\}$

C is the set of real numbers in the interval -3 to $+3$.

DISCUSSION. The set C may be expressed concisely as

$$\{z | -3 \leqq z \leqq 3, z \in R\}$$

For P_3:

x	-3	-2	-1	0	1	2
y	-2	-1	0	1	2	3

The graph of P_3 is shown in *Fig. 6-14*.

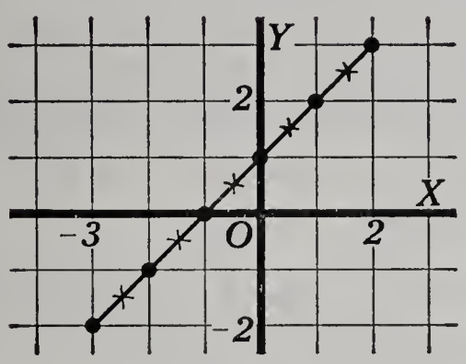


Fig. 6-14

Domain is $\{x| -3 \leqq x \leqq 2, x \in R\}$.

Range is $\{y| -2 \leqq y \leqq 3, y \in R\}$.

which means that z is any real number in the interval between and including -3 and $+3$. Since the replacement set for x and y is an infinite set, the relation determines an infinite set of ordered pairs and the graph an infinite set of points. To obtain the graph, arbitrarily select a convenient number of replacements for x from the set C and construct the corresponding table of values. Since the interval has end values, the table of values should include these. In this table, one of the replacements for x is -3, and one of the replacements for y is $+3$. The end points of the graph are determined by the ordered pairs $(-3, -2)$ and $(2, 3)$.

If a few other points, such as those whose ordered pairs are given in the following table, are tested, it is found that they appear to lie on the line segment joining the points with coordinates $(-3, 2)$ and $(2, 3)$.

x	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
y	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$

The graph is the line segment joining the points with coordinates $(-3, -2)$ and $(2, 3)$. The graph is said to be continuous in this interval.

From the graph it is seen that:

the domain is

$$\{x| -3 \leqq x \leqq 2, x \in R\};$$

the range is

$$\{y| -2 \leqq y \leqq 3, y \in R\}.$$

Example 5. Draw the graph of the relation

$$P_4 = \{(x, y)|y = x + 1,\ x,\ y \in R\}$$

and state its domain and range.

Solution.

$P_4 = \{(x, y)|y = x + 1,\ x,\ y \in R\}$

For P_4:

x	-3	-2	-1	0	1	2
y	-2	-1	0	1	2	3

The partial graph of P_4 is shown in *Fig. 6-15.*

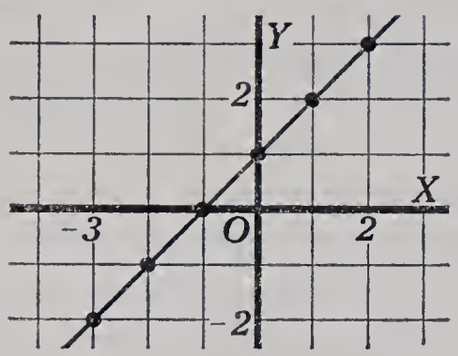


Fig. 6-15

Domain is $\{x|x \in R\}$.

Range is $\{y|y \in R\}$.

DISCUSSION. Make an arbitrary selection of replacements for x from the set R; construct the table of values and plot the corresponding points. The graph of P_4 is the line determined by the plotted points. This is a reasonable assumption since the interval selected for plotting points was chosen arbitrarily. Only partial graphs of relations in $R \times R$ can be drawn.

From the graph it may be seen:
the domain is $\{x|x \in R\}$;
the range is $\{y|y \in R\}$.

Examples 6 and 7 illustrate relations the graphs of which are not straight lines or sets of points which lie on a straight line.

Example 6. Draw the graph of

$P_1 = \{(x, y)|y = x^2,\ x \in A,\ y \in I\}$,

if $A = \{-3, -2, -1, 0, 1, 2, 3\}$.

Solution.

$P_1 = \{(x, y)|y = x^2,\ x \in A,\ y \in I\}$

For P_1:

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

The graph of the relation is shown in *Fig. 6-16* .

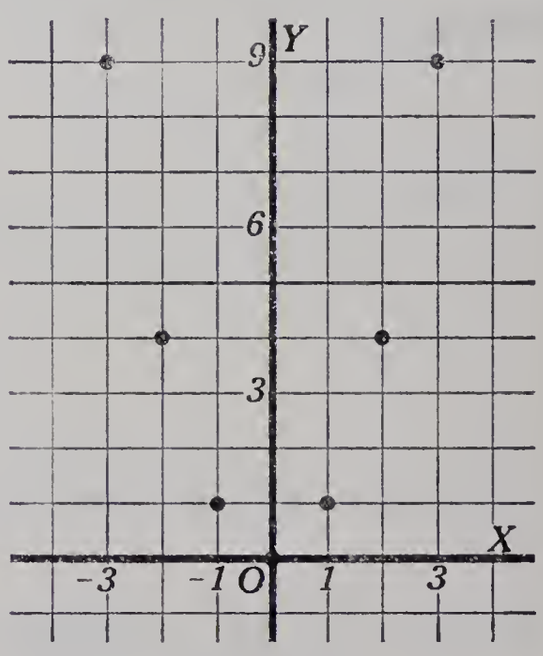


Fig. 6-16

Example 7. Draw the graph of $P_2 = \{x, y | y = x^2, x, y \in R\}$ and state the domain and range of the relation.

Solution.

$P_2 = \{(x, y) | y = x^2, x, y \in R\}$

$\because\ x \in R$, then x^2 (or y) $\in R$.

$\therefore\ P_2 = \{(x, x^2) | x \in R\}$

For P_2:

x	-3	-2	$-.14$	-1
y	9	4	1.96	1

$-.5$	0	.5	1	1.4	2	3
.25	0	.25	1	1.96	4	9

The partial graph is shown in *Fig. 6-17.*

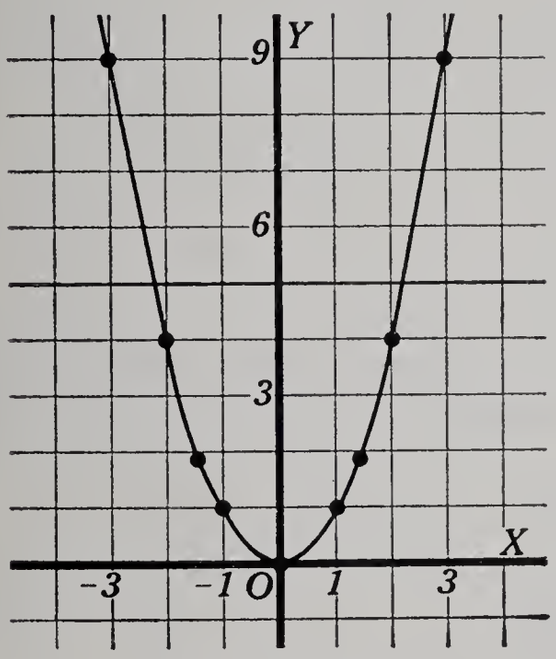


Fig. 6-17

Domain is $\{x | x \in R\}$.

Range is $\{y | y \geqq 0, y \in R\}$.

DISCUSSION. The table of values for P_1 in Example 6 is satisfactory for establishing the pattern of the graph of P_2. In the table given, a few more replacement values are used to illustrate that the smooth curve drawn through the plotted points really is a good approximation to the required graph.

Both the graph and the defining equation, $y = x^2$, suggest that as x assumes all real values in the interval -3 to $+3$, y assumes all real values in the range interval 0 to $+9$. Also, since x may be replaced by any real number, the curve extends indefinitely beyond the points whose coordinates are $(-3, 9)$ and $(3, 9)$.

The graph suggests that:

the domain is $\{x | x \in R\}$;

the range is $\{y | y \geqq 0, y \in R\}$.

Write solutions to the following problems and compare them with those on page 478.

1. Draw the graph of the relation
$M = \{(x, y) | 2x + y = 3, x, y \in R\}$, and state its domain and range.

2. Draw the graph of the relation
$N = \{(x, y) | x = 3, x, y \in R\}$, and state its domain and range.

Exercise 6-5

(A)

1. If $x, y \in R$, and the following are defining equations of relations, express the ordered pairs (x, y) of the relations in terms of the variable x:

(i) $3x = y$ (ii) $x - 2y = 0$ (iii) $x + y = 2$
(iv) $4x - y = 5$ (v) $3x + 4y = 0$ (vi) $2x + 3y = 4$
(vii) $5x + 2y + 6 = 0$ (viii) $ax - y = b$ (ix) $px = qy$
(x) $cx + dy = e$ (xi) $4x^2 - y = 0$ (xii) $x^2 + y - 2 = 0$

2. For $x, y \in I$, find three ordered pairs defined by the following equations:

(i) $2x + y = 3$ (ii) $3x - y = 5$ (iii) $x + 2y = 7$
(iv) $y = 2x^2$ (v) $xy = 12$ (vi) $y = x^2 - 2$

3. Determine which of the following ordered pairs satisfy the relation $A = \{(x, y) | x + 2y = 10, x, y \in I\}$:

(i) $(0, 5)$ (ii) $(2, 5)$ (iii) $(-3, 7)$
(iv) $(-2, 6)$ (v) $(4, 3)$ (vi) $(9, \frac{1}{2})$

(B)

4. For $x, y \in I$, make a table of values showing three ordered pairs defined by each of the following equations:

(i) $x + y = 5$ (ii) $x - y = 5$ (iii) $2x - y = 6$
(iv) $x + 3y = 0$ (v) $2x + y = 1$ (vi) $y = x^2$
(vii) $x + 2y = 8$ (viii) $3x - y = -2$ (ix) $2y = x^2 - 1$
(x) $2x - 3y = 7$ (xi) $4x + 2y = 5$ (xii) $x = y^2$

5. If $x, y \in I$, draw the graph of each of the following relations:

(i) $R_1 = \{(x, y) | y = x - 3\}$ (ii) $R_2 = \{(x, y) | x = y + 4\}$
(iii) $R_3 = \{(x, y) | y = 3\}$ (iv) $R_4 = \{(x, y) | 2x - y = 12\}$
(v) $R_5 = \{(x, y) | 3y + x - 15 = 0\}$ (vi) $R_6 = \{(x, y) | \frac{1}{3}x - y = 2\}$

6. If $x, y \in R$, draw the graph of each of the following relations:

(i) $P_1 = \{(x, y) | x + 2y = 5\}$ (ii) $P_2 = \{(x, y) | 2x - y = 4\}$
(iii) $P_3 = \{(x, y) | 2x + y = 1\}$ (iv) $P_4 = \{(x, y) | x = 4\}$
(v) $P_5 = \{(x, y) | x - \frac{1}{2}y = 2\}$ (vi) $P_6 = \{(x, y) | x = y^2\}$

7. For $x, y \in R$, draw the graph of $\{(x, y) | x - y = 2\}$ and determine the ordinates of points whose abscissas are 4 and -2.

8. (i) Using the same axes, draw the graphs of the relations
$A = \{(x, y)|x + y = 7, x, y \in R\}$ and
$B = \{(x, y)|x - y = 5, x, y \in R\}$.

(ii) Name the coordinates of the points where each line crosses the axes and of the point where the two lines intersect.

(C)

9. The cost of printing tickets for a dance is given by the equation $x = 300 + y$ where x represents the cost in cents and y represents the number of tickets printed. Draw the graph of the relation defined by this equation and from it determine the cost of printing 600 tickets.

***6·7* Linear relations.** It has been observed that the graphs of some relations are sets of distinct points, some are line segments, some are straight lines, and some are curved lines. The segments, lines, or curves are always associated with domains and ranges which are either the set of real numbers or some subset of the real numbers.

Experience leads us to the conclusion that the graph of a relation with a defining sentence such as $3x + 2y = 6$, $x, y \in R$, is a straight line.

The equation $3x + 2y = 6$ involves the two variables, x and y, the two coefficients of the variables, 3 and 2, and the constant term, 6. If we represent the coefficients of the variables by a, b, and the constant term by c, all such equations of the first degree in two variables may be represented by $ax + by = c$.

This equation is called a *general equation* because it represents all equations which have this form. Comparing $3x + 2y = 6$ with the general equation $ax + by = c$, we see that $a = 3$, $b = 2$, $c = 6$ in this case.

The study we have made of the graphs of relations suggests that the following general statement is true.

If $a, b, c \in R$, then any sentence of the form

$$ax + by = c, x, y \in R$$

is the defining sentence of a relation on R whose graph is a straight line.

If the defining sentence of a relation is an equation of the first degree in two variables, then the relation is called a *linear relation.*

***6·8* The x- and y-intercepts of the graph of a linear relation.** The graph of the linear relation $L = \{(x, y)|3x - 2y = 6, x, y \in R\}$ is illustrated in *Fig. 6-18.*

x	0	2	4	6
y	-3	0	3	6

It is seen that the graph crosses the x-axis at a directed distance of $+2$ units from the origin.

The directed distance from the origin to the point where the graph crosses the x-axis is called the x-intercept of the graph.

The directed distance from the origin to the point where the graph crosses the y-axis is called the y-intercept.

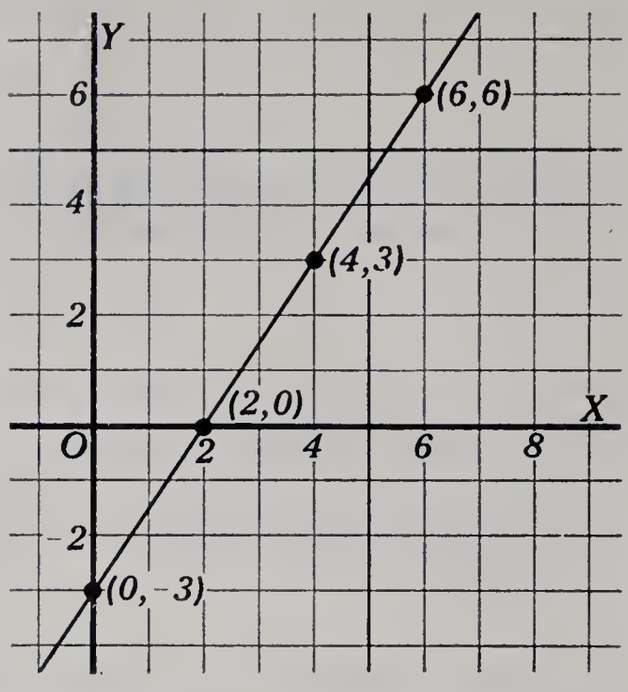


Fig. 6-18

Thus, for the graph of this relation:

(i) the x-intercept is 2;

(ii) the y-intercept is -3.

Each intercept determines a point on the graph:

(i) the x-intercept, 2, determines the point with coordinates (2, 0);

(ii) the y-intercept, -3, determines the point with coordinates $(0, -3)$.

Since any line is determined by two points on it, the graph in *Fig. 6-18* could have been drawn by the following procedure:

(i) find the x- and y- intercepts;

(ii) plot the two points determined by them;

(iii) draw the straight line determined by the points.

This method of drawing the graph is called the *intercept method.*

To find the x-intercept of the graph from the defining equation, we note that the x-intercept is the abscissa of the point on the graph whose ordinate is zero. Therefore, by substituting $y = 0$ in the equation, the corresponding value of x, and hence the x-intercept, is determined.

Thus, in this example in which $3x - 2y = 6$:

$$\text{if } y = 0, \text{ then } 3x = 6$$

$$\therefore x = 2$$

$\therefore$ the x-intercept is 2.

In a similar manner, the y-intercept is determined by substituting $x = 0$ in the equation.

If $x = 0$, then $-2y = 6$

$\therefore y = -3$

$\therefore$ the y-intercept is -3.

Example 1. Use the intercept method to draw the graph of the linear relation $G = \{(x, y) | 2x + 3y = 5, x, y, \in R\}$.

Solution. $2x + 3y = 5$

If $y = 0$, then $2x = 5$

$\therefore x = \frac{5}{2}$

$\therefore$ the x-intercept is $\frac{5}{2}$ or $2\frac{1}{2}$.

If $x = 0$, then $3y = 5$

$\therefore y = \frac{5}{3}$

$\therefore$ the y-intercept is $\frac{5}{3}$ or $1\frac{2}{3}$.

The graph is shown in *Fig. 6-19.*

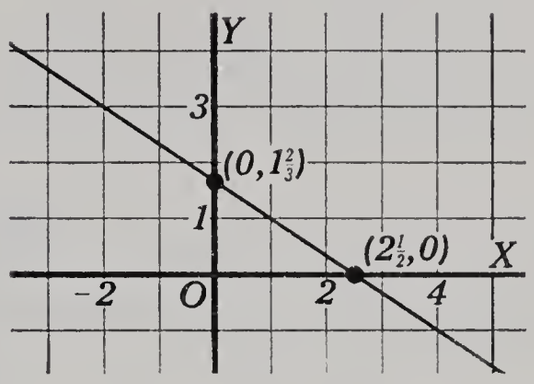


Fig. 6-19

Example 2. Draw the graph of the relation

$H = \{(x, y) | 2x - 3y = 0, x, y \in R\}$.

Solution. $2x - 3y = 0$

If $x = 0$, then $-3y = 0$

$\therefore y = 0$

$\therefore$ the y-intercept is zero.

If $y = 0$, then $2x = 0$

$\therefore x = 0$

$\therefore$ the x-intercept is zero.

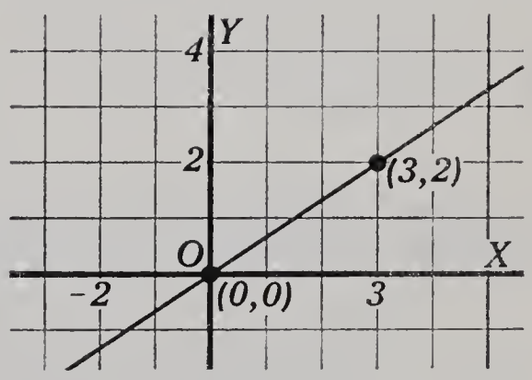


Fig. 6-20.

The two intercepts determine only one point, the point with coordinates (0, 0) on the graph. Thus, a second point must be determined before the graph may be drawn.

If $x = 3$, then $6 - 3y = 0$

$\therefore -3y = -6$

$\therefore y = 2$

(3, 2) corresponds to a point on the graph.

The graph is shown in *Fig. 6-20.*

Exercise 6-6

(A)

1. For each of the linear relations on R defined as follows, find the coordinates of three points on the graph:

(i) $F_1 = \{(x, y) | 3x + y = 7\}$ (ii) $F_2 = \{(x, y) | x + 2y = 4\}$

(iii) $F_3 = \{(x, y) | 2x - y + 5 = 0\}$ (iv) $F_4 = \{(x, y) | 2x + 3y = 5\}$

(v) $F_5 = \{(x, y) | 4x - 2y + 3 = 0\}$ (vi) $F_6 = \{(x, y) | x = 4\}$

(vii) $F_7 = \{(x, y) | y = -2\}$ (viii) $F_8 = \{(x, y) | x + 2y = 0\}$

(B)

2. Find the x-intercept and the y-intercept of the graph of each of the following linear relations on R:

(i) $L_1 = \{(x, y) | 2x + 3y = 6\}$ (ii) $L_2 = \{(x, y) | 4x - y = 5\}$

(iii) $L_3 = \{(x, y) | x - 2y = 0\}$ (iv) $L_4 = \{(x, y) | 3x + 5y + 7 = 0\}$

(v) $L_5 = \{(x, y) | 4y - 5x - 3 = 0\}$ (vi) $L_6 = \{(x, y) | 2x = 3y - 6\}$

Draw the graph of the following relations on R:

3. $A = \{(x, y) | 3x - y = 4\}$ **4.** $B = \{(x, y) | x + 2y = 5\}$

5. $C = \{(x, y) | x - y + 6 = 0\}$ **6.** $D = \{(x, y) | 2x + 4y = 5\}$

7. $E = \{(x, y) | 3x + y + 2 = 0\}$ **8.** $F = \{(x, y) | y = 3\}$

9. $G = \{(x, y) | x = -4\}$ **10.** $H = \{(x, y) | 2x + y = 0\}$

6·9 Linear relations defined by inequations. The graph of the linear relation $P = \{(x, y) | y = x, x, y \in R\}$ is shown in *Fig. 6-21*. This straight

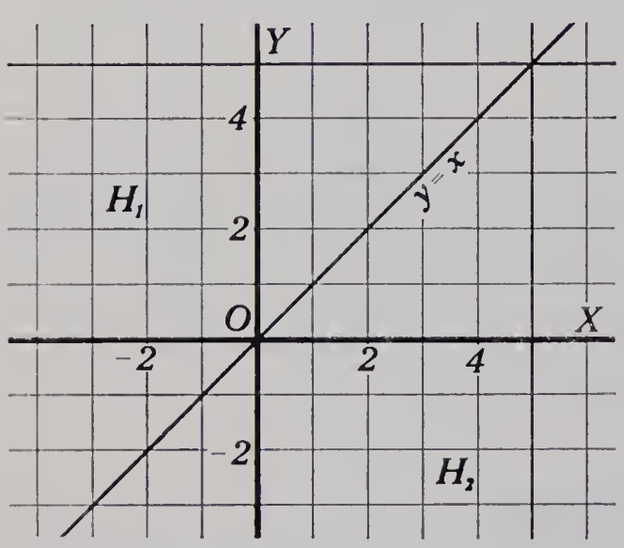


Fig. 6-21

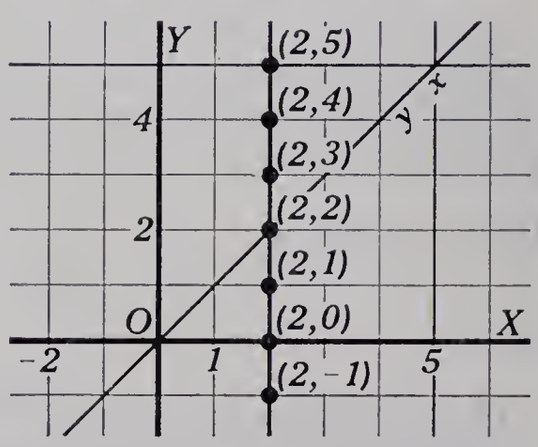


Fig. 6-22

line separates the plane into the two half-planes H_1 and H_2 (*Fig. 6-21*). H_1 is the set of points above the line, and H_2 is the set of points below the line. The points on the line are not in either half-plane. The line, however, is called the *boundary of each half-plane.* Thus, with reference to the line whose equation is $y = x$, the points in the plane may be classified as (i) above (ii) on (iii) below the line.

A defining sentence for each of these three sets of points may be determined as follows.

Consider all the points whose abscissa is 2. These points all lie on the line parallel to the y-axis and passing through the point with coordinates (2, 0).

The intersection of this line (*Fig. 6-22*) with the line whose equation is $y = x$ is the point with coordinates (2, 2). For this point $y = x$. For all points directly above the point with coordinates (2, 2), for example, points with coordinates (2, 3), (2, 4), (2, 5),

$$y > x .$$

For points directly below the point with coordinates (2, 2), for example, points with coordinates (2, 1), (2, 0), (2, $-$ 1),

$$y < x .$$

Since the domain and range are continuous, similar relationships hold with respect to any point on the line for all the other points in the plane. Thus:

(i) for any point on the line, $y = x$;

(ii) for any point above the line, $y > x$;

(iii) for any point below the line, $y < x$.

The set of points on the line is the graph of the relation

$L = \{(x, y) | y = x, x, y \in R\}$.

The set of points above the line, that is the half-plane H_1, is the graph of the relation

$H_1 = \{(x, y) | y > x, x, y \in R\}$.

The set of points below the line, that is the half-plane H_2, is the graph of the relation

$H_2 = \{(x, y) | y < x, x, y \in R\}$.

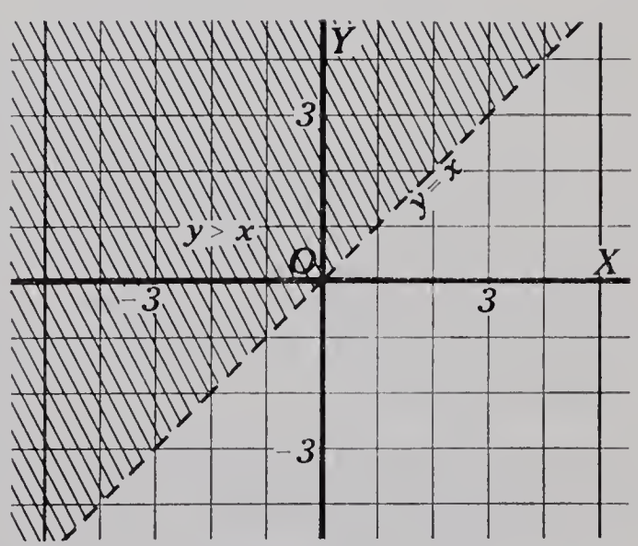


Fig. 6-23

The two sets H_1 and H_2 are defined by inequations and their graphs are described as *regions* of the plane. This is generally true of the graphs of relations defined by inequations in two variables.

The graph of the relation $H_1 = \{(x, y)|y > x, x, y \in R\}$ is shown in *Fig. 6-23*. To draw this graph, the broken line representing the graph defined by $y = x$, $x \in R$ is drawn first. Then the part of the plane above the line is shaded in with solid lines as indicated. The broken line, the boundary of the region, may be considered a construction line. All such lines are drawn as broken lines.

Example 1. Draw the graph of the relation

$$A = \{(x, y)|2x + y \leqq 3, x, y \in R\} .$$

Solution.

The graph of the relation defined by $2x + y = 3$ has x-intercept $\frac{3}{2}$ and y-intercept 3 .

The graph of A consists of:
(i) all the points on the line with equation $2x + y = 3$;
(ii) all the points below this line.

The graph of A is shown in *Fig. 6-24*. Note that in this example the line with equation $2x + y = 3$ is drawn in solid because the points on it are part of the graph of the relation A.

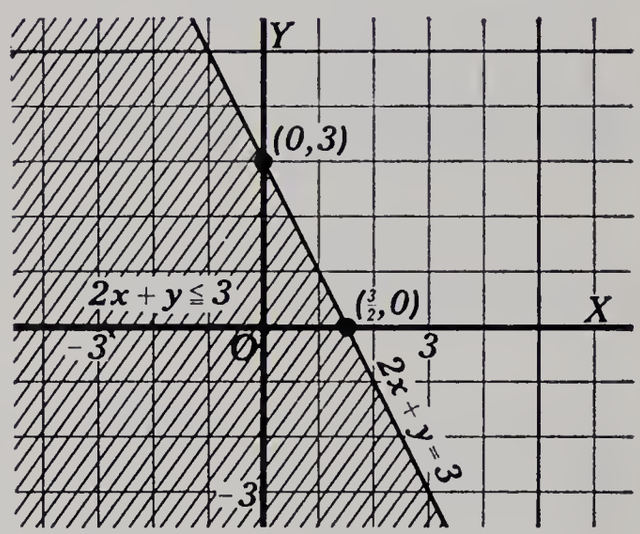


Fig. 6-24

Example 2. Draw the graph of the relation
$B = \{(x, y)|2x - y < 4, x, y \in I\}$
and state its domain and range.

Solution. Since $x, y \in I$, the graph of B will be the set of lattice points in the region *above* the line whose equation is $2x - y = 4$.

The x-intercept of the line is 2 ; the y-intercept of the line is -4 .

The graph of B is shown in *Fig. 6-25*.

Domain is $\{x|x \in I\}$.
Range is $\{y|y \in I\}$.

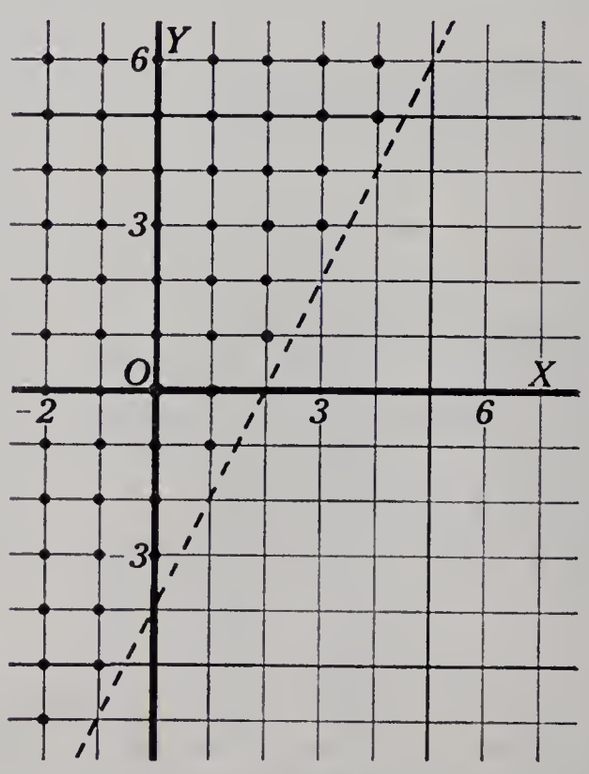


Fig. 6-25

If the defining sentence of a relation is an inequation of the first degree in two variables, the relation is a *linear relation.*

Write a solution to the following problem and compare it with that on page 479.

1. Draw the graph of the relation

$$F = \{(x, y)|x \geqq 3, x, y \in R\}$$

and state its domain and range.

Exercise 6-7

(A)

1. List 5 ordered pairs belonging to each of the relations in R defined by the following sentences:

(i) $y > 2x$ (ii) $y \leqq 3x$ (iii) $y > x - 3$

(iv) $x + y \geqq 2$ (v) $x + 2y < 5$ (vi) $y < 3$

(vii) $2x - y < 3$ (viii) $2x + y \geqq 3$

(B)

Draw the graph of each of the following if $(x, y) \in I \times I$*:*

2. $\{(x, y)|x + y \geqq 3\}$ **3.** $\{(x, y)|2x - y > 0\}$

4. $\{(x, y)|x + 2y \geqq 5\}$ **5.** $\{(x, y)|y \geqq -3\}$

6. $\{(x, y)|x - y < 2\}$ **7.** $\{(x, y)|2x + 2y > 5\}$

8. $\{(x, y)|3x - 2y < 4\}$ **9.** $\{(x, y)|x \leqq 2y - 1\}$

Draw the graph of each of the following if $(x, y) \in R \times R$*:*

10. $\{(x, y)|y > 3x - 1\}$ **11.** $\{(x, y)|x < 4\}$

12. $\{(x, y)|2x - y \geqq 3\}$ **13.** $\{(x, y)|2x \leqq y + 1\}$

14. $\{(x, y)|2x + y \leqq 4\}$ **15.** $\{(x, y)|2x + 3y \geqq 6\}$

(C)

Draw the graph of each of the following if $(x, y) \in R \times R$*:*

16. $\{(x, y)|2x - 3y > 4\}$ **17.** $\{(x, y)|3x + 4y \leqq 2\}$

18. $\{(x, y)|2x - 3y \geqq 5\}$ **19.** $\{(x, y)|y \leqq x^2\}$

20. $\{(x, y)|\ |x| < 3\}$ **21.** $\{(x, y)|x^2 + y^2 \leqq 16\}$

22. $\{(x, y)|x \leqq -2 \text{ and } y \geqq 1\}$

23. $\{(x, y)|2x - y > 3 \text{ and } x + y < 2\}$

6·10 Relations involving the intersection and union of sets.

a. *Relations involving the intersection of sets.*

The graph of $A = \{(x, y)|y = 2x, x, y \in R\}$ and the graph of
$B = \{(x, y)|y = x + 2, x, y \in R\}$
are the two straight lines (broken) illustrated in *Fig. 6-26.*

The two graphs intersect at the point whose coordinates are (2, 4). The ordered pair (2, 4) is an element of set A and also of set B and is thus the *intersection set* of A and B.

$C = \{(2, 4)\}$ is a relation whose domain is $\{2\}$ and whose range is $\{4\}$.

C is the relation defined as the intersection of relations A and B.

Symbolically, this relation may be expressed in any one of the following ways:

(i) $C = A \cap B$ (read, "A cap B" or "A intersect B");

(ii) $C = (A$ and $B)$;

(iii) $C = \{(x, y)|y = 2x, x, y \in R\} \cap \{(x, y)|y = x + 2, x, y \in R\}$;

(iv) $C = \{(x, y)|y = 2x$ and $y = x + 2, x, y \in R\}$;

(v) $C = \{(x, 2x)|x \in R\} \cap \{(x, x + 2)|x \in R\}$.

Form (v) may be used in this case because y (or $2x$) from A and y (or $x + 2$) from B are real numbers for all $x \in R$; the domain of each of the given relations is $\{x|x \in R\}$, and the range of each is $\{y|y \in R\}$.

The graph of C is the point whose coordinates are (2, 4) shown in *Fig. 6-26.* Note the use of broken lines as construction lines in the determination of the point which is the graph of C.

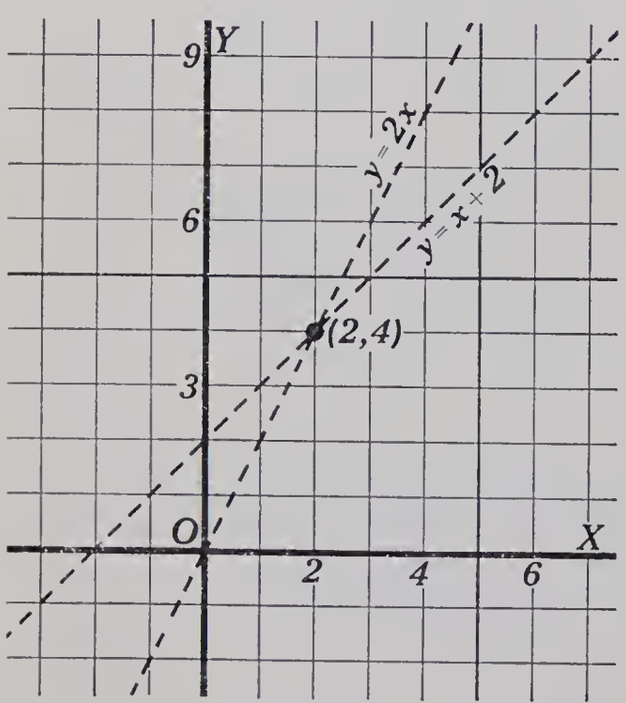


Fig. 6-26

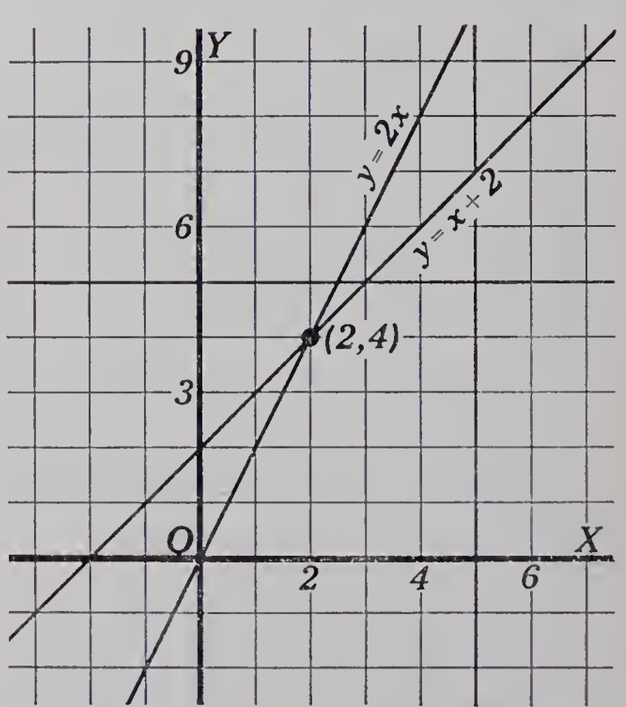


Fig. 6-27

b. *Relations involving the union of two sets.*

The graph of $A = \{(x, y)|y = 2x,\ x,\ y \in R\}$ and the graph of

$$B = \{(x, y)|y = x + 2,\ x,\ y \in R\}$$

are the two straight lines (solid) drawn in *Fig. 6-27.*

The ordered pairs which are the coordinates of any point on either of the two straight lines belong to either set A or to set B or, in the case of (2, 4), to both sets. Thus, this set of ordered pairs is the *union set* of set A and set B.

The relation D which determines this set of ordered pairs has domain $\{x|x \in R\}$ and range $\{y|y \in R\}$.

Symbolically, this relation may be expressed in any one of the following ways:

(i) $D = A \cup B$ (read, "A cup B" or "A union B");

(ii) $D = (A \text{ or } B)$;

(iii) $D = \{(x, y)|y = 2x,\ x,\ y \in R\} \cup \{(x, y)|y = x + 2,\ x,\ y \in R\}$;

(iv) $D = \{(x, y)|y = 2x \text{ or } y = x + 2,\ x,\ y \in R\}$;

(v) $D = \{(x, 2x)|x \in R\} \cup \{(x, x + 2)|x \in R\}$.

Unless otherwise specified, relations in the following examples and exercises are relations on the set of real numbers.

Example. Draw the graphs of the relations:

(i) $M = \{(x, y)|x + y = 10 \text{ and } y = x + 2\}$;

(ii) $P = \{(x, y)|x + y = 10 \text{ or } y = x + 2\}$;

and state the domain and range of each.

Solution.

(i) Let

$M_1 = \{(x, y)|x + y = 10\}$

and

$M_2 = \{(x, y)|y = x + 2\}$.

Then

$M = M_1 \cap M_2$.

For the graph of M_1:
the x-intercept is 10;
the y-intercept is 10.

For the graph of M_2:
the x-intercept is -2:
the y-intercept is 2.

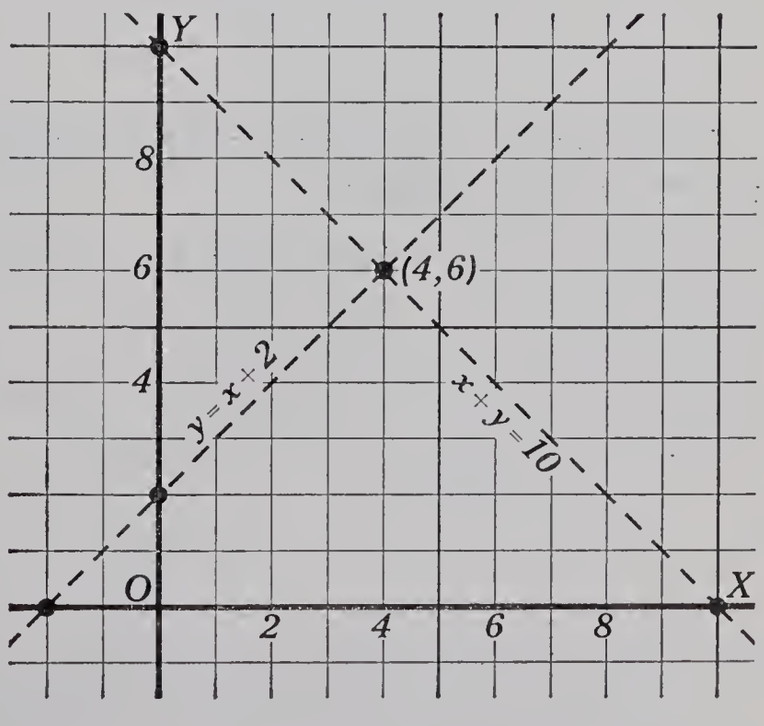


Fig. 6-28

The graphs of M_1 and M_2 are used as construction lines to determine the point of intersection with coordinates (4, 6) which is the graph of M (*Fig. 6-28*).

Thus $M = \{(4, 6)\}$; its domain is $\{4\}$; its range is $\{6\}$.

(ii) Since P is the union of the two sets M_1 and M_2, $P = M_1 \cup M_2$.

The graph of P is shown in *Fig. 6-29* and consists of all the points on the graph of both M_1 and M_2. The set of ordered pairs of P cannot be listed. From the graph, or otherwise, the domain of P is $\{x|x \in R\}$; the range of P is $\{y|y \in R\}$.

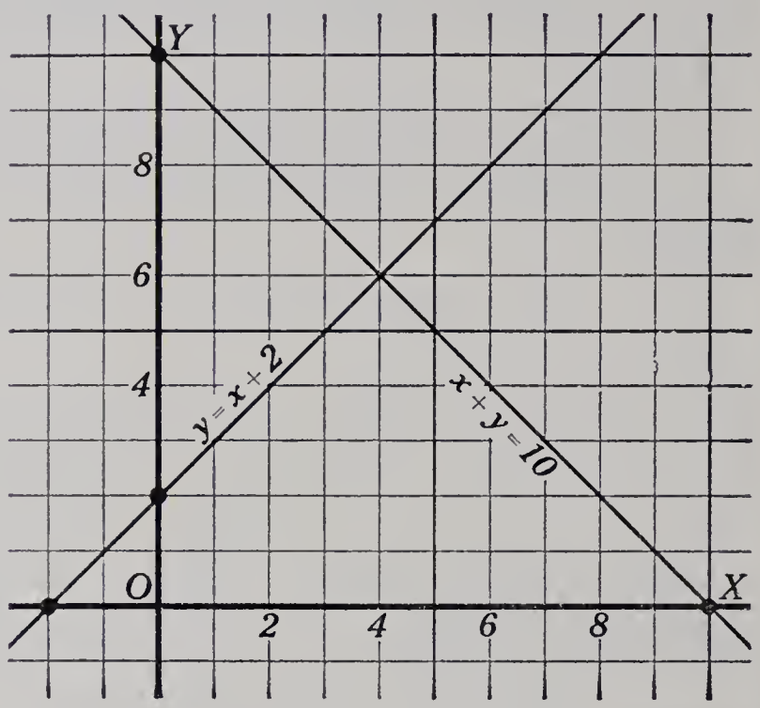


Fig. 6-29

Exercise 6-8

(A)

Classify each of the following as an intersection or a union of two relations and state these two relations:

1. $A = \{(x, y)|x + 2y = 8 \text{ and } 2x - y = 6\}$
2. $B = \{(m, n)|m = 3n \text{ or } 2m = n\}$
3. $C = \{(p, q)|3p - q = 4 \text{ and } 2p + q = 6\}$
4. $D = \{(a, b)|2a + 3b = 6 \text{ or } a + 4b = 8\}$

(B)

5. Draw the graphs of the relations in questions 1 to 4. Where possible, list the ordered pairs of the relation. State the domain and range in each case.

Find graphically:

6. $\{(x, y)|2x = y + 3\} \cap \{(x, y)|x = 2y + 4\}$
7. $\{(x, y)|y = x + 1\} \cup \{(x, y)|y = -x - 3\}$
8. $\{(a, b)|a + b = 0\} \cap \{(a, b)|a - b = 0\}$
9. $\{(m, n)|m + n = 4\} \cup \{(m, n)|n = 4\}$

10. Given the following sets:
$A = \{(x, y)|x + y = 6\}$ $\quad B = \{(x, y)|x - y = 4\}$
$C = \{(x, y)|2x + 2y = 7\}$ $\quad D = \{(x, y)|3x + 3y = 18\}$
find graphically (i) $A \cap B$ (ii) $A \cap C$ (iii) $A \cap D$. If two sets of ordered pairs are determined by linear relations, how many members may be included in their intersections?

(C)

11. Determine graphically:
$\{(x, y)|x - y = 6 \text{ and } x + 5y = -12 \text{ and } 3x + 2y = 3\}$

6·11 Relations involving the intersection and union of sets defined by inequations.

Example 1. Graph the relations:
(i) $A = \{(x, y)|y > 3 - x \text{ and } y > 2x + 2\}$;
(ii) $B = \{(x, y)|y > 3 - x \text{ or } y > 2x + 2\}$;
state the domain and range of each.

Solution.

(i) Let $A_1 = \{(x, y)|y > 3 - x\}$
and $A_2 = \{(x, y)|y > 2x + 2\}$.

For A_1 consider $y = 3 - x$;
the x-intercept of the graph is 3 ;
the y-intercept of the graph is 3 .

For A_2 consider $y = 2x + 2$;
the x-intercept of the graph is -1 ;
the y-intercept of the graph is 2 .

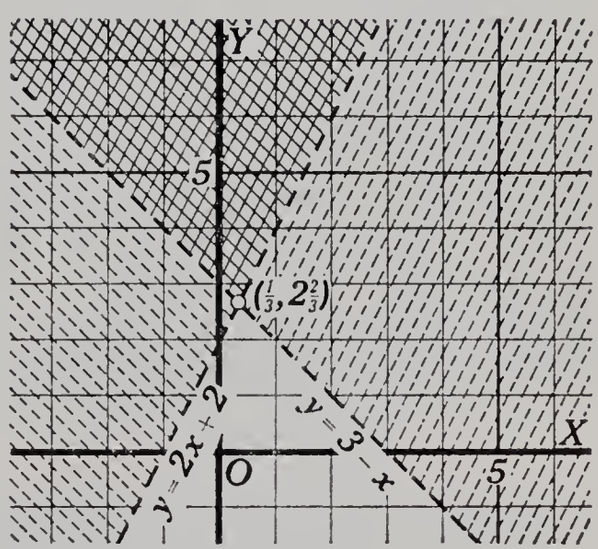


Fig. 6-30

The graph of A_1 is all the points in the region above the line with equation $y = 3 - x$ in *Fig. 6-30.*

The graph of A_2 is all the points in the region above the line with equation $y = 2x + 2$ in *Fig. 6-30.*

Since $A = A_1 \cap A_2$ the graph of A is the set of points common to the

two regions. This is marked with solid lines in *Fig. 6-30.* (Construction lines are broken lines.)

The intersection point of the graphs of A_1 and A_2 has coordinates $(\frac{1}{3}, 2\frac{2}{3})$. This point is not on the graph of A.

The domain of A is $\{x|x \in R\}$; the range of A is $\{y|y > 2\frac{2}{3}, y \in R\}$.

(ii) Since $B = A_1 \cup A_2$, the union of A_1 and A_2, therefore the graph of B is all the points in the region marked with solid lines in *Fig. 6-31.* Note that the point with coordinates $(\frac{1}{3}, 2\frac{2}{3})$ is not on the graph of B.

The domain of B is $\{x|x \in R\}$; the range of B is $\{y|y \in R\}$.

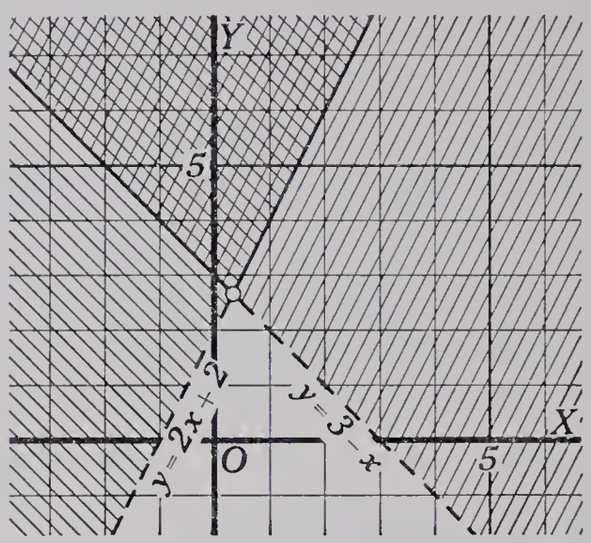


Fig. 6-31

Example 2. Using intersection or union of sets, define the relations for which the following are the graphs:

(i)

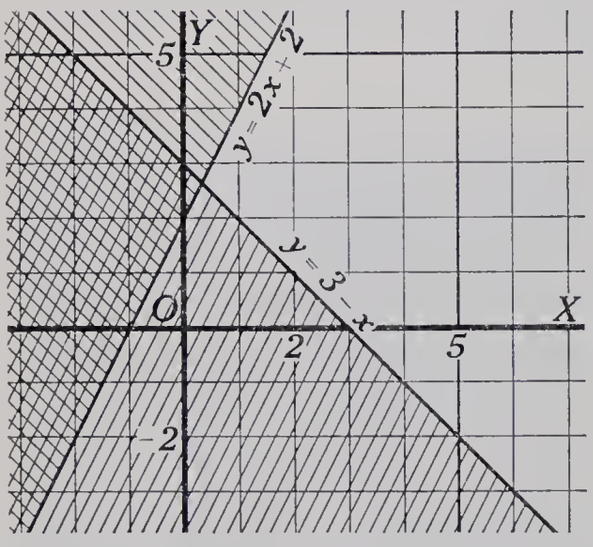


Fig. 6-32

(ii)

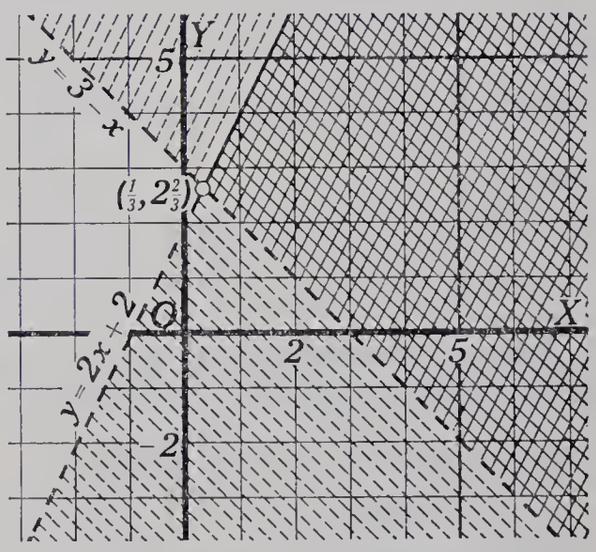


Fig. 6-33

Solution. (i) The graph includes in part all the points in the region above the line whose equation is $y = 2x + 2$ together with all the points on the line. These points are the graph of the relation

$$C_1 = \{(x, y)|y \geqq 2x + 2\}.$$

The graph also includes all the points in the region below the line whose equation is $y = 3 - x$ together with all the points on the line. These points are the graph of the relation

$$C_2 = \{(x, y)|y \leqq 3 - x\}.$$

Therefore, the total graph is the graph of the relation

$$C = C_1 \cup C_2 \text{ or } C = \{(x, y)|y \geqq 2x + 2 \text{ or } y \leqq 3 - x\} .$$

The domain of C is $\{x|x \in R\}$, and the range is $\{y|y \in R\}$.

(ii) The graph includes only those points on the line whose equation is $y = 2x + 2$ and in the region below this line which are also in the region above the line whose equation is $y = 3 - x$.

Thus, the required relation D is the intersection of the two relations D_1 and D_2, where

$$D_1 = \{(x, y)|y \leqq 2x + 2\};$$
$$D_2 = \{(x, y)|y > 3 - x\} .$$

Thus, $D = \{(x, y)|y \leqq 2x + 2 \text{ and } y > 3 - x\}$.

The point whose coordinates are $(\frac{1}{3}, 2\frac{2}{3})$ is not included in the graph.

The domain of D is $\{x|x > \frac{1}{3}, x \in R\}$; the range of D is $\{y|y \in R\}$.

Exercise 6-9

(B)

Graph the following relations; state the domain and range of each:

1. $A = \{(x, y)|2x - y \geqq 0 \text{ or } x + 2y \geqq 5, x, y \in R\}$

2. $B = \{(x, y)|2x - y < 0 \text{ and } x + 2y \geqq 5, x, y \in R\}$

3. $C = \{(x, y)|2x + y = 0 \text{ and } x + 2y \leqq 5, x, y \in R\}$

4. $D = \{(x, y)|2x + y \geqq 0 \text{ and } x + 2y \geqq 5\}$
or $\{(x, y)|2x + y \leqq 0 \text{ and } x + 2y = 5\}$

5. If $A = \{(x, y)|y \geqq -3\}$ $B = \{(x, y)|x \leqq 5\}$
$C = \{(x, y)|y \leqq x + 1\}$
graph $A \cap B \cap C$ (the intersection of A, B, and C).

6. Graph $\{(x, y)|y > 3\} \cup \{(x, y)|x < -2\}$.

7. Graph $\{(x, y)|2x + y \geqq 3\} \cap \{(x, y)|2x + y \leqq 5\}$.

8. If $M = \{(x, y)|y < 4\}$ $N = \{(x, y)|y > 1\}$
$P = \{(x, y)|x < 3\}$ $Q = \{(x, y)|x > -2\}$
graph $M \cap N \cap P \cap Q$.

9. If $D = \{(x, y)|y < 2x - 2\}$ $E = \{(x, y)|y < -x + 5\}$
$F = \{(x, y)|y > -1\}$
graph (i) $D \cap E \cap F$ (ii) $D \cup E \cup F$.

10. Graph $\{(x, y)|y = x^2 \text{ and } x = 2\}$.

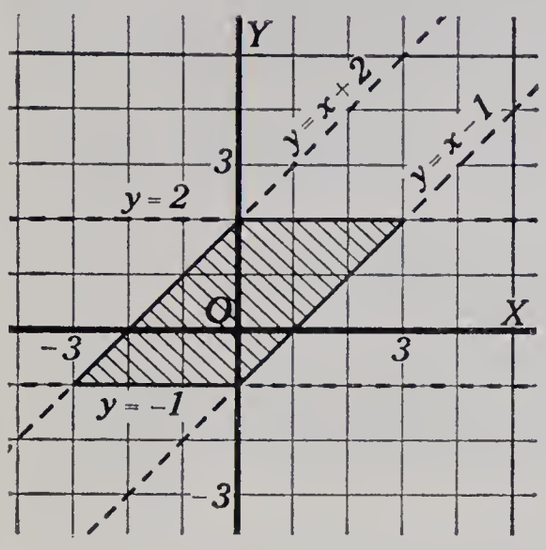
11. Graph $\{(x, y)|y > x^2 \text{ and } x < 2\}$.

12. Graph $\{(x, y)|y \leqq x^2 \text{ and } y \leqq 3\}$.

13. Graph $\{(x, y)|y \geqq x^2 \text{ and } x \geqq -1 \text{ and } x \leqq 2\}$.

Using intersection of sets, define the relations for which the following are the graphs:

14.



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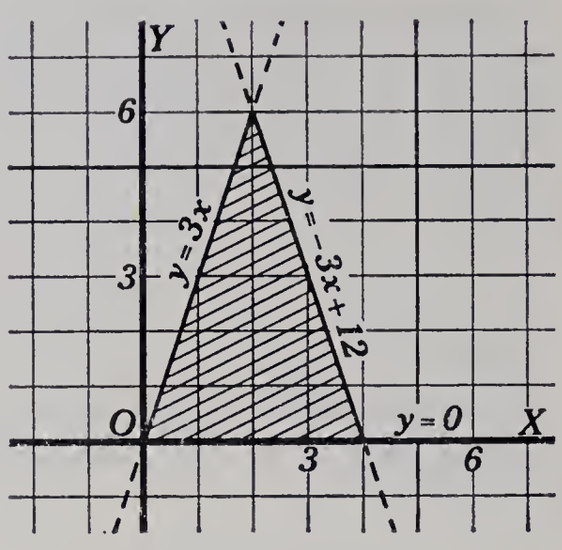

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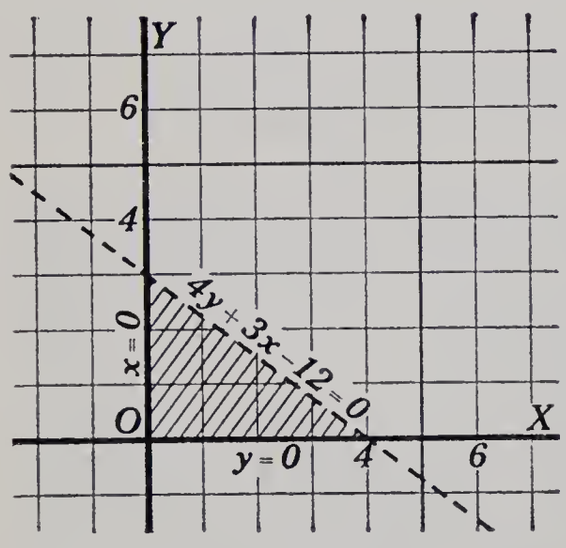

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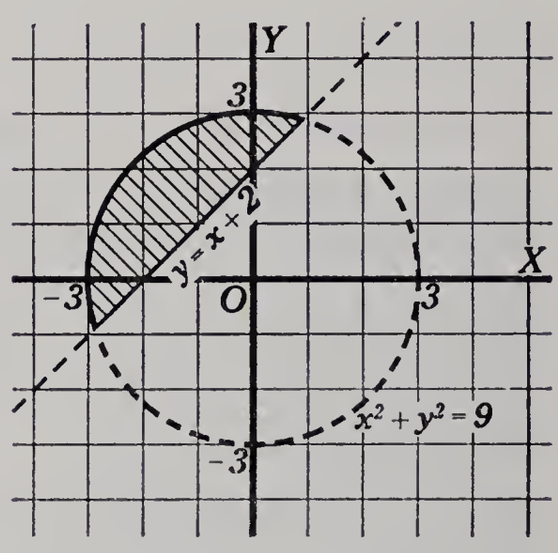

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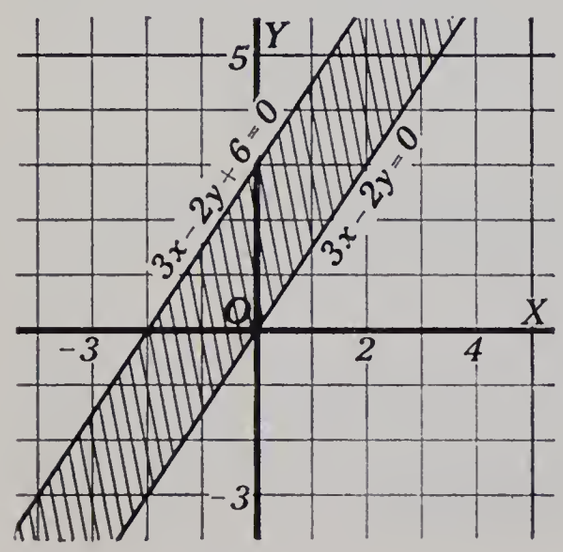

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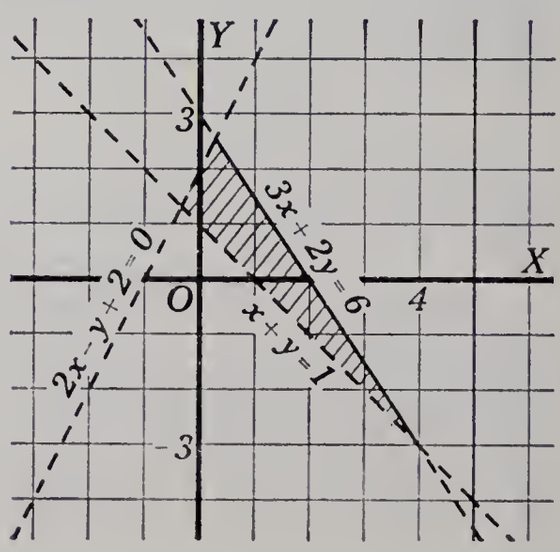


Practice Exercise 6-10

(B)

Draw the graphs of the following relations; state the domain and range of each:

1. $P=\{(x, y)|y > |x| \text{ and } 2y = x + 4\}$
2. $Q=\{(x, y)|y > |x| \text{ or } 2y = x + 4\}$
3. $R=\{(x, y)|y \geqq |x| \text{ and } 2y \leqq x + 4\}$
4. $S=\{(x, y)|y - 2x = 5 \text{ and } |x| = 3\}$
5. $T=\{(x, y)|y - 2x \leqq 5 \text{ and } x \leqq 3\}$

(C)

6. $M=\{(x, y)|x^2 + y^2 = 25 \text{ and } 3x - 4y = 0\}$
7. $N=\{(x, y)|x^2 + y^2 \leqq 25 \text{ and } 3x - 4y \leqq 0\}$
8. Find graphically the ordered pair which has the greatest value of the range of the relation:

$C=\{(x, y)|x > 0 \text{ and } y > 0 \text{ and } y \leqq 5x + 3 \text{ and } 4x + y - 6 \leqq 0\}$

***6·12* Linear relations in Physics (supplementary).** Many laws in Physics express a relation between two or more variables. For example, the relation between potential difference, current strength, and resistance in an electrical circuit is expressed by the equation,

$$V = IR$$

where V represents the potential difference in volts,

I represents the current strength in amperes,

and R represents the resistance in ohms.

Example 1. (i) Draw a graph relating potential difference and current strength for an electrical circuit with a resistance of 25 ohms if the current strength varies from 0 to 6 amperes.

(ii) From the graph, find (a) the potential difference for a current strength of 3.4 amperes (b) the current strength for a potential difference of 130 volts.

Solution. (i) Since the resistance is 25 ohms, the relation involving potential difference and current strength is

$$A=\{(I, V)|V = 25I, 0 \leqq I \leqq 6, I, V \in R\}\ .$$

The defining sentence of the relation contains a first degree equation, $V = 25I$, in the variables I and V, and therefore the graph is a line segment.

For this relation:

I	0	2	4
V	0	50	100

The graph, shown in *Fig. 6-34*, lies in the first quadrant since I and V are always positive or zero.
(ii) From the graph:

(a) the potential difference for a current strength of 3.4 amperes is 85 volts (ordinate of P_1), and

(b) the current strength for a potential difference of 130 volts is 5.2 amperes (abscissa of P_2).

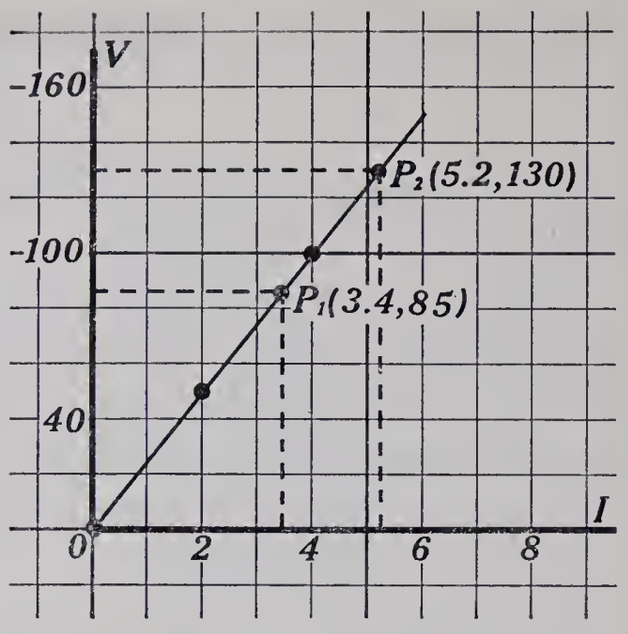


Fig. 6-34

The process of estimating a value of the range of a relation for a given value of the domain, or vice-versa, by a procedure other than using the defining sentence (using a graph, for example) is called *interpolation*. The result obtained is approximate.

Example 2. Two cars A and B start at the same time, on the same road, in the same direction. Car B starts 30 miles ahead of car A. Car A travels at 60 m.p.h., and car B travels at 50 m.p.h.

With reference to the same set of axes, draw a graph relating time and distance for each car. From the graph find (i) the time taken for car A to overtake car B (ii) the distance between the cars after 2 hours (iii) the distance between the cars after 5 hours.

The relation between time, speed, and distance for a moving object is expressed by the equation

$$d = vt,$$

where d represents the distance travelled,

v represents the speed,

and t represents the time,

where d, v, t are expressed in related units.

Solution. The relation involving time and distance for car A is

$$R_1 = \{(t, d) | d = 60t,\ t, d \geqq 0,\ t, d \in R\}\ .$$

Since car B starts 30 miles ahead of car A, the time-distance relation for car B is $R_2 = \{(t, d) | d = 30 + 50t,\ t, d \geqq 0,\ t, d \in R\}\ .$

For car A:

t	0	2	4
d	0	120	240

For car B:

t	0	2	4
d	30	130	230

The graph, shown in *Fig. 6-35*, is often referred to as a *distance-time graph.*

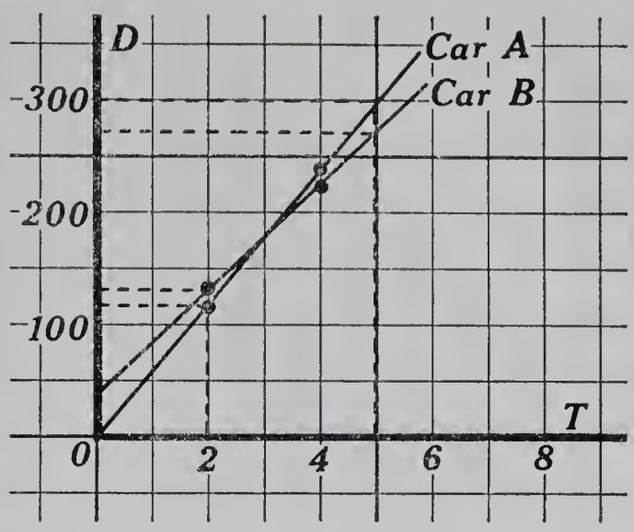


Fig. 6-35

(i) Since the coordinates of the point of intersection are (3, 180), therefore car A will overtake car B in 3 hours.

(ii) From the graph, by interpolation, car A is 120 miles from its starting point, and car B is 130 miles from the same point after 2 hours. Thus, the distance between the cars at the end of two hours is (130 − 120) miles, or 10 miles.

(iii) From the graph, car A is 300 miles from its starting point, while car B is 280 miles from the same point after 5 hours. Thus, the distance between the cars after 5 hours is (300 − 280) miles or 20 miles.

In this case, each of the ordinates obtained (300 and 280) is greater than those used in obtaining the table of values. However, the graph can be extended indefinitely in the first quadrant because the domain and range of the relation have no upper bound, and the coordinates of points on this extended part can be obtained by *extrapolation.*

Example 3. The following graphs, *Fig. 6-36*, represent the time-distance relations of two racing cars. Using set-builder notation, write a sentence which defines the relation in each case.

Solution. (i) Because the origin lies on the graph, *Fig. 6-36* (*a*), and the graph is a line segment, the defining sentence contains an equation of the form $d = vt$, where v represents a constant.

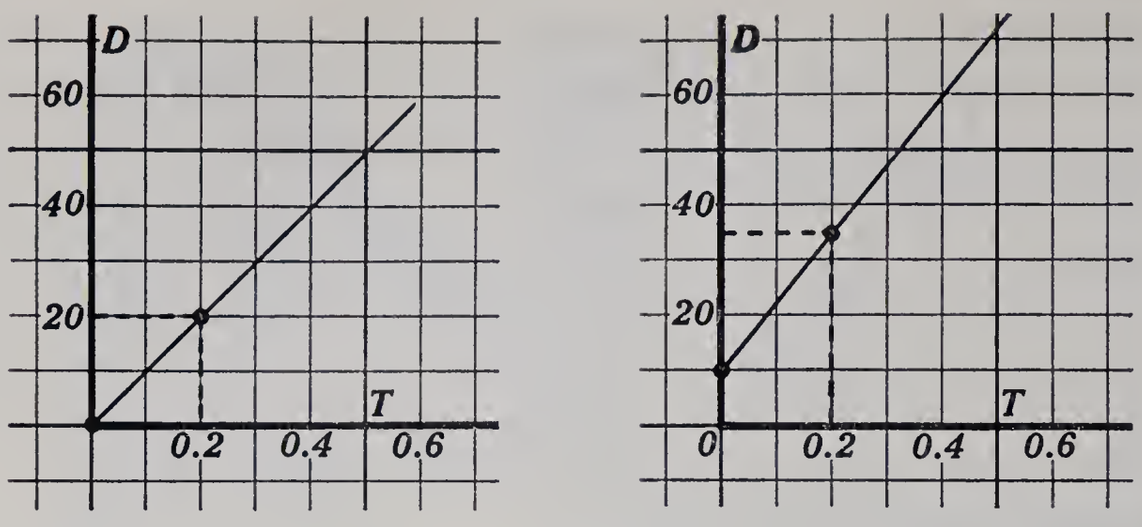
Because the point with coordinates (.20, 20) lies on the graph,

$$20 = v(.20)$$

$$\therefore\ 100 = v$$

$\therefore$ the defining sentence is

$$d = 100t, \quad t, d \geqq 0, \quad t, d \in R\,.$$



(a) (b)

Fig. 6-36

Fig. 6-36(a) is the graph of the relation

$$P_1 = \{(t, d) | d = 100t,\ t, d \geqq 0,\ t, d \in R\} .$$

(ii) Because the point with coordinates (0, 10) lies on the graph, *Fig. 6-36 (b)*, and the graph is a line segment, the defining sentence contains an equation of the form $d = vt + 10$, where v represents a constant.

Because the point with coordinates (.20, 35) lies on the graph,

$$35 = v(.20) + 10$$
$$\therefore \quad 25 = .20v$$
$$\therefore \quad 125 = v$$
$$\therefore$$ the defining sentence is
$$d = 125t + 10,\ t, d \geqq 0,\ t, d \in R .$$

Fig. 6-36(b) is the graph of the relation

$$P_2 = \{(t, d) | d = 125t + 10,\ t, d \geqq 0,\ t, d \in R\} .$$

Example 4. From a standing start, a car can attain a speed of 60 m.p.h. in 20 seconds.

(i) Calculate the acceleration, assuming it to be uniform.

(ii) Draw a speed-time graph and define the relation using set-builder notation. Consider only the first 20 seconds.

Solution. (i) Because the car attains a speed of 60 m.p.h. in 20 seconds, then the car would attain a speed of 3 m.p.h. in 1 second. Thus, the acceleration is 3 m.p.h. each second.

(ii) For this relation:

t	0	1	2	3	5	10	20
d	0	3	6	9	15	30	60

Because the graph is a line segment passing through the origin

(*Fig. 6-37*), the defining sentence contains an equation of the form

$$v = at$$

where
a represents constant acceleration,
v represents the speed, and
t represents the time
in related units.

In part (i) the acceleration was found to be 3 m.p.h. each second.

Thus the relation can be written

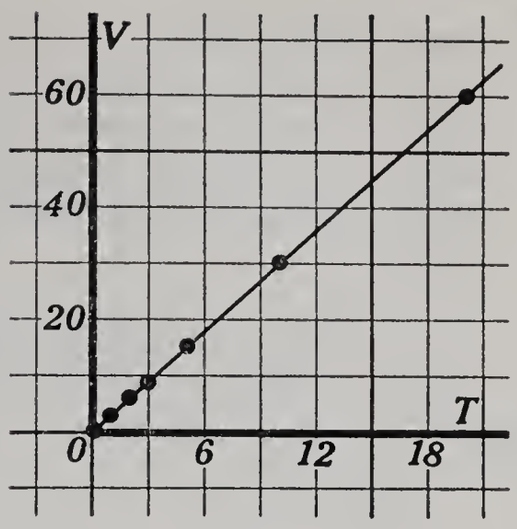


Fig. 6-37

$$S = \{(t, v) | v = 3t,\ 0 \leqq t \leqq 20,\ \text{t},\ v \in R\}\ .$$

Exercise 6-11

(B)

1. Using the equation $V = IR$, draw a graph relating potential difference and resistance for a circuit in which the current strength is always 8.2 amperes, and the resistance varies from 1 ohm to 10 ohms.

2. (a) Draw a graph relating the variables T and V, where
$V = 1089 + 2T$, and
V represents the speed of sound in air in feet per second,
T represents the air temperature in degrees centigrade,
and the air temperature varies from 0°C. to 60°C.

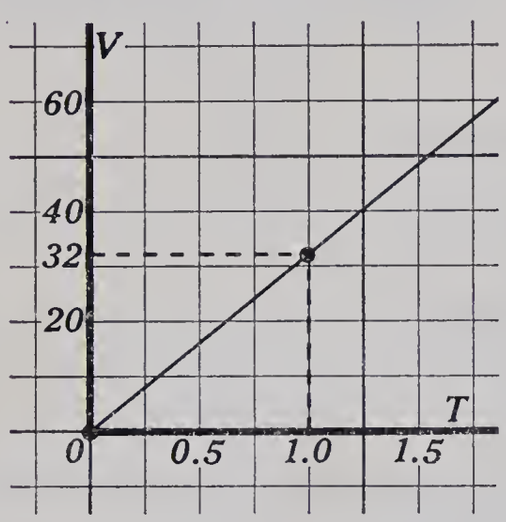
(b) From the graph, find:
(i) the speed of sound in air for an air temperature of 26°C.;
(ii) the air temperature necessary for the speed of sound to be 1,100 feet per second.

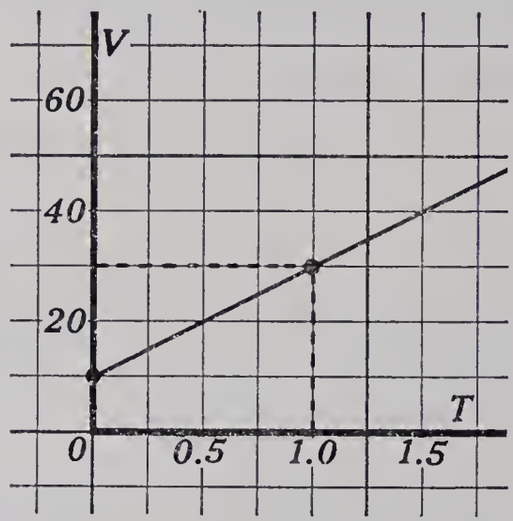
3. (a) Draw a graph relating temperature readings in degrees Fahrenheit (F.), with the corresponding readings in degrees centigrade (C.), where C. $= \frac{5}{9}$(F. $-$ 32) and F. varies from $-$ 50 to $+$ 100.

(b) From the graph, find:
(i) the equivalent temperature in degrees Fahrenheit to $+$ 15°C.;
(ii) the equivalent temperature in degrees centigrade to 10°F.;
(iii) the temperature at which the centigrade and Fahrenheit readings are the same.

4. For a sound wave, $V = n\lambda$, where V represents the velocity of the sound wave in cm. per sec., n represents the frequency in vibrations per sec., and λ represents the wave-length in cm., draw a graph relating wave-length and velocity for a sound wave of frequency 256 vibrations per sec., for the wave-length varying from 100 cm. to 150 cm.

5. The following graphs represent the speed-time relations of two moving objects. Using set-builder notation, write a sentence which defines the relation in each case.





6. Two cars, A and B, pass the same point at the same time, travelling in the same direction. Car A is travelling at a constant rate of 40 m.p.h.; car B is travelling at 10 m.p.h. as it passes the point and accelerates 5 m.p.h. each second thereafter.

(i) From a speed-time graph, find how much time it takes for car B to overtake car A.

(ii) Using set-builder notation, write sentences defining the speed-time relation for each car.

Chapter VII

SYSTEMS OF LINEAR EQUATIONS

In Chapter VI we discussed sets of ordered pairs defined by

(i) a first degree equation in two variables, and

(ii) the intersection and union of two sets.

The solution sets were determined by a graphical method to illustrate the ideas involved. In this chapter we will discover algebraic means of determining solution sets defined by a system of two linear equations or inequations of the first degree.

7·1 Solving a system of linear equations. The defining equations of the relation

$$A = \{(x, y) | x + y = 2 \text{ and } x + 2y = 6,\ x, y \in R\}$$

are:

$$\begin{cases} x + y = 2 & (1) \\ x + 2y = 6 & (2) \end{cases}$$

Each equation defines a set of ordered pairs, the graph of which is a straight line (*Fig. 7-1*), and the relation A determines the one ordered pair $(-2, 4)$ which is common to the two sets and whose graph is the point of intersection of the two lines (*Fig. 7-1*).

The pair of equations

$$\begin{cases} x + y = 2 \\ x + 2y = 6 \end{cases}$$

is called a *system of two equations.*

$\{(-2, 4)\}$ is called the *solution set* of the system.

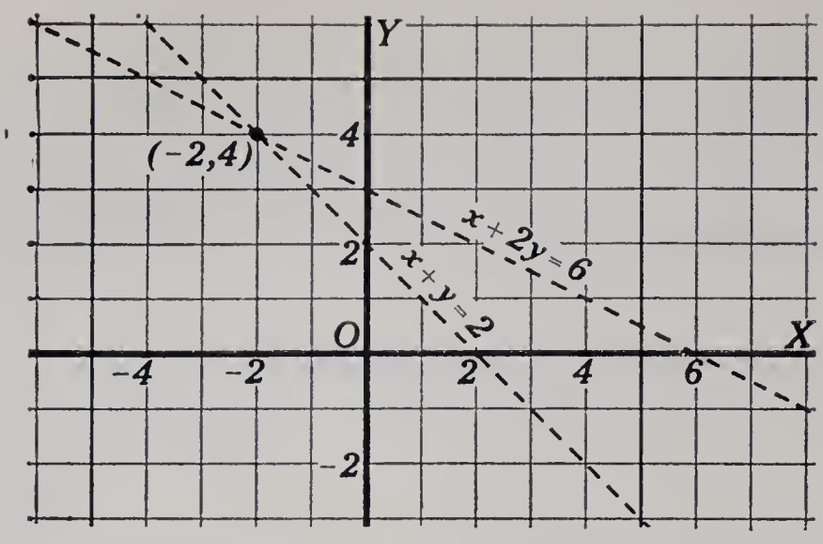


Fig. 7-1

The solution set of a system of equations in two variables, x and y, is the set of all ordered pairs (x, y) which are common to the solution sets defined by each of the given equations.

It may be seen from *Fig. 7-2* that the systems

$$\text{(i)}\quad \begin{cases} x + 2y = 6 \\ x + y = 2 \end{cases}$$

$$\text{(ii)}\quad \begin{cases} x = -2 \\ x + 2y = 6 \end{cases} \qquad \text{(iii)}\quad \begin{cases} x = -2 \\ x + y = 2 \end{cases}$$

$$\text{(iv)}\quad \begin{cases} y = 4 \\ x + 2y = 6 \end{cases} \qquad \text{(v)}\quad \begin{cases} y = 4 \\ x + y = 2 \end{cases}$$

$$\text{(vi)}\quad \begin{cases} x = -2 \\ y = 4 \end{cases}$$

have the same solution set $\{(-2, 4)\}$.

To find the solution set of the given system algebraically, it is necessary to find a way to obtain system (vi) from system (i).

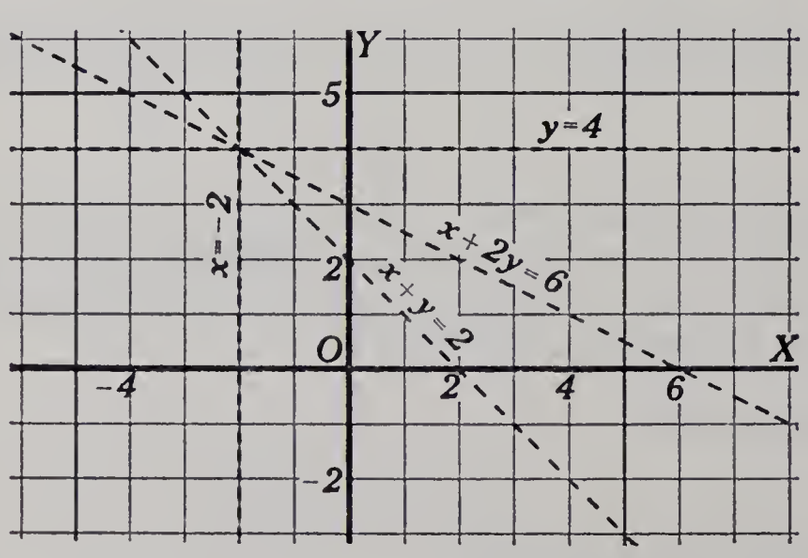


Fig. 7-2

The relation A consists of the intersection set of the two relations $A_1 = \{(x, y)|x + y = 2, x, y \in R\}$ and $A_2 = \{(x, y)|x + 2y = 6, x, y \in R\}$ which may be expressed as:

$$A_1 = \{(x, 2 - x)|x \in R\} \text{ and } A_2 = \left\{\left(x, \frac{6 - x}{2}\right)|x \in R\right\}$$

since $2 - x$, $\dfrac{6 - x}{2} \in R$ by the law of closure.

Comparing the ordered pairs of A_1 and A_2, it is seen that the ordered pairs which belong to both A_1 and A_2 must be such that

$$2 - x = \frac{6 - x}{2}, \quad x \in R$$

or	$4 - 2x = 6 - x$	(Multiplication property)
or	$-x = 2$	(Addition property)
or	$x = -2$.	(Division property)

Thus, to obtain the solution set we solve either of the two simpler systems:

(ii) $\begin{cases} x = -2 \\ x + 2y = 6 \end{cases}$ or (iii) $\begin{cases} x = -2 \\ x + y = 2 \end{cases}$

This may be done simply by substitution, whence

$-2 + 2y = 6$	or	$-2 + y = 2$
$\therefore 2y = 8$		$\therefore y = 4$
$\therefore y = 4$		

and the system $\begin{cases} x = -2 \\ y = 4 \end{cases}$ is obtained.

It is readily seen that the steps of the argument are reversible, and, therefore, the systems are *equivalent* and the solution set is $\{(-2, 4)\}$.

The pattern of algebraic solution and the reason for each step is shown in the following:

	$\begin{cases} x + y = 2 \\ x + 2y = 6 \end{cases}$	(1) (2)	
From (1)	$y = 2 - x$	(3)	(Addition property)
From (2)	$y = \dfrac{6 - x}{2}$	(4)	(Subtraction and division)
From (3) and (4)	$2 - x = \dfrac{6 - x}{2}$	(5)	(Transitive property)
	$\therefore 4 - 2x = 6 - x$		(Multiplication property)
	$\therefore -x = 2$		(Addition and subtraction)
	$\therefore x = -2$	(6)	(Division property)

$\because\ y = 2 - x$	(3)	or	$\because\ y = \dfrac{6 - x}{2}$	(4)
$\therefore\ y = 2 - (-2)$			$\therefore\ y = \dfrac{6 - (-2)}{2}$	
$\therefore\ y = 4$	(7)		$\therefore\ y = 4$	(7)

The solution set is $\{(-2, 4)\}$.

Verification.

L.S.(1)	$= -2 + 4$	L.S.(2)	$= -2 + 2(4)$
	$= 2$		$= 6$
R.S.(1)	$= 2$	R.S.(2)	$= 6$

The algebraic process of determining the solution set of a system of equations is called *solving the system.* The essential feature of the procedure in the previous example is that by *comparison* of equations (3) and (4), equation (5), which does not contain the variable y, is formed. By eliminating y in this way, an equation in one variable is obtained and this can be solved easily. This method of solving is referred to as the method of *elimination by comparison.*

The system $\begin{cases} x + y = 2 & (1) \\ x + 2y = 6 & (2) \end{cases}$

may also be written

$$\begin{cases} x = 2 - y & (3) \\ x = 6 - 2y & (4) \end{cases}$$

whence by *comparison*

$$2 - y = 6 - 2y$$

or $$y = 4.$$

Thus, to obtain the solution set we must solve either of the two simpler systems:

(iv) $\begin{cases} y = 4 \\ x + 2y = 6 \end{cases}$ or (v) $\begin{cases} y = 4 \\ x + y = 2 \end{cases}$

which may be done by substitution, as before, to obtain

$$\begin{cases} x = -2 \\ y = 4. \end{cases}$$

In the worked examples of the text, the authorities which are most significant in the development are given in brackets. When writing a solution you should think the authorities, but it is not necessary to include them in your solution.

Unless otherwise stated, the variables are real numbers.

Write solutions for the following problems and compare them with those on page 479.

1. Find the solution set defined by the following system of equations; verify: $\begin{cases} x + y = 3 \\ 2x + 3y = 8 \end{cases}$
2. Find $\{(x, y)|\frac{1}{4}x + \frac{1}{2}y = 7 \text{ and } \frac{1}{3}x + \frac{1}{4}y = 6\}$.

Exercise 7-1

(A)

Express each of the following in terms of x:

1. $\{(x, y)|3x + y = 8\}$
2. $\{(x, y)|4x + 3y = 7\}$
3. $\{(x, y)|5x - y = 3\}$
4. $\{(x, y)|x + 5y = -2\}$
5. $\{(x, y)|x + \frac{1}{3}y = 5\}$
6. $\{(x, y)|2x - 3y = 5\}$

By expressing y in terms of x, or x in terms of y, state a system of equations equivalent to the given systems for each of the following:

7. $\begin{cases} 2y - x = 0 \\ y + 2 = x - 2 \end{cases}$
8. $\begin{cases} x + y = 3 \\ x - y = 1 \end{cases}$
9. $\begin{cases} 2x + y = 3 \\ 5x + 2y = 7 \end{cases}$

Use the method of comparison to obtain a system equivalent to the given system for each of the following:

10. $\begin{cases} y = x \\ y = 3x - 2 \end{cases}$
11. $\begin{cases} y = x - 1 \\ y = 2x - 5 \end{cases}$
12. $\begin{cases} x = \dfrac{y - 2}{3} \\ x = \dfrac{2y - 7}{5} \end{cases}$

(B)

Use the method of comparison to determine the ordered pairs (x, y) defined by the following systems of equations; verify:

13. $\begin{cases} 5x + 2y = 11 \\ 3x + 4y = 1 \end{cases}$
14. $\begin{cases} 3x + 2y = 11 \\ 4x - 3y = 9 \end{cases}$
15. $\begin{cases} y = 2x + 3 \\ x + y = 6 \end{cases}$
16. $\begin{cases} 3y = -2 - 5x \\ y + 3x + 6 = 0 \end{cases}$
17. $\begin{cases} \frac{1}{3}x - \frac{1}{5}y = 2 \\ \frac{2}{3}x + \frac{3}{5}y = 9 \end{cases}$
18. $\begin{cases} \frac{1}{2}y - 3x = 8 \\ 3x + y = -2 \end{cases}$

Find, using the method of comparison:

19. $\{(x, y)|9x = 7y \text{ and } x - y = -2\}$
20. $\{(x, y)|3y + 4x = 5 \text{ and } 12x + 6y = 13\}$
21. $\{(x, y)|4x + 3y = 24 \text{ and } 3x + 4y = 25\}$
22. $\left\{(x, y)\middle|\dfrac{x}{3} - \dfrac{y}{6} = \dfrac{1}{2} \text{ and } \dfrac{x}{5} - \dfrac{3}{10}y = \dfrac{1}{2}\right\}$

Find, using the method of comparison:

23. $\{(x, y)|5x + \frac{1}{5}y = -4 \text{ and } 5y + \frac{1}{5}x = 4\}$

24. $\{(x, y)|x + \frac{1}{3}y = y \text{ and } y + \frac{1}{4}x = x + 3\}$

(C)

Solve each of the following systems:

25. $3x - 4y = 5x + y = 23$

26. $\begin{cases} \dfrac{x-2}{3} = \dfrac{y+5}{2} \\ \dfrac{13-y}{6} = \dfrac{2x-7}{3} \end{cases}$

27. $\begin{cases} \frac{1}{3}x + \frac{1}{4}y = 6 \\ y - \frac{1}{4}(x - y) = 7 \end{cases}$

7·2 Elimination of a variable by substitution. The following example illustrates a method of eliminating a variable from a system of equations by *substitution.*

The relation $B = \{(x, y)|x + y = 3 \text{ and } x + 2y = 4\}$ consists of the intersection set of

$B_1 = \{(x, y)|x + y = 3\}$ and $B_2 = \{(x, y)|x + 2y = 4\}$.

The graphical solution in *Fig. 7-3* indicates that the solution set is $\{(2, 1)\}$ and that the following systems have the same solution set.

(i) $\begin{cases} x + y = 3 \\ x + 2y = 4 \end{cases}$

(ii) $\begin{cases} x = 2 \\ x + y = 3 \end{cases}$ (iii) $\begin{cases} x = 2 \\ x + 2y = 4 \end{cases}$ (iv) $\begin{cases} y = 1 \\ x + y = 3 \end{cases}$ (v) $\begin{cases} y = 1 \\ x + 2y = 4 \end{cases}$

(vi) $\begin{cases} x = 2 \\ y = 1 \end{cases}$

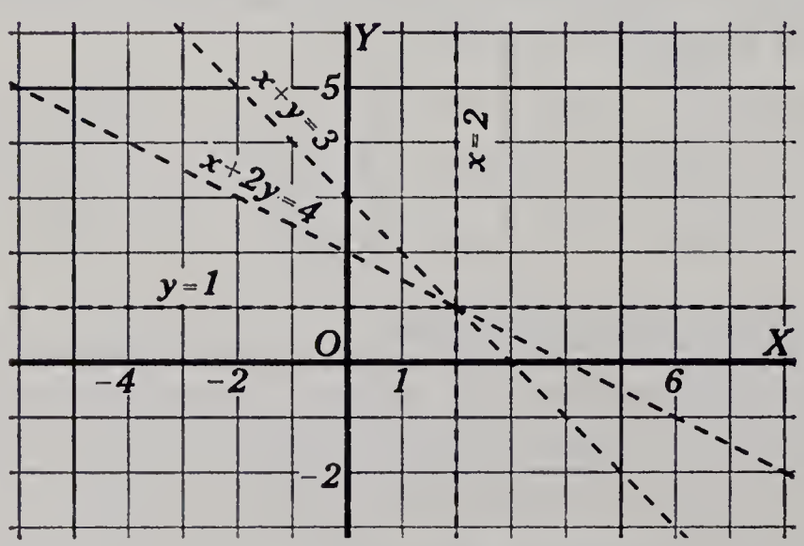


Fig. 7-3

To obtain system (vi) from system (i) algebraically, we note that B_1 may be described as

$$B_1 = \{(x, 3 - x) | x \in R\} .$$

From the set of ordered pairs $(x, 3 - x)$, we are required to select those ordered pairs that belong to B_2: that is, those which satisfy the equation

$$x + 2y = 4 .$$

These may be found by *substituting* $(x, 3 - x)$ in the equation and solving to find the replacements for x which permit this.

On substitution $x + 2(3 - x) = 4$

$\therefore x + 6 - 2x = 4$ (D)

$\therefore -x = -2$ (D, Subtraction property)

$\therefore x = 2$ (Multiplication property)

Thus, to obtain the solution set we must solve either of the two simpler systems:

(ii) $\begin{cases} x = 2 \\ x + y = 3 \end{cases}$ or (iii) $\begin{cases} x = 2 \\ x + 2y = 4 \end{cases}$

This may be done simply by substitution, whence

$2 + y = 3$ or $2 + 2y = 4$

$\therefore y = 1$ $\therefore 2y = 2$

$\therefore y = 1$

and the system $\begin{cases} x = 2 \\ y = 1 \end{cases}$ is obtained.

It is readily seen that the steps of the argument are reversible, and, therefore, the systems are equivalent and the solution set is $\{(1, 2)\}$.

The pattern of algebraic solution and the reason for each step is shown as follows:

$\begin{cases} x + y = 3 & (1) \\ x + 2y = 4 & (2) \end{cases}$

From (1) $y = 3 - x$ (3) (Addition property)

Substitute in (2) $x + 2(3 - x) = 4$ (Substitution)

$\therefore x + 6 - 2x = 4$ (D)

$\therefore -x = -2$ (D, Subtraction property)

$\therefore x = 2$ (4) (Multiplication property)

Substitute in (3) $y = 3 - 2$

$\therefore y = 1$ (5)

The solution set is $\{(2, 1)\}$

Verification.

L.S.(1) $= 2 + 1$ L.S.(2) $= 2 + 2$

$= 3$ $= 4$

R.S.(1) $= 3$ R.S.(2) $= 4$

It should be noted that the above solution could have been obtained by starting in several ways. For example,

from (1) $x = 3 - y$

from (2) $x = 4 - 2y$ and $y = \dfrac{4 - x}{2}$.

Substitution in the other equation would have provided one or other of the systems illustrated in *Fig. 7-3*.

Exercise 7-2

(A)

By solving one of the equations of the system for y in terms of x, obtain a system equivalent to the given system in each of the following:

1. $\begin{cases} 3y = x + 2 \\ x + y = 6 \end{cases}$ **2.** $\begin{cases} 2x + y = 3 \\ 3x - y = 2 \end{cases}$ **3.** $\begin{cases} y + 3 = x + 2 \\ 2y + 1 = 2x + 5 \end{cases}$

Use the method of substitution to obtain a system of equations equivalent to the given system for each of the following:

4. $\begin{cases} y = x - 3 \\ x + y = 7 \end{cases}$ **5.** $\begin{cases} x = 2 - y \\ 3x + 2 = y + 5 \end{cases}$ **6.** $\begin{cases} m = 2 + p \\ p + m = 7 \end{cases}$

(B)

Solve each of the following systems, using the method of substitution; verify:

7. $\begin{cases} x + y = 3 \\ 2x + 3y = 8 \end{cases}$ **8.** $\begin{cases} 5x - 2y = 18 \\ 2x - y = 7 \end{cases}$ **9.** $\begin{cases} 5x - y = 8 \\ 3x + 2y = 10 \end{cases}$

10. $\begin{cases} 2x + 3y = 7 \\ x + 2y = 4 \end{cases}$ **11.** $\begin{cases} \frac{1}{6}x + y = 3 \\ x + \frac{1}{2}y = 7 \end{cases}$ **12.** $\begin{cases} 3x - y = 23 \\ \frac{x}{3} + \frac{y}{4} = 4 \end{cases}$

Solve each of the following systems using any method:

13. $\begin{cases} 4x - y = 6 \\ 3x - 2y = 7 \end{cases}$ **14.** $\begin{cases} 4x + 3y = 15 \\ 6x - 5y = -6 \end{cases}$ **15.** $\begin{cases} 3y + 9x = 42 \\ 2y - 4x = -2 \end{cases}$

16. $\begin{cases} x = \frac{3}{2}y \\ 4y - 2x = 4 \end{cases}$ **17.** $\begin{cases} .2x + .5y = 10 \\ 3x + 4y = 108 \end{cases}$ **18.** $\begin{cases} 7(y-5) = x - 2 \\ \frac{4x - 3}{2} = \frac{y + 10}{6} \end{cases}$

Find, using the method of substitution:

19. $\{(x, y)|3y - 2x = 11 \text{ and } 13x - 5y = 1\}$

20. $\{(x, y)|3x = 2y + 1 \text{ and } 3y = -2x + 18\}$

21. $\{(x, y)|\frac{1}{2}y - 3x = 8 \text{ and } 3x + y = -2\}$

22. $\{(x, y)|x + \frac{1}{3}y = y \text{ and } y + \frac{1}{4}x = x + 3\}$

23. (a) Find the solution set defined by the system: $\begin{cases} x + 2y = 5 \\ 2x + y = 1 \end{cases}$

(b) Show that the elements of the solution set for (a) satisfy each of the following equations:

(i) $2(x + 2y - 5) + (2x + y - 1) = 0$

(ii) $3(x + 2y - 5) + 5(2x + y - 1) = 0$

(iii) $x + 2y - 5 - 2(2x + y - 1) = 0$

7·3 Elimination of a variable by addition or subtraction. The following example illustrates a third method of solving a system of equations. The fact that the systems developed in the procedure are equivalent systems is illustrated graphically.

The graph of $A = \{(x, y)|x + y = 3 \text{ and } x - y = 1\}$ is shown in *Fig. 7-4*. The solution set is $\{(2, 1)\}$.

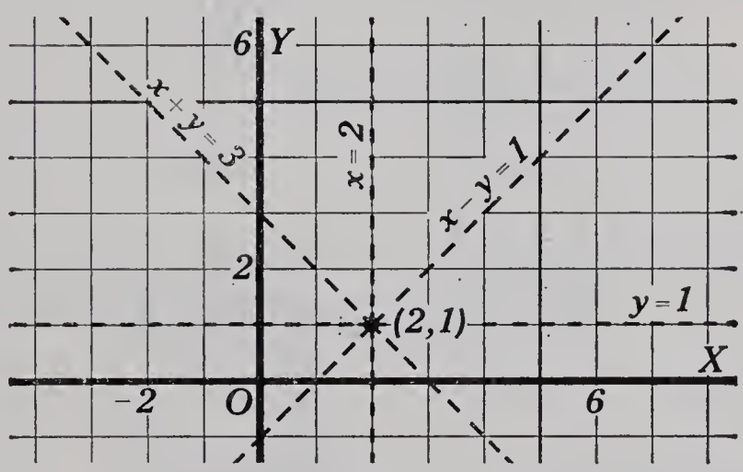


Fig. 7-4

From the graph in *Fig. 7-4* it can be seen that the systems of equations

(i) $\begin{cases} x + y = 3 \\ x - y = 1 \end{cases}$

(ii) $\begin{cases} x = 2 \\ x + y = 3 \end{cases}$ (iii) $\begin{cases} x = 2 \\ x - y = 1 \end{cases}$ (iv) $\begin{cases} y = 1 \\ x + y = 3 \end{cases}$ (v) $\begin{cases} y = 1 \\ x - y = 1 \end{cases}$

(vi) $\begin{cases} x = 2 \\ y = 1 \end{cases}$

have the same solution set $\{(2, 1)\}$.

The equation $x = 2$ may be obtained by eliminating y from the given equations by addition.

$$\begin{cases} x + y = 3 & (1) \\ x - y = 1 & (2) \end{cases}$$

$$\therefore 2x = 4$$

$$\therefore x = 2 \quad (3)$$

The equation $y = 1$ may be obtained by eliminating x from the given equations by subtraction.

$$\begin{cases} x + y = 3 & (1) \\ x - y = 1 & (2) \end{cases}$$

$$\therefore 2y = 2$$

$$\therefore y = 1 \quad (4)$$

If the original system is now replaced by one of the simpler systems, for example:

$$\begin{cases} x = 2 & (3) \\ x + y = 3 & (1) \end{cases}$$

Then by substitution,

$$2 + y = 3$$

$$\therefore y = 1 \quad (4)$$

or

$$\begin{cases} y = 1 & (4) \\ x - y = 1 & (2) \end{cases}$$

Then by substitution,

$$x - 1 = 1$$

$$\therefore x = 2 \quad (3)$$

The system in each case is equivalent to the system

$$\begin{cases} x = 2 \\ y = 1 \end{cases}$$

from which the solution set $\{(2, 1)\}$ is obtained.

The solution is usually arranged as follows:

$$\begin{cases} x + y = 3 & (1) \\ x - y = 1 & (2) \end{cases}$$

(1) + (2) $\quad 2x = 4$

$$\therefore x = 2 \quad (3)$$

Substitute in (1) $\quad 2 + y = 3$

$$\therefore y = 1 \quad (4)$$

The solution set is $\{(2, 1)\}$.

Verification. L.S.(1) $= 2 + 1 = 3$ $\qquad$ L.S.(2) $= 2 - 1 = 1$

R.S.(1) $= 3$ $\qquad$ R.S.(2) $= 1$

Write a solution for the following problem and compare it with that on page 480.

1. Determine the solution set defined by the system of equations; verify:

$$\begin{cases} 3x + 2y = 13 \\ 5x - 2y = 3 \end{cases}$$

Example. Find $\{(x, y) | 5x + 2y = 11 \text{ and } 3x + 4y = 1\}$.

Solution. Since addition or subtraction of equations (1) and (2) will not eliminate a variable, the multiplication property of equality is used

to write equivalent equations for each in which the coefficients of x or y are equal in absolute value.

$$\begin{cases} 5x + 2y = 11 & (1) \\ 3x + 4y = 1 & (2) \end{cases}$$

$2 \times (1)$ $\quad \begin{cases} 10x + 4y = 22 & (3) \\ 3x + 4y = 1 & (2) \end{cases}$

$(3) - (2)$ $\quad 7x = 21$

$\therefore x = 3 \qquad (4)$

Substitute in (2) $\quad 9 + 4y = 1 \qquad (2)$

$\therefore 4y = -8$

$\therefore y = -2 \qquad (5)$

The solution set is $\{(3, -2)\}$.

Write solutions for the following problems and compare them with those on page 480.

2. Solve the following system and verify:

$$\begin{cases} 5x + 2y = 9 \\ 2x + 3y = 8 \end{cases}$$

3. Find $\{(x, y) | \frac{1}{2}x - \frac{1}{3}y = 0 \text{ and } \frac{1}{4}x - \frac{3}{2}y = -4\}$; verify.

Exercise 7-3

(A)

Eliminating a variable by addition or subtraction, find the solution set of each of the following systems of equations:

1. $\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$ **2.** $\begin{cases} 2x + y = 14 \\ 3x - y = 16 \end{cases}$ **3.** $\begin{cases} 2x - 3y = 3 \\ x - 3y = 1 \end{cases}$

4. $\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$ **5.** $\begin{cases} 3x + 2y = 19 \\ 3x - 5y = 5 \end{cases}$ **6.** $\begin{cases} x - 2y = 5 \\ x + 6y = -5 \end{cases}$

(B)

Determine the solution set defined by the following systems of equations; verify:

7. $\begin{cases} 2x + y = 8 \\ x + y = 5 \end{cases}$ **8.** $\begin{cases} 2x + 3y = 13 \\ 2x + y = 7 \end{cases}$ **9.** $\begin{cases} x + y = 2 \\ 3x + 2y = 5 \end{cases}$

10. $\begin{cases} 6x + 3y = -9 \\ 2x + 3y = -1 \end{cases}$ **11.** $\begin{cases} 5x + y = 15 \\ 2x - 3y = 6 \end{cases}$ **12.** $\begin{cases} 2x + 4y = 32 \\ 2x - 5y = 5 \end{cases}$

13. $\begin{cases} x + y = 11 \\ \frac{1}{2}x + \frac{1}{3}y = 5 \end{cases}$

14. $\begin{cases} \frac{x}{2} + \frac{y}{3} = 1 \\ \frac{x}{2} - \frac{2}{3}y = 3 \end{cases}$

15. $\begin{cases} \frac{x}{4} - \frac{5y}{3} = -9 \\ \frac{7x}{4} - 6y = -29 \end{cases}$

Find:

16. $\{(x, y) | 5x - 6y = 31 \text{ and } 6x - 3y = 33\}$

17. $\{(x, y) | 3x + 9y = 42 \text{ and } 2x - 4y = -2\}$

18. $\{(x, y) | 2x + 3y = 1 \text{ and } 5x - 4y = 14\}$

19. $\{(x, y) | 7x + 3y = 27 \text{ and } 2x + 5y = 16\}$

20. $\{(x, y) | -2x + 3y = 6 \text{ and } 6x + 5y = 30\}$

21. $\left\{(x, y) \middle| \frac{x}{5} + \frac{y}{6} = 3 \text{ and } \frac{x}{10} - \frac{y}{3} = -1\right\}$

22. $\left\{(x, y) \middle| \frac{2x}{3} - \frac{3y}{2} = -4 \text{ and } \frac{3x}{2} - \frac{y}{8} = 4\right\}$

23. $\left\{(x, y) \middle| \frac{1}{4}(x + y) = 4\frac{1}{2} \text{ and } \frac{x - y}{3} = 4\right\}$

24. $\left\{(x, y) \middle| \frac{x}{4} - \frac{y}{5} = 2 \text{ and } \frac{x}{8} - \frac{y}{3} = -\frac{5}{2}\right\}$

Solve:

25. $\begin{cases} 9x = 7y \\ x - y = -2 \end{cases}$

26. $\begin{cases} 2x - 3y + 1 = 0 \\ 5x - 2y - 14 = 0 \end{cases}$

27. $y - 3x = 2y - x = 20$

(C)

Find the solution set defined by each of the following systems:

28. $6x + 3 = 2x - 2y = 3y$

29. $\begin{cases} 3(y + 7) - 4(x + 6) = 0 \\ 7(y + 5) = 2(x + 10) \end{cases}$

7·4 Solving systems of equations in more than two variables (supplementary). We learned in Chapter VI that one equation in two variables has an unlimited number of solutions or defines an unlimited number of ordered pairs. Two equations of the first degree in two variables usually define a *single ordered pair*.

To solve systems of two linear equations the method is:

(i) *eliminate variables to obtain equations with fewer variables;*

(ii) *obtain at each step a system of equations equivalent to the previously obtained system.*

This method is a general one and may be applied to solving a system of any number of equations. For example, consider the following problem.

Determine three integers such that:

(i) the sum of the first two is 10;

(ii) the sum of the second two is 8;

(iii) the sum of the first and third is 12 .

Since there are three integers to be determined, we may say that for any three integers represented by x, y, and z, the facts of the problem are represented by the system of equations:

$$\begin{cases} x + y = 10 & (1) \\ y + z = 8 & (2) \\ x + z = 12 & (3) \end{cases}$$

If these facts are sufficient to define the three numbers, then these three equations, when solved, will provide whatever sets of three integers satisfy the problem.

One way to solve the system is as follows:

$$\begin{cases} x + y = 10 & (1) \\ y + z = 8 & (2) \\ x + z = 12 & (3) \end{cases}$$

$$\begin{array}{lll} & x + y = 10 & (1) \\ & y + z = 8 & (2) \\ \hline (1) - (2) & x - z = 2 & (4) \\ & x + z = 12 & (3) \\ \hline (4) + (3) & 2x = 14 & \\ & \therefore\ x = 7 & (5) \\ (4) - (3) & -2z = -10 & \\ & \therefore\ z = 5 & (6) \\ \text{Substitute in (1)} & 7 + y = 10 & \\ & \therefore\ y = 3 & (3) \end{array}$$

It may be verified that the three integers are 3, 5, and 7.

The equivalent systems of equations in the solution are:

$$\begin{array}{ccc} \text{(i)} & \text{(ii)} & \text{(iii)} \\ \begin{cases} x + y = 10 \\ y + z = 8 \\ x + z = 12 \end{cases} & \begin{cases} x - z = 2 \\ x + z = 12 \\ x + y = 10 \end{cases} & \begin{cases} x = 7 \\ z = 5 \\ y = 3 \end{cases} \end{array}$$

We see that this system of three first degree equations in three variables (note that all three variables did not appear in each equation in this example) defines only one set of three numbers.

A second method of solving the system of equations suggests that the particular pattern of elimination does not always need to be followed.

Adding the three equations:

(1) + (2) + (3)	$2x + 2y + 2z = 30$	(Addition property)
	$\therefore\ x + y + z = 15$	(4)
	$x + y \quad\ = 10$	(1)
(4) − (1)	$z = 5$	
	$x + y + z = 15$	(4)
	$y + z = 8$	(2)
(4) − (2)	$x = 7$	(5)
	$x + y + z = 15$	(4)
	$x \quad + z = 12$	(3)
(4) − (3)	$y = 3$	(6)

The three integers are 3, 5, and 7.

One should always study the equations carefully before proceding to solve them, so as to take advantage of any pattern which will simplify the amount of work required.

Write solutions for the following examples and compare them with those on page 481.

1. Solve the following system; verify:

$$\begin{cases} x + 2y + 3z = 26 & (1) \\ 4x - 5y + 6z = 29 & (2) \\ 7x + 8y - 9z = -16 & (3) \end{cases}$$

2. Solve the following system:

$$\begin{cases} 5 + 4z = y + 3 & (1) \\ x + y = 3z & (2) \\ 7z = x + 2 & (3) \end{cases}$$

3. Solve the following system:

$$\begin{cases} a + b = 3 & (1) \\ b + c = 5 & (2) \\ c + d = 7 & (3) \\ a + 3d = 7 & (4) \end{cases}$$

The graphical interpretation of an equation in three variables is a set of points in three dimensional space. Three coordinates are required to

determine a point in three dimensional space with reference to three axes. The equation

$$2x + y + 3z = 6$$

defines the set of points of the plane containing $\triangle ABC$ in *Fig. 7-5*. Such graphs are hard to draw and do not provide a convenient method for determining the solution of three equations in three variables.

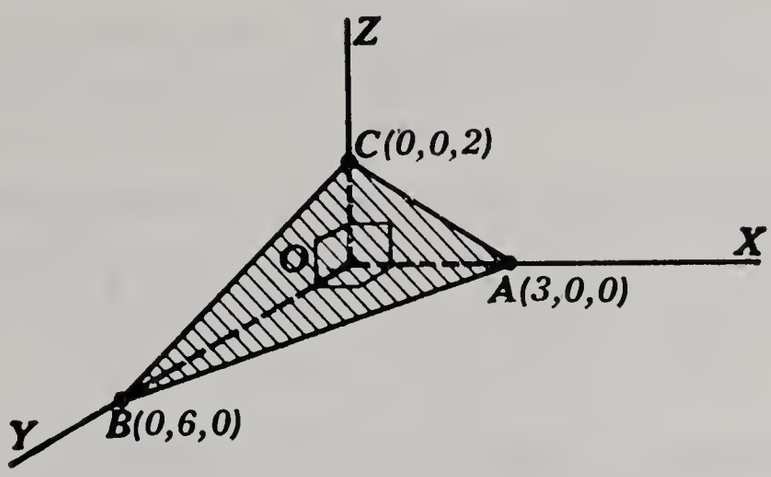


Fig. 7-5

Exercise 7-4

(B)

Solve and verify:

1. $\begin{cases} 2x - y + 3z = 9 \\ 3x + y - 4z = -7 \\ 6x - y + 5z = 19 \end{cases}$

2. $\begin{cases} x + 2y + 3z = 20 \\ x + 3y + 4z = 27 \\ x + 4y + 6z = 38 \end{cases}$

3. $\begin{cases} x + y - z = 0 \\ x + 2y + z = 11 \\ x - y + z = 2 \end{cases}$

4. $\begin{cases} x + y = 14 \\ y + z = 9 \\ z + x = 11 \end{cases}$

Solve:

5. $\begin{cases} x + y + z = 21 \\ x - z = 10 \\ y - z = 8 \end{cases}$

6. $\begin{cases} 3x + 2y + z = 72 \\ -x + 3y + 2z = 48 \\ 3x + 2z = 60 \end{cases}$

7. $\begin{cases} 3x + y - 2z = 1 \\ 3x + 3y + z = 15 \\ 3x + 2y + 2z = 13 \end{cases}$

8. $\begin{cases} 3x - y + z = 17 \\ 5x + 3y - 2z = 10 \\ 7x + 4y - 5z = 3 \end{cases}$

Solve:

9. $\begin{cases} 5x + 2y + 10z = 5 \\ 3x - y + 5z = -6 \\ 2x - 3y - 7z = 5 \end{cases}$

10. $\begin{cases} x - 3y + z = 2 \\ 2x + y - 2z = 3 \\ 4x - 2y - 3z = 4 \end{cases}$

11. $\begin{cases} 7x - 3y = 30 \\ 9y - 5z = 34 \\ x + y - z = 7 \end{cases}$

12. $\begin{cases} 5 + 4z = y + 3 \\ x + y = 3z \\ 7z = x + 2 \end{cases}$

(C)

Solve:

13. $\begin{cases} a + 2b - c + d = 4 \\ 2a - b - c - d = 7 \\ 3a + 4b + c + 2d = 1 \\ 4a - 2b + 3c - 3d = -2 \end{cases}$

14. $\begin{cases} m + n + p + q = 2 \\ \frac{1}{2}m + p = -\frac{5}{2} \\ m + \frac{1}{2}n + 2p + \frac{1}{2}q = 5 \\ 2m - n - p - 2q = -3 \end{cases}$

7·5 Solving problems by means of a system of equations. In Chapter III we studied the analysis and solution of problems whose facts we were able to express in terms of one letter symbol. Problems involving two or more quantities to be determined may be expressed in terms of two or more letter symbols. To solve the problem, the facts of the problem may be expressed by a system of equations which consists of as many equations as there are representative letter symbols. If the number of letter symbols involved is two, the analysis of the problem requires the development of two equations. The following examples illustrate this.

Example 1. Find two numbers such that their sum is 18, and 4 times the first number exceeds 6 times the second by 2.

Solution. Represent the first number by x
and the second number by y.
Their sum is $x + y$.
But this is given equal to 18.
$\therefore\ x + y = 18 \qquad (1)$
4 times the first exceeds 6 times the second by $4x - 6y$.
But this is given equal to 2.

$$\therefore\ 4x - 6y = 2 \qquad (2)$$

$$\begin{cases} x + y = 18 & (1) \\ 4x - 6y = 2 & (2) \end{cases}$$

$4 \times (1)$ $\quad 4x + 4y = 72 \qquad (3)$

$(3) - (2)$ $\quad 10y = 70$

$$\therefore\ y = 7 \qquad (4)$$

Substitute in (1) $x + 7 = 18$

$\therefore\ x = 11$

The numbers are 11 and 7.

Verification.

	11	$4 \times 11 = 44$
	7	$6 \times 7 = 42$
Sum	18	Difference 2

Example 2. A merchant wishes to mix tea worth 75¢ per pound with tea worth 95¢ per pound to make 100 pounds of a mixture worth 86¢ a pound. How many pounds of each should he use?

Solution. Represent the number of pounds of 75¢ tea by x and the number of pounds of 95¢ tea by y.

The value of x pounds at 75¢ per pound is $75x$¢.

The value of y pounds at 95¢ per pound is $95y$¢.

Total value of this mixture is $(75x + 95y)$¢.

But this is given equal to (100×86)¢ or 8,600¢.

$\therefore\ 75x + 95y = 8{,}600$

or $15x + 19y = 1{,}720$ (1)

Total amount of tea in the mixture is $(x + y)$ pounds.

But this is given equal to 100 pounds.

$\therefore\ x + y = 100$ (2)

$$\begin{cases} 15x + 19y = 1{,}720 & (1) \\ x + y = 100 & (2) \end{cases}$$

From (2) $x = 100 - y$ (3)

Substitute in (1) $15(100 - y) + 19y = 1{,}720$

$\therefore\ 1500 - 15y + 19y = 1{,}720$

$\therefore\ 4y = 220$

$\therefore\ y = 55$ (4)

Substitute in (3) $x = 100 - 55$

$\therefore\ x = 45$ (5)

The required mixture is 45 pounds of 75¢ tea and 55 pounds of 95¢ tea.

Verification. Number of pounds in mixture $= 45 + 55 = 100$

45 pounds at 75¢ per pound cost \$33.75

55 pounds at 95¢ per pound cost \$52.25

Total value \$86.00

Example 3. An aircraft travelled 1,500 miles with the wind in 3 hours, but took $3\frac{3}{4}$ hours to return against the wind. Find, in miles per hour, the airspeed of the plane in still air and the wind velocity.

Solution. The relationship between distance d, speed v, and time t, is given by the equation

$$d = vt .$$

Represent the airspeed by x m.p.h. and the windspeed by y m.p.h.

TRIP	SPEED IN m.p.h.	TIME IN HOURS	DISTANCE IN MILES
Going	$x + y$	3	$3(x + y)$
Returning	$x - y$	$\frac{15}{4}$	$\frac{15}{4}(x - y)$

But it is given that the distance each way is 1,500 miles.

$$\therefore \begin{cases} 3(x + y) = 1{,}500 & (1) \\ \frac{15}{4}(x - y) = 1{,}500 & (2) \end{cases}$$

(1) $\div 3$ $\qquad x + y = 500 \qquad (3)$

(2) $\times \frac{4}{15}$ $\qquad x - y = 400 \qquad (4)$

(3) + (4) $\qquad 2x = 900$

$\therefore x = 450 \qquad (5)$

(3) − (4) $\qquad 2y = 100$

$\therefore y = 50 \qquad (6)$

Airspeed is 450 m.p.h. and windspeed is 50 m.p.h.

Verification. Rate going is 500 m.p.h.

$$\text{Time taken} = \frac{1{,}500}{500} \text{ hrs.} = 3 \text{ hrs.}$$

Rate returning is 400 m.p.h.

$$\text{Time taken} = \frac{1{,}500}{400} \text{ hrs.} = 3\tfrac{3}{4} \text{ hrs.}$$

Example 4 which follows deals with the digits of a number. Before attempting the problem we should recall that in our decimal place-value system the three-digit numeral 745 means

$$7 \times 10^2 + 4 \times 10 + 5$$

not $7 \times 4 \times 5$.

In a problem requiring the determination of the digits of a two-digit numeral, if the tens digit is represented by x and the units digit by y, the numeral is represented by

$$x \times 10 + y \qquad \text{or} \qquad 10x + y$$

but not by xy which, in algebra, means the product of x and y .

The numeral 45 represents $4 \times 10 + 5$. If the digits are reversed, the numeral becomes $5 \times 10 + 4$ or 54. If the digits are reversed in the numeral $10x + y$, it becomes $10y + x$.

Example 4. A number has two digits. If 45 is added to the number, the digits are reversed. The sum of the number and the number formed by reversing the digits is 99. Find the number.

Solution. Represent the tens digit by x
and the units digit by y.
Then the number is represented by $10x + y$.
The number plus 45 is $10x + y + 45$.
But this is given equal to $10y + x$.

$$\therefore \quad 10x + y + 45 = 10y + x$$
$$\text{or} \qquad 9x - 9y = -45$$
$$\text{or} \qquad x - y = -5 \qquad (1)$$

The sum of the number and the number reversed is

$$10x + y + 10y + x = 11x + 11y.$$

But this is given equal to 99.

$$\therefore \quad 11x + 11y = 99$$
$$\text{or} \qquad x + y = 9 \qquad (2)$$

$$\begin{cases} x - y = -5 & (1) \\ x + y = 9 & (2) \end{cases}$$

(1) + (2) $\qquad 2x = 4$

$$\therefore \quad x = 2 \qquad (3)$$

(2) − (1) $\qquad 2y = 14$

$$\therefore \quad y = 7 \qquad (4)$$

The number is 27.

Verification. Number plus 45 $= 27 + 45 = 72$
Number reversed $= 72$
Number plus number reversed $= 27 + 72 = 99$

Exercise 7-5

(B)

1. Find two numbers whose difference is 21, such that one-fifth of their sum is 11.
2. Find two numbers such that one-half their sum is 11 and one-third their difference is 2.
3. Find two numbers such that 3 times the first added to 4 times the second is 44, and 4 times the first less 3 times the second is 17.
4. Add the smaller of two numbers to one-half the greater and the sum is 11. Subtract the smaller from one-quarter of the greater and the difference is − 1. Find the numbers.

5. Find two numbers such that 2 times the larger added to 3 times the smaller is 88, and one-half the larger diminished by one-third the smaller is 9.
6. If in an archery contest a bull's eye counts 5 and an inner counts 4, find how many of each of these a marksman makes in scoring 52 with 12 shots.
7. If 4 boys and 3 men earn a total of \$51 for 1 day's work, and 7 boys and 2 men earn \$60 for 1 day's work, what are the daily wages of a boy and of a man?
8. If 5 lb. of coffee and 10 lb. of tea together cost \$14.00, and 4 lb. of coffee and 6 lb. of tea together cost \$9.50, find the price of each per lb.
9. Three lb. of coffee and 5 lb. of tea cost \$8.10. If the coffee were reduced 20% in price and the tea increased 10%, the cost would be \$7.83. Find the original price of each per lb.
10. A grocer bought oranges, some at 54¢ a dozen and some at 60¢ a dozen. He paid \$20.10 for them altogether. He sold them all at 75¢ a dozen and cleared \$6.15. How many oranges did he buy at each price?
11. The perimeter of a rectangular field is 54 rods. If the length is decreased by 5 rods and the width increased by 2 rods, the area is decreased by 40 square rods. Find the original dimensions of the field.
12. A right-angled triangle has its hypotenuse 8 inches longer than its shortest side. If the sum of the two shorter sides is 17 inches and the perimeter of the triangle is 30 inches, find the lengths of the sides of the triangle.
13. A ton of anthracite coal occupies 34 cubic feet, and a ton of bituminous coal, 42 cubic feet. If a man has 3,200 cubic feet of space available for storing coal and he needs 3 times as many tons of bituminous as of anthracite, find the number of tons he should purchase of each.
14. A man invests \$6,000, part at 3% per annum and part at 5% per annum. If his total income from these investments is \$220, find the amount invested at each rate.
15. A man who has \$30,000 invested, part at $4\frac{3}{4}\%$ per annum and the remainder at $5\frac{1}{4}\%$ per annum, receives \$65 per annum more income from the former than from the latter. How much is invested at $5\frac{1}{4}\%$?
16. A man has \$12,000 to invest. He invests part in $3\frac{1}{4}\%$ bonds and part in $4\frac{1}{2}\%$ bonds. If his total annual income is \$490, how much does he invest in each bond issue?
17. A commercial airliner travelled 2,200 miles with the wind in $7\frac{1}{3}$ hours and took 10 hours to return the same distance against the wind. Find its airspeed in still air and the wind velocity.

18. Two persons, A and B, who are 30 miles apart meet after 5 hours if they walk towards each other. However, if they walk in the same direction from their starting points, it takes A 15 hours to overtake B. Find their rates of walking.

19. A certain two-digit number is equal to four times the sum of its digits. If 18 be added to the number, the digits are reversed. Find the number.

20. The difference between the two digits of a number is 3. The sum of the number and the number formed by reversing the digits is 121. Find the number.

21. A certain number between 10 and 100 is 7 times the sum of its digits. If 27 is subtracted from the number, the digits are reversed. Find the number.

22. The sum of the digits of a two-digit number is 10. The difference between the number and the number formed by reversing the digits is 54. Find the number.

23. A two-digit number has its tens digit 5 more than its units digit. If 8 times the sum of the digits is equal to the number, find the number.

(C)

24. In an informal ballot a resolution was adopted by a majority of 10 votes. In the formal ballot, one-fourth of those who had voted for the resolution now voted against it, and it was lost by 6 votes. Find the number of voters.

25. Find a fraction such that, if 4 is added to the numerator it becomes $\frac{7}{8}$, but if 4 is added to the denominator, it becomes $\frac{3}{4}$.

26. When 23 is added to a certain number expressed by two digits, the result is 5 greater than the number expressed by the digits in reverse order. If the original number is increased by $\frac{6}{7}$ of itself, the result is the same as if the tens digit had been replaced by 6. Find the original number.

7·6 Classification of systems of linear equations (supplementary). Systems of two linear equations are classified as follows.

a. *Consistent systems.* A system is *consistent* if the equations have a unique common solution.

$$\{(x, y) | 2x - 3y = 8 \text{ and } 3x + y = 1\} = \{(1, -2)\}$$

is defined by the system of equations

$$\begin{cases} 2x - 3y = 8 & (1) \\ 3x + y = 1 & (2) \end{cases}$$

which have the ordered pair $(1, -2)$ as a unique common solution. The

graphical solution, *Fig. 7-6*, indicates the point common to the two lines defined by the individual equations. This system of equations is said to be *consistent*. It is also referred to as *independent*.

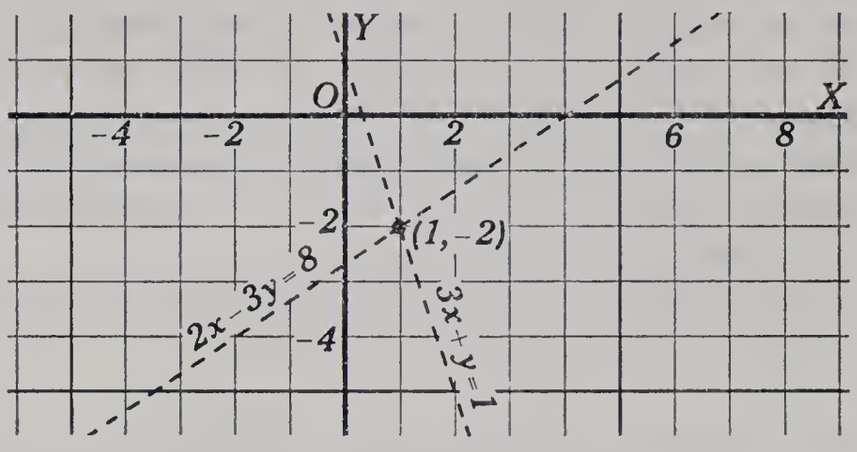


Fig. 7-6

b. *Dependent systems.* A system is *dependent* if each equation has the same solution set.

$$\{(x, y) | 2x - 3y = 8 \text{ and } 6x - 9y = 24\}$$

is defined by the system of equations:

$$\begin{cases} 2x - 3y = 8 & (1) \\ 6x - 9y = 24 & (2) \end{cases}$$

It is seen that equation (2) is equivalent to equation (1): that is, (2) is obtained from (1) by multiplying by 3 . Thus, any ordered pair which satisfies the equation (1) will satisfy equation (2) .

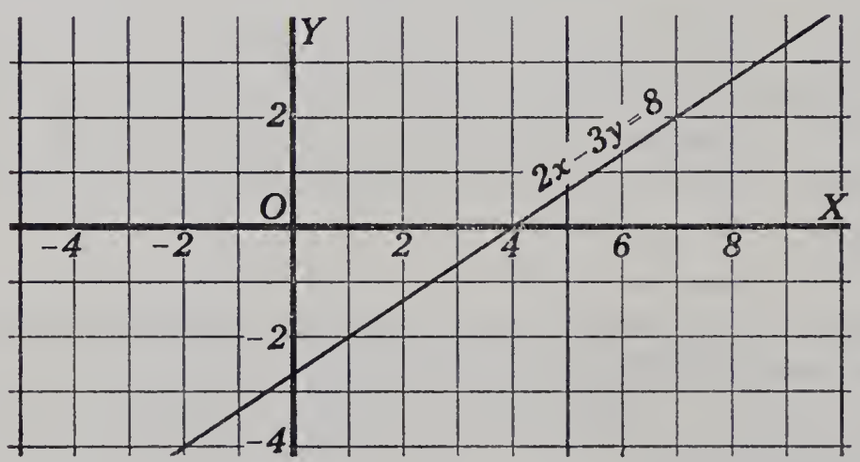


Fig. 7-7

The graph of the solution set of each equation is the same straight line. This system of equations has an unlimited number of solutions. Each equation is said to be *dependent* on the other, and the system is referred to as a *dependent system.*

c. *Inconsistent systems.* A system is *inconsistent* if the two equations have no solution in common.

$$\{(x, y) | 2x - 3y = 8 \text{ and } 4x - 6y = 12\} = \emptyset$$

is defined by the system of equations:

$$\begin{cases} 2x - 3y = 8 & (1) \\ 4x - 6y = 12 & (2) \end{cases}$$

$$2 \times (1) \quad 4x - 6y = 16 \quad (3)$$

$$(2) - (3) \quad 0x + 0y = -4 \quad (4)$$

There are no ordered pairs (x, y) which satisfy this equation. Thus, the solution set is $\emptyset$. The graph of the solution set of each equation indicates that the lines are parallel and have no point in common. This system is said to be *inconsistent.*

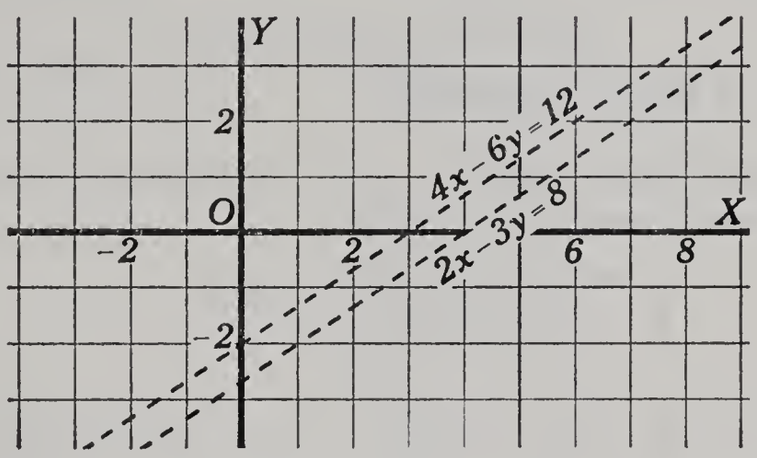


Fig. 7-8

Exercise 7-6

(B)

Draw the graph of each of the following systems of equations and then classify each as consistent, inconsistent, or dependent:

1. $\begin{cases} x + y = 3 \\ 2x + y = 4 \end{cases}$ **2.** $\begin{cases} 2x + 3y = 6 \\ 4x + 6y = 7 \end{cases}$ **3.** $\begin{cases} x - y = 3 \\ 3x - 3y = 9 \end{cases}$

7·7 Solution of systems of linear equations with literal coefficients (supplementary).

Example 1. If $x, y \in R$, and b, c represent real constants, find the solution set of the following system of equations:

$$\begin{cases} bx + y = b^2 + c \\ x + cy = b + c^2 \end{cases}$$

While any one of the three methods of eliminating a variable may be used, it is often simplest to use the method of elimination by addition or subtraction.

Solution.

$$\begin{cases} bx + y = b^2 + c & (1) \\ x + cy = b + c^2 & (2) \end{cases}$$

$(2) \times b$ $\quad bx + bcy = b^2 + bc^2$ (3)

$\quad bx + y = b^2 + c$ (1)

$(1) - (3)$ $\quad y - bcy = c - bc^2$

$\therefore\ y(1 - bc) = c(1 - bc)$

$\therefore\ y = c\,,\ 1 - bc \neq 0$ (4)

Substitute in (2) $\quad x + c^2 = b + c^2$

$\therefore\ x = b$ (5)

The solution set is $\{(b, c)\}$.

If $1 - bc = 0$, then y is undefined.

Example 2. If $x,\ y \in R$, and $a,\ b,\ c,\ d$ represent real number constants, solve the following system for x and y:

$$\begin{cases} ax + by = cd \\ cx - dy = ab \end{cases}$$

Solution.

$$\begin{cases} ax + by = cd & (1) \\ cx - dy = ab & (2) \end{cases}$$

$c \times (1)$ $\quad acx + bcy = c^2d$ (3)

$a \times (2)$ $\quad acx - ady = a^2b$ (4)

$(3) - (4)$ $\quad bcy + ady = c^2d - a^2b$

$\therefore\ y(bc + ad) = c^2d - a^2b$

$$\therefore\ y = \frac{c^2d - a^2b}{bc + ad} \qquad (5)$$

Since the substitution of $\dfrac{c^2d - a^2b}{bc + ad}$ for y in either equation (1) or (2) is cumbersome, x may best be determined by eliminating y from the two given equations.

$d \times (1)$ $\quad adx + bdy = cd^2$ (6)

$b \times (2)$ $\quad bcx - bdy = ab^2$ (7)

$(5) + (6)$ $\quad adx + bcx = cd^2 + ab^2$

$\therefore\ x(ad + bc) = cd^2 + ab^2$

$$\therefore\ x = \frac{cd^2 + ab^2}{ad + bc} \qquad (8)$$

The solution is $\left(\dfrac{cd^2 + ab^2}{ad + bc}\,,\ \dfrac{c^2d - a^2b}{bc + ad}\right)$.

Exercise 7-7

(B)

Solve for x and y:

1. $\begin{cases} rx - sy = r^2 - s^2 \\ x + y = r + s \end{cases}$

2. $\begin{cases} 2x - ay = c \\ x + by = c \end{cases}$

3. $\begin{cases} mx + ny = 4mn \\ mx - ny = 0 \end{cases}$

4. $\begin{cases} ax - by = a^2 + b^2 \\ x + y = 2a \end{cases}$

5. $\begin{cases} hx = ky \\ 3hx - 2ky = hk \end{cases}$

6. $\begin{cases} ax + by = c \\ -bx + ay = c \end{cases}$

7. $\begin{cases} x + y = 1 \\ a_1x + b_1y = c_1 \end{cases}$

8. $\begin{cases} ax + by = 2 \\ a^2x - b^2y = a - b \end{cases}$

9. $\begin{cases} ax + by = 2ab \\ bx - ay = b^2 - a^2 \end{cases}$

10. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 2 \\ ax - by = a^2 - b^2 \end{cases}$

Practice Exercise 7-8

(B)

Solve each of the following systems using the method of comparison; verify:

1. $\begin{cases} x + 2y = 4 \\ 2x + 3y = 7 \end{cases}$

2. $\begin{cases} 4x - y = 5 \\ 3x + 2y = 12 \end{cases}$

3. $\begin{cases} 2x_1 + x_2 = 23 \\ 4x_1 - x_2 = 19 \end{cases}$

4. $\begin{cases} 2x + 3y = 12 \\ 3x + 2y = 13 \end{cases}$

5. $\begin{cases} x - y = 2 \\ \dfrac{x}{5} + \dfrac{y}{3} = 2 \end{cases}$

6. $\begin{cases} \dfrac{x}{3} = \dfrac{y}{2} \\ x - \dfrac{5}{6}y = 4 \end{cases}$

Find, using the method of comparison:

7. $\{(x, y) | x - 3y = -5 \text{ and } 2x + y = -3\}$

8. $\{(x, y) | 2x = 9y \text{ and } 7x + 27 = 18y\}$

9. $\{(x, y) | 12x + 6y = 13 \text{ and } 4x + 3y = 5\}$

10. $\{(x_1, x_2) | 2x_1 + 1 = 5x_2 \text{ and } 4x_2 - x_1 - 4 = 5x_2 - 2x_1\}$

11. $\left\{(x, y) \middle| \dfrac{x}{3} - \dfrac{y}{6} = 1 \text{ and } \dfrac{x}{5} - \dfrac{3}{10}y = 1\right\}$

12. $\left\{(x, y) | 3y - 7x = x + 1 \text{ and } \dfrac{3x - 2}{y} = 1\right\}$

Practice Exercise 7-9

(B)

Solve each of the following systems using the method of substitution; verify:

1. $\begin{cases} x + 3y = 38 \\ 3x - y = 24 \end{cases}$
2. $\begin{cases} 2x - 3y = -5 \\ 4x - y = 5 \end{cases}$
3. $\begin{cases} 2x_1 - x_2 + 5 = 0 \\ 3x_1 + 2x_2 - 3 = 0 \end{cases}$
4. $\begin{cases} 13 + 2a_2 = 9a_1 \\ 3a_2 = 7a_1 \end{cases}$
5. $\begin{cases} \frac{2}{3}x + y = 8 \\ x + \frac{1}{4}y = 7 \end{cases}$
6. $\begin{cases} \frac{3}{4}x - \frac{1}{3}y = 7 \\ \frac{2}{3}x + \frac{2}{3}y = 12 \end{cases}$

Find, using the method of substitution:

7. $\{(x, y) | x = 3y - 2 \text{ and } 9y = 4x - 7\}$
8. $\{(y_1, y_2) | 3y_2 - 4y_1 = 10 \text{ and } 2y_2 + 6y_1 = 11\}$
9. $\{(x, y) | 3x + 2y = 118 \text{ and } x + 5y = 191\}$
10. $\{(x, y) | 7x + 6y = 71 \text{ and } 5x - 8y = -23\}$
11. $\left\{(x, y) | x = 3y \text{ and } \frac{x}{3} + 2y = 9\right\}$
12. $\left\{(x, y) | \frac{3}{5}x + \frac{y}{3} = 0 \text{ and } x - \frac{2}{3}y = 11\right\}$

Practice Exercise 7-10

(B)

Solve each of the following systems using the method of elimination by addition or subtraction; verify:

1. $\begin{cases} 3x + 2y = 13 \\ 3x - 2y = 5 \end{cases}$
2. $\begin{cases} 7x + 3y = 10 \\ 35x - 6y = 1 \end{cases}$
3. $\begin{cases} 2b_1 - 3b_2 + 5 = 0 \\ 4b_1 + 3b_2 + 1 = 0 \end{cases}$
4. $\begin{cases} 7x - 5y = 45 \\ 2x + 3y = 4 \end{cases}$
5. $\begin{cases} \frac{5}{6}x - y = -4 \\ x - \frac{5}{3}y = -9 \end{cases}$
6. $\begin{cases} \frac{x}{4} - \frac{y}{5} = 2 \\ \frac{x}{2} + \frac{y}{3} = 15 \end{cases}$
7. $2x + 5y = y - 3x = 34$
8. $3y - 8x = 3x - y - 1 = 0$

Find, using elimination either by addition or subtraction:

9. $\{(x, y) | 11y - 6x = 36 \text{ and } 7y + 24x = 9\}$
10. $\{(x, y) | 5y - 3x = 85 \text{ and } 5x + 12y = 21\}$
11. $\{(x, y) | 7x + 5y = 1 \text{ and } 5x + 7y = 11\}$

12. $\{(x, y)|9x + 8y = 7 \text{ and } 21x + 10y = 12\}$

13. $\left\{(x, y)\left|\frac{x}{4} - \frac{y}{5} = 2 \text{ and } \frac{x}{2} + \frac{y}{3} = 15\right.\right\}$

14. $\left\{(x, y)\left|\frac{x}{3} + \frac{y}{8} = 26 \text{ and } 2x - 3y = 36\right.\right\}$

Solve each of the following systems using any method:

15. $\begin{cases} \frac{3}{5}x + \frac{y}{3} = 0 \\ x - \frac{2}{3}y = 11 \end{cases}$

16. $\begin{cases} \frac{x}{13} - \frac{y}{7} = 0 \\ 6x - 10y = 8 \end{cases}$

17. $\frac{x}{5} - \frac{y}{3} = \frac{x}{10} + \frac{y}{6} = 4$

18. $\begin{cases} 2x - \frac{2y + 7}{3} = 7 \\ \frac{2y + 1}{3} - \frac{10 - x}{2} = y - 5 \end{cases}$

19. $\begin{cases} 5(x + y) - 7(x - y) = 26 \\ \frac{3x + 7y}{4} - \frac{6x - y}{3} = 0 \end{cases}$

Practice Exercise 7-11

(B)

Solve and verify:

1. $\begin{cases} y + z = 7 \\ z + x = 9 \\ x + y = 12 \end{cases}$

2. $\begin{cases} x + y + z = 6 \\ 2x + 3y + z = 13 \\ 4x + 9y + z = 31 \end{cases}$

3. $\begin{cases} 2x - y + z = 1 \\ 5x + y + 3z = 2 \\ 2x - 3y + 4z = -2 \end{cases}$

4. $\begin{cases} x - 3y + 2z = -1 \\ 2x - 7y + 4z = -4 \\ 3x - 5y - z = -9 \end{cases}$

Solve:

5. $\begin{cases} 2x + 4y + 3z = 11 \\ 2x - 2y - 3z = -2 \\ x - 3y - 3z = -1 \end{cases}$

6. $\begin{cases} 2x_1 + 3x_2 + 5x_3 = 21 \\ -3x_1 + 2x_2 + 4x_3 = 11 \\ 3x_1 + x_2 + 7x_3 = 20 \end{cases}$

7. $\begin{cases} 5x - 9y + z = -22 \\ x - 2y + z = 0 \\ 3x - 7y + 4z = 1 \end{cases}$

8. $\begin{cases} 4x - 3y = -1 \\ 7x - 3y = 5 \\ 8z - 9y = 5 \end{cases}$

9. $\begin{cases} x + 2y + z = -1 \\ 3x + y + 6z = 13 \\ x + y + z = 1 \end{cases}$

10. $\begin{cases} 2x - 3y + 4z - 11 = 0 \\ 3x + 2y - z - 3 = 0 \\ 4x - y - 3z - 6 = 0 \end{cases}$

Practice Exercise 7-12

(B)

1. Find two numbers such that 7 times the first is greater than 3 times the second by 21, and 3 times the first together with twice the second is 55.

2. If 5 lb. of coffee and 3 lb. of sugar cost \$6.30, and 4 lb. of coffee and 6 lb. of sugar cost \$5.40, find the cost of each per lb.

3. Find two numbers such that $\frac{1}{5}$ of the greater is equal to $\frac{1}{2}$ of the smaller, while $\frac{2}{3}$ of the greater exceeds $\frac{3}{4}$ of the smaller by 22.

4. A man bought two pieces of land paying a total of \$2,500. He sold one property at a loss of 8% and the other at a gain of 10%, receiving in all \$2,480. Find the cost of each piece of land.

5. A grocer buys 50 lb. of tea and 120 lb. of coffee for \$126. By selling the tea at a profit of $33\frac{1}{3}\%$ and the coffee at a profit of 25% he gains \$34. Find the grocer's cost price for each per lb.

6. A grocer bought tea at 70¢ a lb. and coffee at 80¢ a lb. at a total cost of \$166. He sold the tea at 90¢ per lb. and the coffee at \$1.10 per lb. and gained \$56. How many pounds of each did he buy?

7. A bill of \$19.50 was paid in half-dollars and quarters. Four times the number of quarters exceeded twice the number of half-dollars by 12. How many were there of each?

8. If 6 men and 8 women earn \$384 in 4 days, and 2 men and 5 women earn \$138 in 3 days, find the daily wages of a man and of a woman.

9. Find two numbers such that if the first is increased by 9 it will be twice the second, and if the second is increased by 18 it will be twice the first.

10. When the sum of two numbers is divided by 3 the quotient is 26, and when the difference is multiplied by 4 the product is 24. Find the numbers.

11. Two persons, 18 miles apart, setting out at the same time are together in $4\frac{1}{2}$ hours if they walk in the same direction, but in 2 hours if they walk towards each other. Find their rates of walking.

12. A two-digit number is six times the sum of its digits. If 9 is subtracted from the number, the result obtained is the number formed by reversing the digits. Find the original number.

INTRODUCTION TO GEOMETRY

I·1 **The mathematical method.** The popular idea of mathematics as a precise but static body of knowledge to which modern man may add little ended abruptly with the almost unbelievable achievements of the atomic age. The creation of new mathematics has been a necessary contribution made by mathematicians to the achievements of our era.

Creating new mathematics means first discovering new mathematical relationships and then proving them. To discover new mathematics, the mathematician uses his knowledge, his curiosity, his imagination, and his ingenuity. He experiments, he guesses, he uses analogies, he builds theoretical and physical models. When he has formulated a new relationship which satisfies his intuition or which seems reasonable in the real world, he calls the relationship a *conjecture.* He tries to prove the conjecture by showing that it follows logically from accepted mathematical statements and ideas. If he succeeds in proving the conjecture and then succeeds in having his proof accepted generally, the new relationship becomes a *theorem* which is added to the body of mathematical knowledge.

Mathematical knowledge grows in a logical, orderly way. The mathematical method involves *discovery, proof,* and the *organization* of knowledge into a *logical system.* Of course, mathematicians must agree on the meaning of the words "follow logically" and on what constitutes a "proof".

In the study of geometry, a fundamental aim is to learn the mathematical method: that is, to learn how to discover relationships, to make conjectures, to reason logically, and to organize knowledge into a logical system.

I·2 **Why study geometry?** Logical reasoning plays so vital a part in our thinking that it is essential for us to understand the nature of logical reasoning. It is equally important that we are able to apply this understanding to everyday, critical thinking. Our study of geometry will help us improve our thinking habits generally.

Knowledge of geometry and the mathematical method are of great importance in the study of physics, chemistry, engineering, pure mathematics, astronomy, statistics, architecture, many of the biological sciences, and even in some branches of economics and psychology. In fact, geometry is essential to almost all the physical and social sciences.

Geometric design is found in nature, art, and industry. The knowledge of geometry enhances our insight into the worlds of nature, science, and industry.

In the present age of science, it is necessary for every educated person to have some understanding of how scientists think and of how science is developing. The study of the mathematical method as it applies to geometry is a step towards this understanding.

I·3 The story of geometry.

a. *Origin.* The word "geometry" is derived from the Greek *ge*, meaning earth, and *metria*, meaning measure. Hence, the original meaning of the word was "land measurement".

It is believed that skill in land measurement was first developed in Egypt about 5,000 years ago when the Nile River flooded periodically, obliterating existing landmarks. The land-surveyors who had to re-measure the land were called *Harpedonaptae*, or rope stretchers, because they used a rope with twelve equally spaced knots, stretched as shown in *Fig. I-1*, for making a right angle.

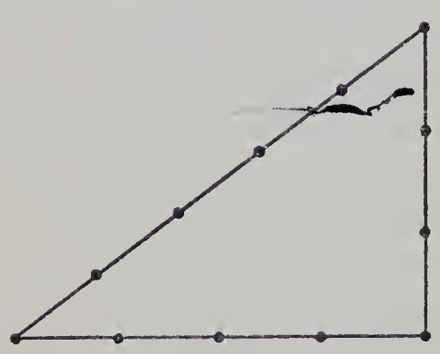


Fig. I-1

Archaeologists have discovered that ancient buildings and monuments—the pyramids, for example—reveal the remarkable knowledge of practical geometry possessed by their builders. Without the aid of a magnetic compass, the ancient builders laid foundations due north and south; corners were square, and errors of measurement of lengths and angles were extremely small.

b. *Great geometers of antiquity.* Thales, who was a statesman, an engineer, as well as a man of commerce, lived in Asia Minor during the sixth century B.C. He is credited with the introduction of theoretical geometry and with making astronomy a more exact science. One of his chief interests was the measurement of inaccessable heights and distances.

Pythagoras (582-507 B.C.) was a pupil of Thales. He developed still further the ideas of theoretical geometry based on axioms, postulates, definitions, and proofs. He is best known for the theorem concerning right triangles which bears his name. He influenced Greek thought profoundly, and his influence is still felt in the fields of mathematics, science, music, religion, and philosophy.

Plato (427-347 B.C.) was one of the greatest philosophers of antiquity. After studying under Socrates for eight years, he became associated with a sect of the Pythagoreans in Athens and formed his own school of philosophy. The inscription over the gateway to the olive grove where his school was held, read, "Let no one destitute of geometry enter my doors".

c. *The Alexandrian era.* In the fourth century B.C., Alexander the Great founded the city of Alexandria on the Nile delta. When he died in 323 B.C., his successor, Ptolmey (the statesman), made Alexandria a centre of literary and scientific study. As a result, the greatest of the ancient world's universities developed there. It had a library of 600,000 scrolls before it had been in existence forty years. The university flourished for nearly 1,000 years, until it was destroyed in 642 A.D. during an Arab invasion of North Africa. During its long history, the university had many famous professors of geometry who helped lay the foundations on which modern geometries are built.

Of these men, Euclid (*ca.* 365 B.C.) is, perhaps, the best known because his name was, for centuries, almost synonymous with the word "geometry". In his *Elements*, a treatise on geometry consisting of thirteen books, he organized the existing knowledge of geometry into a logical system which set the pattern of the study of geometry for the next two thousand years. Of Euclid himself little is known, not even his date of birth, his date of death, or his nationality.

Archimedes (287-212 B.C.) came to Alexandria from Sicily. He was a practical man who invented irrigation systems and war machines. His work on levers and buoyancy is well known in elementary science. He claimed that if he had another world to stand on and a place to put a fulcrum, he could move the earth with a lever. He is remembered for his studies of circles, spheres, cylinders, parabolas, and spirals.

Appolonius (*ca.* 225 B.C.) was known by his contemporaries as the "Great Geometer". He wrote eight books on geometry, concentrating on the parabola, the ellipse, and the hyperbola. He wrote for pure intellectual interest, and it was not until eighteen hundred years later that Kepler saw the amazing practical applications of his knowledge in the laws relating to the orbits of planets as they revolve around the sun.

Important contributions to the development of geometry were made, as well, by Eratosthenes (276-194 B.C.) who was chief librarian at the Uni-

versity of Alexandria; Ptolmey, the astronomer (85-165 A.D.); Heron (*ca.* 150 A.D.); Menelaus (1st century A.D.); and Diophantus (250-275 A.D.).

d. *Medieval and modern developments.* With the rise of the Roman Empire, unfortunately interest in mathematical discovery waned. Rome's contributions were more in the realms of government, warfare, and architecture.

After Rome fell to the barbarians from the north in the fifth century A.D., learning was at a very low ebb for almost one thousand years. During this time, the Arabs are credited with most of the mathematical discoveries. With the coming of the Renaissance in the fourteenth century, interest in mathematics was revived in Europe.

In 1637 René Descartes, a French mathematician, showed that the methods of algebra could be applied to geometry. Points were located in relation to a pair of axes, usually at right angles. Number pairs were associated with the points, and equations with lines and curves. The work of Descartes led to the invention of calculus by Newton and Leibnitz in the seventeenth century.

In the nineteenth century, Gauss, Lobachevski, Bolyai, Klein, and others, showed that geometries other than Euclid's were possible. They showed that, by beginning with different sets of postulates, completely new geometries (non-Euclidian geometries) could be developed logically. In 1899 the German mathematician, Hilbert, published the first really complete treatment of these geometries. Klein, in 1872, showed that many of Euclid's own propositions could be applied, not only to ordinary space, but also to space-time. This idea was eventually used by Einstein in the Theory of Relativity.

In recent times, geometry has had its applications extended to kinematics (study of motion), crystallography (study of properties of crystals, grouping, etc.), statistics, botany, topology, and many other fields.

From this brief account of the development of geometric knowledge, it is apparent that we have been left a valuable heritage. It is because of the labours of devoted mathematicians through the ages that we have access to that branch of human knowledge known as *geometry*.

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Review C

GENERAL PRINCIPLES

C·1 **Basic ideas of geometry.** The basic element of geometry is a *point.* A point is an idea associated with the notion of *position* in *space.* It is not a physical thing, like the point of a pin or the dot made by a pencil on a page. Each of these physical representations of a point has dimensions: length, width, and depth. A point has no dimensions; it exists only in our minds. *Space* is thought of as the set of all points or positions. A *plane* is a special subset of the points in space. It is thought of as having the property of flatness, like a table top, but extending on and on in an unlimited fashion. A *line* is a special subset of the points in a plane. There are many kinds of lines: straight lines, curved lines, broken lines, and others. When the word "line" is used, it means straight line, an idea with which we are all familiar. Thus, geometry is the study of relations between points and sets of points, a study of ideas which exist only in the mind. In the study of numbers, symbols, called *numerals,* have been used to represent the number ideas. Similarly, representations of geometric figures, called *models* or *diagrams,* are drawn to represent the geometric ideas we have in mind, but they must not be confused with the idea itself.

C·2 **Definitions and undefined terms.** In any study or sensible discussion, it is necessary to begin by agreeing on precise meanings for the terms used. The meaning of each term is given in a sentence (statement) called a *definition.* The definitions provide the beginning statements from which deductions are made by logical reasoning. In defining a term, we attempt to express its meaning by using other terms or words which have already been defined. However, there are some ideas which are so fundamental

that they defy description in simpler terms. Thus, a beginning must be made with some concepts which are accepted as undefined.

In geometry some of the basic concepts or undefined terms are *set*, *point*, *line*, *plane*, *belongs to*, *lies on*, *is equal to*.

The following are the characteristics of a correct definition.

1. A definition is a sentence.
2. The subject of the sentence is the name of the thing being defined.
3. The predicate of the sentence:
   (i) places the thing in a known set to which it belongs;
   (ii) states the property which distinguishes the thing from the other members of the set in which it has been placed.
4. The sentence uses only terms which have already been defined or accepted as undefined.
5. The sentence is reversible: that is, the meaning of the sentence is the same if the subject and predicate are interchanged.

In section $C\cdot3$, the definitions of some geometric figures are discussed. They should be tested to see that each of the above requirements applies. In some cases you are asked to make definitions after considering the ideas involved.

***C·3* Definitions of basic geometric figures.** Geometric figures are sets of points related in some particular way. The definition of a particular geometric figure tells how these points are selected from the set of all points. If the geometric figures are plane figures, then the points are selected from the points in a plane.

a. *Line.* Straight line is an undefined term. We think of it as a special set of points in a plane. It extends on and on without limit and has no end points. These properties are indicated by the arrows in *Fig. C-1*.

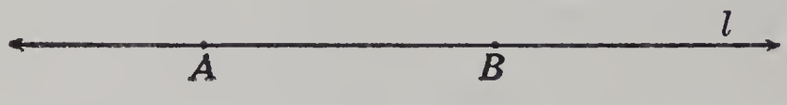


Fig. C-1

It is customary to name a line:

(i) by associating a letter (l) with it and calling it *line l* (*Fig. C-1*), or

(ii) by selecting two points on it, naming them with capital letters, such as A and B, and calling it *line AB* or *line BA* (*Fig. C-1*).

b. *Line segment.* A *line segment* (*Fig. C-2*) is a subset of the points on a line. The subset must be clearly defined.

DEFINITION: *A line segment is the point set consisting of two distinct points on a line and all the points between them.*

Fig. C-2

c. *Half-line and ray.* If a point A is taken on a line (*Fig. C-3*), three disjoint point sets are determined:

(i) $\{A\}$;

(ii) the set of all points on one side of A;

(iii) the set of all points on the other side of A.

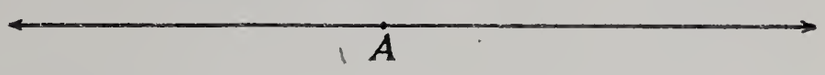


Fig. C-3

The sets in (ii) and (iii) do not include the point A and are called *half-lines.*

DEFINITION: *A half-line is the point set consisting of all points on a line which are on one side of a given point on the line.*

The point A is referred to as the *boundary point* of the half-line.
The half-line with boundary point A is said to be the half-line from A.
The union of sets (i) and (ii), or (i) and (iii), is called a *ray.*

DEFINITION: *A ray is the point set consisting of a given point on a line and all the points on the line on one side of the given point.*

or

A ray is the point set consisting of the union of a point on a line and one of the half-lines determined by the point.

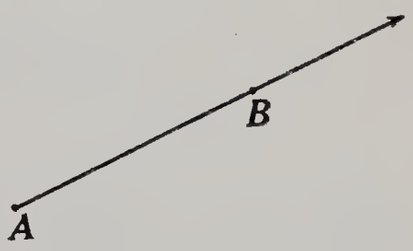


Fig. C-4

Fig. C-4 is a diagram of a ray from point A. If a second point B is selected on the ray, then the ray is named *ray* AB.

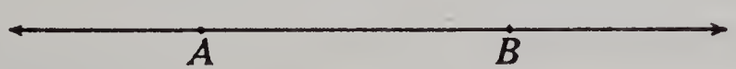


Fig. C-5

In *Fig. C-5*, the ray AB on a line is illustrated. The ray BA is also shown. In naming a ray, the first letter designates the end point.

d. *Angle.* If two distinct rays are drawn from a common point, the figure is called an *angle* (*Fig. C-6*).

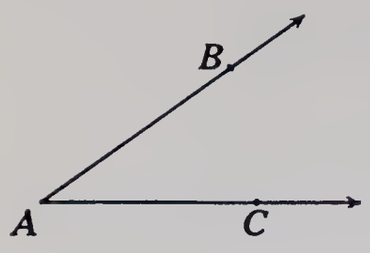


Fig. C-6

DEFINITION: *An angle is the set of all points on two distinct rays with a common end point.*

or

An angle is the point set consisting of the union of two distinct rays having a common end point.

In *Fig. C-6*, the two rays AB and AC compose the angle. It is customary to name the angle $\angle BAC$ or $\angle CAB$, or simply, $\angle A$. Each ray is called a *side* or *arm* of the angle. The common end point of the two rays is called the *vertex* of the angle.

e. *Triangle.*

DEFINITION: *A triangle is the set of all points on the line segments determined by three points which are not all on one line.*

or

A triangle is a point set which consists of the union of the three line segments determined by three points not in the same line.

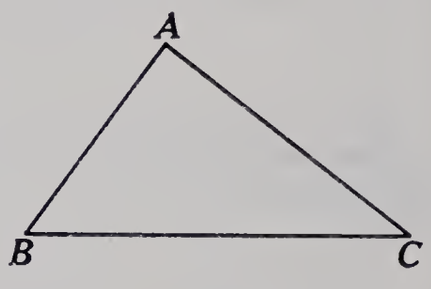


Fig. C-7

In *Fig. C-7*, line segments AB, BC, and CA are called the *sides* of the triangle, and the points A, B, and C are called the *vertices* of the triangle.

Discovery Exercise C-1

(B)

1. For each of the following terms:

(a) plane geometry (b) dime (c) lead pencil

(d) equation (e) inequation (f) monomial

(i) name a set to which the term belongs;

(ii) state the characteristics which distinguish the term from other members of the set.

2. Consider the definition for:

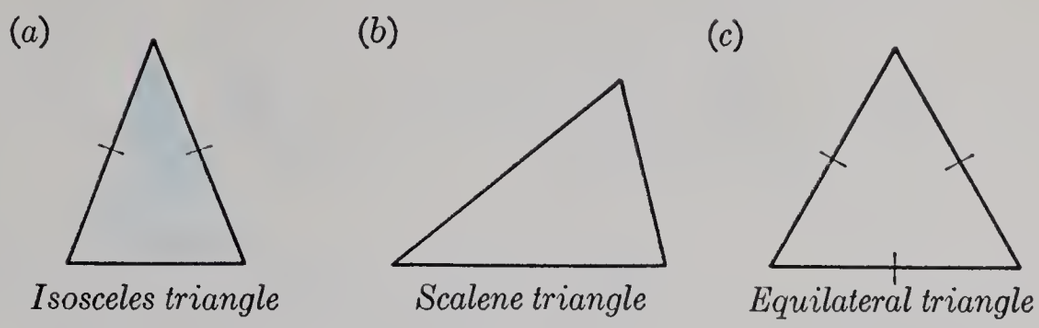
(a) line segment (b) ray (c) half-line

(d) angle (e) triangle

(i) name the set to which each defined term belongs;

(ii) write each definition with subject and predicate interchanged.

3. Examine the diagram which represents each of the following geometric figures:



(i) State a set to which the figure indicated belongs.

(ii) State the characteristic which distinguishes the figure from other members of the set.

(iii) Formulate a definition for each figure.

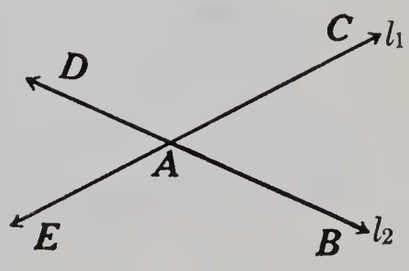
(iv) Write each definition with subject and predicate interchanged.

4. A is a point on line l.

(i) What is the intersection set of the two rays from A on l?

(ii) What is the intersection set of the two half-lines from A on l?

5. l_1 and l_2 are two lines intersecting at A.



(i) Name the four rays determined by the two lines and point A.

(ii) Name the four half-lines determined by A and the lines l_1 and l_2.

(iii) Name the six angles.

(iv) Name the intersection set of (a) the four rays (b) the four half-lines.

(v) Name the union set of the four rays.

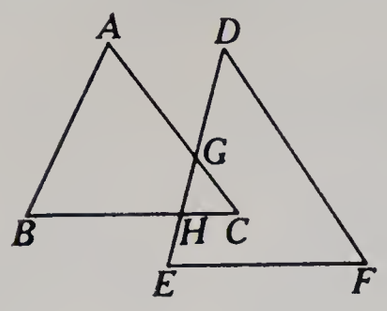
(vi) Name the intersection set of (a) $\angle CAB$ and $\angle BAE$ (b) $\angle DAE$ and $\angle CAB$.

(vii) What is the union set of (a) $\angle CAD$ and $\angle DAE$ (b) $\angle CAD$ and $\angle BAE$?

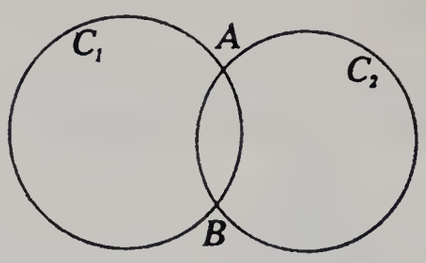
(viii) What is the union set of the half-lines from A?

6. What are the intersection sets in the following figures?

(i)



(ii)



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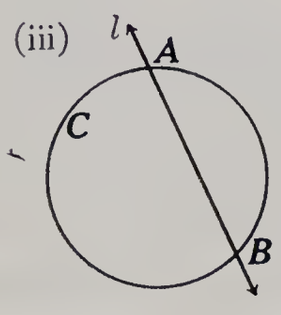
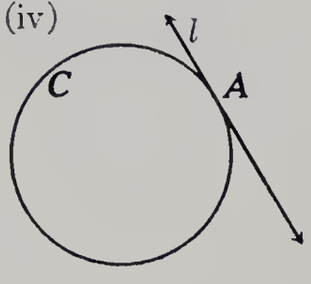
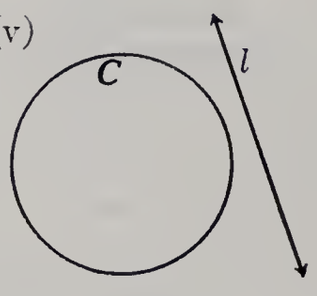


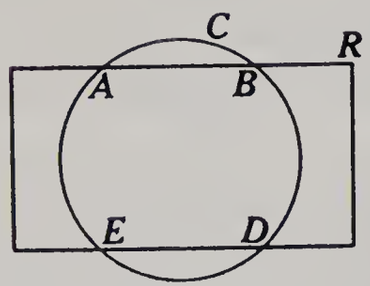
(iv)



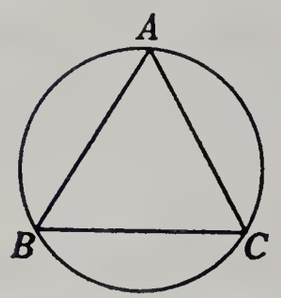
(v)



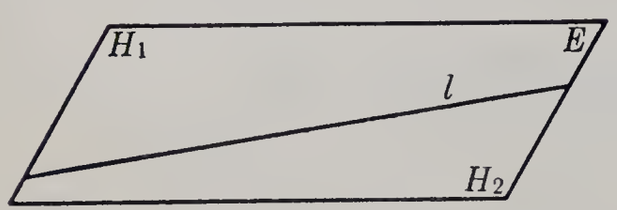
(vi)



(vii)



7. The figure represents a plane E and a line l on it.



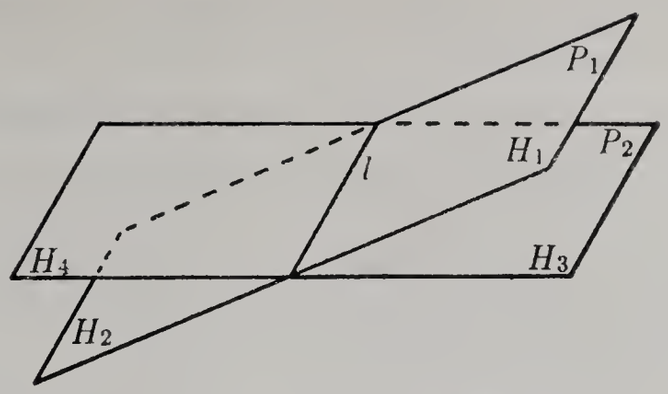
(i) Describe the three disjoint point sets determined by the line l on the plane E.

(ii) The two sets in (i) (excluding the line l) are called half-planes. Write a definition of a half-plane. The line l is called the *boundary* or *edge* of the half-plane.

8. The figure at top of page 237 represents two intersecting planes P_1 and P_2.

(i) What is the intersection set of P_1 and P_2?

(ii) What is the intersection set of the four half-planes H_1, H_2, H_3, H_4? The angle consisting of the line l and the two distinct half-planes H_1 and H_3 is called a *dihedral angle*. l, the common edge of the two half-planes, is called the *edge of the dihedral angle*. The union of the *edge* and either half-plane is called a *face* or *side* of the angle.



(iii) Write a definition for dihedral angle.

(iv) How many dihedral angles are determined by the two intersecting planes and the line l?

(v) Name the intersection set of these angles.

(vi) Name the union set of these angles.

(vii) Name the intersection set of (a) $\angle H_4 l H_1$ and $\angle H_1 l H_3$
(b) $\angle H_4 l H_1$ and $\angle H_2 l H_3$.

(viii) Name the union set of (a) $\angle H_4 l H_2$ and $\angle H_2 l H_3$
(b) $\angle H_4 l H_2$ and $\angle H_1 l H_3$.

(ix) Name the intersection set of the half-planes H_1, H_2, H_3, H_4.

***C·4* Postulates.** Just as the development of definitions depends on the acceptance of certain undefined terms or ideas, the development and proof of geometric conclusions depend on basic accepted properties called *assumptions* or *postulates*. Postulates make precise the starting point of discussion and describe the basic properties of points, lines, and planes.

The exercises which follow lead to the formulation of postulates, new definitions, and the clarification of basic ideas. The postulates are numbered, and the numbers correspond to those in the list of postulates on page 452. In geometry, it is essential to be aware of the ideas involved in the postulates, but it is not necessary to memorize statements of them.

Discovery Exercise C-2
(Formulation of basic postulates)

Compare your statements of postulates with those on page 452; *compare your definitions with those on page* 449; *compare other answers with those on page* 538.

(B)

1. Mark a point on a page (plane) and name it A.

(i) Draw a line on A.

(ii) Draw five other lines on A.

(iii) How many lines can be drawn on A?

(iv) In your diagram how did you indicate that the lines are endless in both directions?

(v) The set of all lines on A is sometimes referred to as a family of lines. What is the common characteristic or property of the members of this family? These lines are said to be *concurrent*.

(vi) Mark and name one other point on each of the six lines drawn. Is it possible to draw a second member of the family on point A and any one of the other points named?

(vii) How many lines lie on any two given points? How many points are required to determine a line?

(viii) Formulate a postulate (1) which expresses the basic idea concerning the number of lines on any point.

(ix) Formulate a postulate (2) which expresses the idea concerning the number of lines on any two points.

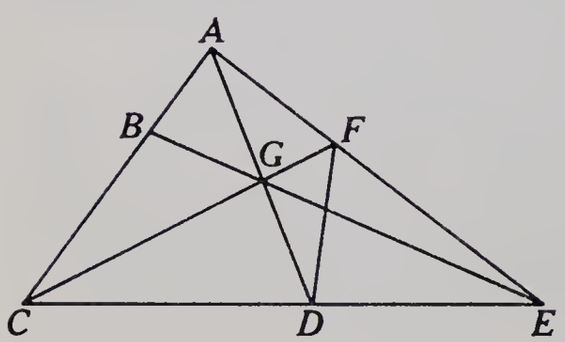
2. If two distinct lines l_1 and l_2 intersect at a point A, state the number of other points of intersection of l_1 and l_2. Formulate a postulate (3) to express this basic property of two intersecting lines.

3. Draw a diagram of two distinct lines which do not have any point in common.

(i) What term is used to describe these lines?

(ii) Write a definition of this term.

4. Draw a line l and name five points A, B, C, D, E on it.



(i) Name five pairs of points which determine the line l.

(ii) Name five points which lie on the line l.

(iii) The points A, B, C, D, and E are *collinear points*. Write a definition for collinear points.

(iv) List the sets of collinear points named in the accompanying figure.

(v) List the line segments in the figure which are concurrent.

5. A three-legged stool will always sit firmly on any floor, while a four-legged stool or table may wobble.

(i) What does this suggest concerning the number of planes containing three distinct points?

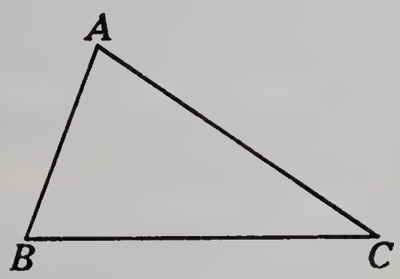
(ii) How many points are required to determine a plane?

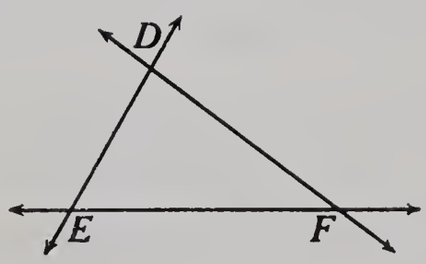
(iii) Name a set of three points which will not determine a plane.

(iv) Formulate a postulate (4) concerning the minimum number of points necessary to determine a plane, and a postulate (5) concerning any three non-collinear points.

6. Draw a line on a page and mark a point not on the line.
   (i) How many planes can contain this line?
   (ii) What does this family of planes have in common?
   (iii) Formulate a postulate (6) concerning two intersecting planes.
   (iv) How many planes contain the line and the given point?
   (v) If a plane contains two given points, does it contain all the points of the line through the two points? Formulate a postulate (7) expressing this idea. This postulate really implies that a plane is flat; if it contains part of a line, it contains all of the line and hence contains many more than three points.
7. Two straight lines, l_1 and l_2, intersect on a point A. How many planes contain both straight lines? Formulate a postulate (8) which expresses this basic idea.
8. In how many points does a line intersect a plane not containing it? Formulate a postulate (9) which expresses this basic idea.
9. On the line l, pictured below, are the two points A and B.

In the diagram, name:
   (i) a line
   (ii) a line segment
   (iii) a ray
   (iv) a ray opposite to ray BA
10. With reference to $\triangle ABC$:





   (i) Name all of the rays determined by the line segments AB, BC, and AC.
   (ii) From the rays listed in (i), name pairs of rays which have a common end point.
   (iii) Explain why the rays on BA and AC do not form an angle.
   (iv) Name the angles formed by the pairs of rays listed in (ii).
   (v) The angles named in (iv) are called the angles of $\triangle ABC$. $\triangle ABC$ does not contain these angles, but it does determine the angles. Name the angles of $\triangle DEF$ in the diagram at the right above.

C·5 Measuring a line segment.

a. *Measuring a physical model of a line segment.* This is done by first associating the number one, 1, with a line segment which is taken as the unit of length; then the length of any given line segment is obtained by counting the number of times the unit length is contained in it.

Fig. C-8

In *Fig. C-8*, if the line segment AB is the unit of length, then the length of CD is obtained by determining the number of times AB is contained in CD. The symbol CD represents:

(i) the line segment CD (the set of points), or

(ii) the length of the line segment CD (a number of units).

A length involves two things, a number and a unit. If the length of line segment CD is 7 inches, we say, $CD = 7$ inches,

or the length of CD is 7 inches,
or the measure of CD in inches is 7,
or the distance between C and D is 7 inches.

Since the choice of unit determines the number of units which is the length of the segment, it is essential in practical applications to have standard units of length. Those in common usage are the *yard* and the *metre*. In order to determine the length of a line segment more precisely, the yard is divided into sub-units: the foot (ft.), the inch (in.), $\frac{1}{10}$ inch, etc; the metre is divided into sub-units: centimetres (cm.) and millimetres (mm.), etc.

Regardless of the unit selected, it is impossible to measure a line segment exactly; every physical measurement is only an approximation.

b. *The length of a line segment.* In theoretical geometry, line segments are sets of points which cannot be physically measured but with which we associate the idea of length.

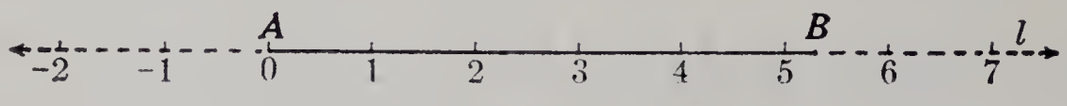


Fig. C-9

Any line segment, AB, *Fig. C-9*, determines a line l. If a unit length is chosen and the number zero associated with the point A, then the points on the ray AB may be placed in *one-to-one correspondence* with the positive

real numbers. Since B is a point on ray AB, then a real number corresponds to point B. This number is the coordinate of point B and is defined to be the length of the line segment AB in terms of the unit used.

Thus, because there is a one-to-one correspondence between the points on a line and the real numbers (postulate 13), there is a unique positive real number associated with a unit which corresponds to any line segment. This is important because it means that line segments may be added and subtracted according to the properties of the real numbers: for example, if $AB = 6$ in., $BC = 10$ in.

$$\therefore AB + BC = (6 + 10) \text{ in.} = 16 \text{ in.}$$

These properties are reviewed in section 8·9. This suggests:

LENGTH OF LINE SEGMENT POSTULATE (14)

To every line segment there corresponds a unique positive real number of units called the length of the line segment, or the distance between its end points.

On some diagrams, for convenience, we express a length simply as a positive real number, and it is understood to be a number of linear units.

C·6 Betweeness. In *Fig. C-10*, point B is between A and C.

Fig. C-10

To say that a point B is between any two other points A and C means:

(i) all three points lie on the same line;

(ii) they are arranged as in *Fig. C-10*.

Item (ii) may be stated mathematically as follows:

B is between A and C if $AB + BC = AC$.

DEFINITION: *B is between A and C if (i) A, B, C are distinct points on a line and (ii) $AB + BC = AC$.*

The following completion postulate (19) for line segments results from this idea of betweeness and the properties of real numbers.

If B is a point between A and C on a line segment, then

$$AC = AB + BC.$$

It follows that,

(i) $AB = AC - BC$

(ii) $BC = AC - AB$

(iii) $AC > AB$

(iv) $AC > CB$.

Discovery Exercise C-3

(B)

1. A real number coordinate system is set up on a line. What is the distance between the pairs of points with coordinates as listed below?

(i) 3, 6 (ii) $2\frac{1}{2}$, $3\frac{1}{4}$ (iii) -2, -7

(iv) 2.5, $-$ 3.6 (v) $-$ 1.253, 1.472 (vi) 3.54, $-$ 4.627

(vii) x_1, x_2 (viii) $5a$, $-3a$ (ix) -6, 0

2. Two distinct points on a line have coordinates 3 and 9 with respect to the coordinate system.

(i) If the coordinate system were shifted four units to the right, what would be the coordinates of the two points?

(ii) Would the distance between the two points with respect to this coordinate system have changed?

(iii) Suppose a new coordinate system with a different unit were used. Would the distance between the two points with respect to this new coordinate system be different from the previous distance?

(iv) How many coordinate systems can be used at one time if a discussion involving length is to make sense?

3. It is known that the coordinate of a point A is less than that of a point B, and that of B is less than that of a point C. State which point is between the other two, if all three are on the same line. Give the reasons for your answer.

4. If M, N, and P are three points on a line with coordinates a, b, c, respectively, and $a > b > c$, argue that N is between M and P.

5. The following sets of coordinates refer to three points on a line. State the distance between each pair of points, and state which point is between the other two.

(i) 3, 5, 7 (ii) 10, 5, $-$ 3

6. M, N, and P are three points on a line. What must be true of the lengths of the segments MN, NP, and MP if:

(i) N is between M and P? (ii) M is between N and P?

(iii) P is between M and N?

7. M, N, and P are three points on a line.

(i) What does it mean to say N is between M and P?

(ii) What does it mean to say M is between N and P?

(iii) By comparing these statements show that statements (i) and (ii) are not consistent (cannot both be true at the same time).

(iv) Formulate a betweeness postulate (16).

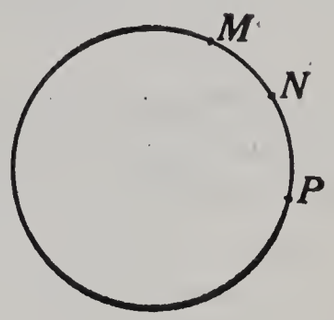
8. (i) Using compasses draw a circle.

(ii) Define a circle as a point set.

In speaking of a circle with centre at point O and radius r, it is customary to think of r as representing either the set of points on a line segment or the length of the given segment. The context determines the meaning that is applicable.

9. M, N, and P are three points on a circle.

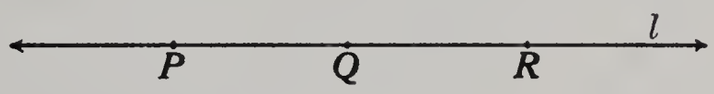
(i) Which point appears to be between the other two points on the circle?



(ii) Think of the points as beads on a circular ring. Shift their positions about the circle so that it appears that (a) M is between N and P (b) P is between M and N.

(iii) Why is betweeness on a circle not so simple a problem as betweeness on a line?

10. PQ and QR on the line l each have length 5 units. If the coordinate of Q is 6:



(i) What are the coordinates of P and R?

(ii) How is Q situated with respect to P and R?

(iii) Is it possible to have more than one midpoint on a given line segment?

(iv) State a postulate (17) to this effect.

(v) Define: midpoint of a line segment.

(vi) Define: bisect.

11. In the accompanying diagram, A and B are two points on a line.

(i) How many points are there on the line between A and B?

(ii) What is the least number of points that could lie between two points A and B on a line? (Recall that a point has no dimensions.)

(iii) Formulate a postulate (18) that states the assumption in (ii).

12. A straight line may be drawn through the locations of three towns A, B, and C. If B is 10 miles from A, and C is 17 miles from B:

(i) draw a diagram to illustrate the possible positions of the towns with respect to each other;

(ii) what additional information would ensure that A is between B and C?

C·7 The measurement of an angle. An angle was defined in section C·3 to be the point set consisting of the union of two distinct rays having a common end point.

Fig. C-11

In theoretical geometry, angles are sets of points which cannot be physically measured, but with which we do associate the idea of size. The measurement of an angle describes, in a sense, the relative position of its rays.

The measurement of an angle is obtained in a manner similar to that of obtaining the length of a line. A unit angle is selected and, on the basis of this unit, a positive real number may be associated with each angle.

In measuring models of angles, the protractor is used. The unit is based on the *straight angle* (*Fig. C-12*) in which the two arms of the angle are opposite rays.

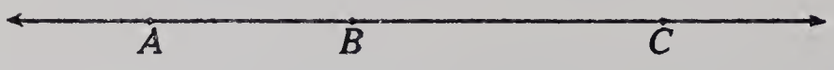


Fig. C-12

The unit angle is called a *degree* and is defined to be $\frac{1}{180}$ of a straight angle.

Thus, if $\angle ABC = 60°$, we say that the measurement of $\angle ABC$ is 60 degrees or the degree measure of $\angle ABC$ is 60.

The symbol $\angle ABC$ (*Fig. C-13*) is used (i) to name the angle ABC (the set of points) or (ii) to indicate the measurement of the angle (a number of degrees), according to the context.

It is accepted intuitively (postulate 15) that to each geometric angle there corresponds a unique positive real number of degrees; the degree measure of an angle is greater than zero and equal to or less than 180.

Since the definition of an angle in geometry involves two distinct rays, there can be no zero angle.

The straight angle ABC, *Fig. C-12*, and the straight line AC are the same set. A straight angle does differ from a straight line in that a specific point has been selected as its vertex.

In *Fig. C-13*, $\angle ABC$ has a measurement between zero and 180°. We may associate with the angle ABC the measurement $(360° - \angle ABC)$

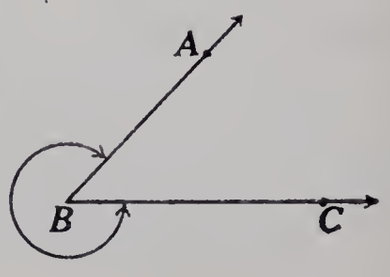


Fig. C-13

indicated by the double-headed arrow. When we are concerned with this latter measurement, we speak of the *reflex angle ABC*.

The idea of an angle as a point set consisting of the union of two distinct rays having a common end point is sufficient for the purposes of this study of geometry. Later you will think of an angle in other ways. For example, an angle may be thought of in terms of the rotation of a ray from an initial position to a terminal position. In this context, angles may have measurements greater than 180°.

Since the measurement of an angle in degrees is a positive real number, angles may be added and subtracted according to the properties of the real numbers: for example,

$$\text{if } \angle ABC = 70°,\ \angle DEF = 50°, \text{ then}$$
$$\angle ABC + \angle DEF = (70 + 50)° = 120°.$$

Also, 120° is said to be the *angle sum* of $\angle ABC$ and $\angle DEF$.

In *Fig. C-14*, since D is on the same side of line AB as C, and D is on the same side of line BC as A, then ray BD is between ray AB and ray AC. D is said to be in the interior of $\angle ABC$.

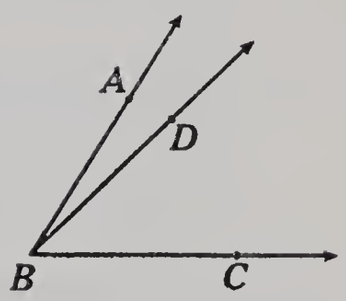


Fig. C-14

The following completion postulate (20) for angles results from this idea of betweeness and the properties of real numbers.

If D is a point in the interior of $\angle ABC$, then

$$\angle ABC = \angle ABD + \angle DBC.$$

It follows that, (i) $\angle ABD = \angle ABC - \angle DBC$
(ii) $\angle DBC = \angle ABC - \angle ABD$
(iii) $\angle ABC > \angle ABD$
(iv) $\angle ABC > \angle DBC$

Discovery Exercise C-4

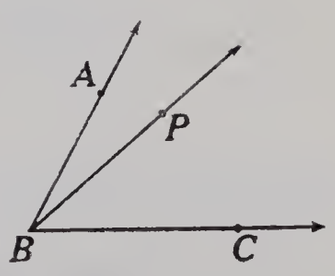
(B)

1. P is a point in the interior of $\angle ABC$ and BP is a ray.

(i) Name the postulate which states $\angle ABC = \angle ABP + \angle PBC$.

(ii) What do the symbols $\angle ABC$, $\angle ABP$, and $\angle PBC$ represent in (i)?

(iii) In what other ways may the statement of (i) be written?



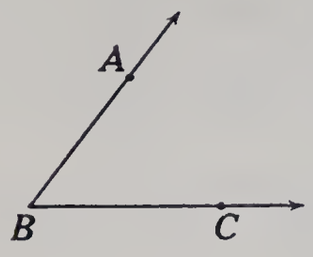
(iv) How many angles exist which are equal to $\angle ABC$?

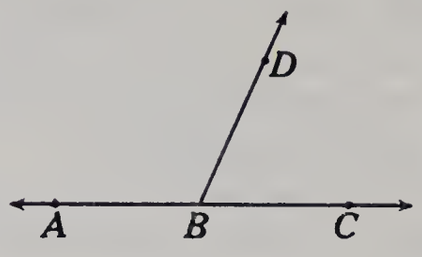
(v) Formulate a postulate (10) which clearly expresses the answer to (iv).

2. In the diagram at the left below, ABC is an angle.

(i) How many rays can be drawn from B which bisect $\angle ABC$?

(ii) Formulate a postulate (11) which states this assumption.





3. $\angle ABC$ is a straight angle (at the right above). BD is any other ray drawn from B.

(i) What is the angle sum of $\angle ABD$ and $\angle DBC$?
Two angles whose angle sum is 180° are called *supplementary angles.*

(ii) Which of the following pairs are measurements of supplementary angles?

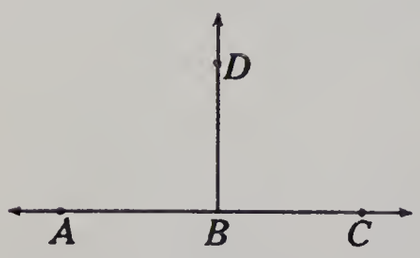
(a) 60°, 30°	(b) 72°, 30°	(c) 90°, 90°
(d) 120°, 50°	(e) 35°, 145°	(f) 110°, 60°

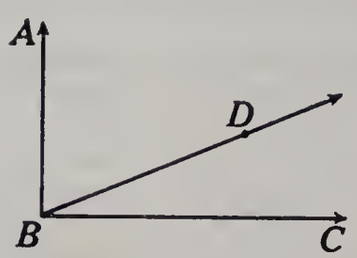
(iii) State the measurement of the angle supplementary to each of the following:
35°, 72°, 105°, 84°, 92°, 115°, 130°

4. (i) How many rays bisect a straight angle?

(ii) What is the measurement of each of the angles ABD and DBC, if BD is a ray bisecting the straight angle ABC? Each angle is called a *right angle.*

(iii) Define a right angle.
The ray BD is said to be *perpendicular* to line AC (left below). This is written, ray $DB \perp$ line AC.





5. $\angle ABC$ is a right angle, and BD is a ray such that D is in the interior of $\angle ABC$ in the diagram at the right above.

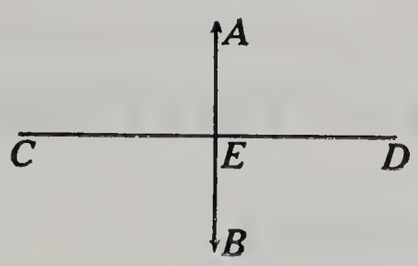
(i) What is the angle sum of $\angle ABD$ and $\angle DBC$?
Two angles whose angle sum is 90° are *complementary angles.*

(ii) Which of the following pairs are measurements of complementary angles?

(a) 45°, 45° (b) 30°, 60° (c) 40°, 60°
(d) 72°, 38° (e) 25°, 65° (f) 80°, 9°

(iii) State the measurement of the angle complementary to each of the following:
70°, 62°, 54°, 42°, 72°, 36°, 18°

6. CD is a line segment, and the line AB passes through the midpoint of CD, and $\angle AED = 90°$. AB is called the *right bisector* of CD.

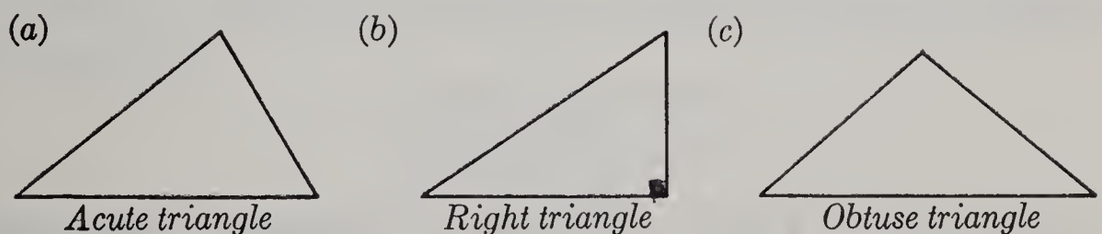


(i) Write a definition of the right bisector of a line segment.

(ii) How many right bisectors does a line segment have?

(iii) Formulate a postulate (12) to this effect.

7. An *acute angle* is an angle with measurement less than 90°.
An *obtuse angle* is an angle with measurement greater than 90° and less than 180°.
By examining the diagram which represents the indicated classification of triangles according to angles:



(i) State a set to which each of the figures indicated belongs.

(ii) State the characteristic which distinguishes the figure from other members of the set.

(iii) Formulate a definition of each figure.

(iv) Write the definition with subject and predicate interchanged.

Review D

FUNDAMENTAL CONSTRUCTIONS

INDUCTIVE REASONING

D·1 **Geometric constructions.** The ideas of geometry have many practical applications. The engineer, the architect, the artist all make use of geometric ideas and the properties of geometric figures in modern design.

In the study of geometry, we use physical drawings, or models, of particular geometric figures to help us discover geometric principles and understand geometric ideas.

D·2 **Fundamental ruler-compasses constructions 1, 2, 3.**

CONSTRUCTION 1

Angle Construction

At a given point in a given straight line to construct an angle equal to a given angle.

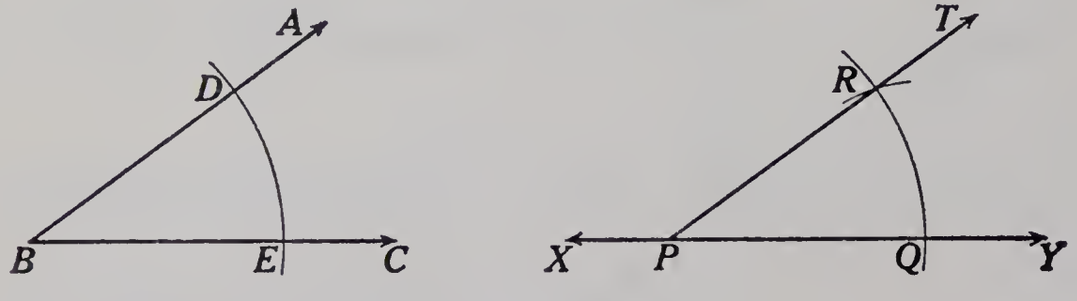


Fig. D-1

Given: P is a point in a line XY and ABC is any angle (*Fig. D-1*).
Required: To construct at P on XY an angle equal to $\angle ABC$.

Construction: With centre B and any radius draw an arc cutting AB, BC on D and E, respectively.
With centre P and the same radius, draw an arc cutting XY on Q.
With centre Q and radius DE, draw an arc cutting the former arc on R.
Draw ray PT on R.
Then $\angle TPY$ is the required angle.
Examine fundamental constructions 2 and 3 which follow. Describe the method of construction and compare your description with that on page 483

CONSTRUCTION 2

To construct the bisector of a given angle.

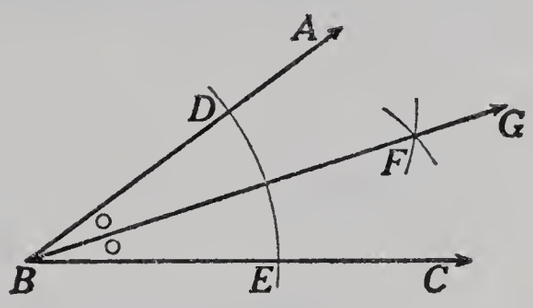


Fig. D-2

Given: An angle ABC (*Fig. D-2*).
Required: To construct the bisector of $\angle ABC$.

CONSTRUCTION 3

To construct a line segment perpendicular to a given line segment at a given point on it.

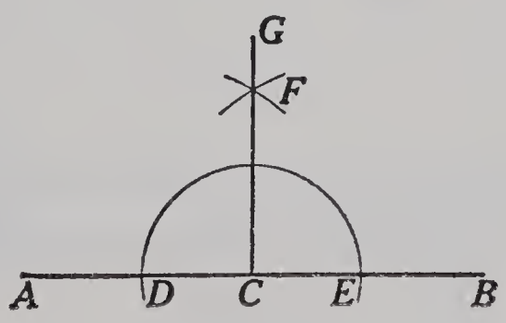


Fig. D-3

Given: A line segment AB and a point C on it (*Fig. D-3*).

Required: To construct a perpendicular line segment at C.

The ray determined by CG bisects the straight angle ACB formed by the rays determined by CA and CB.

Example 1. Construct an angle equal to the angle sum of two given angles.

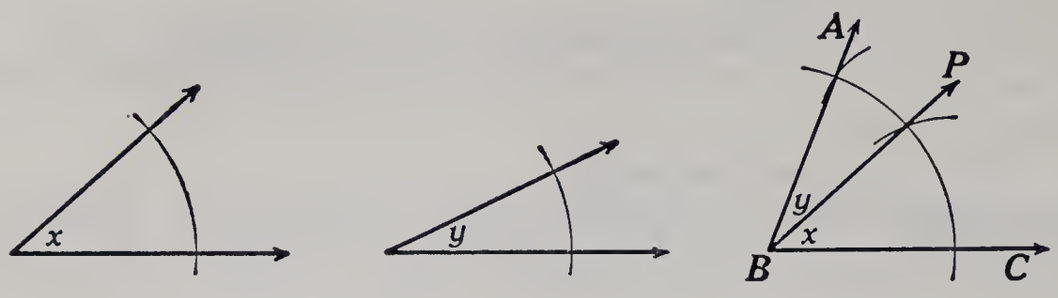


Fig. D-4

Given: Two angles, $\angle x$ and $\angle y$ (*Fig. D-4*).
Required: To construct an angle equal to the angle sum of $\angle x$ and $\angle y$.
Construction: Draw any ray BC.
On BC construct at B, $\angle PBC = \angle x$.
On BP at B construct $\angle ABP = \angle y$.
Then $\angle ABC$ is the required angle.

Example 2. Construct $\triangle ABC$ having $\angle A = 45°$, $\angle B = 90°$, and $AB = 2''$.

PRELIMINARY SKETCH RULER-COMPASSES CONSTRUCTION

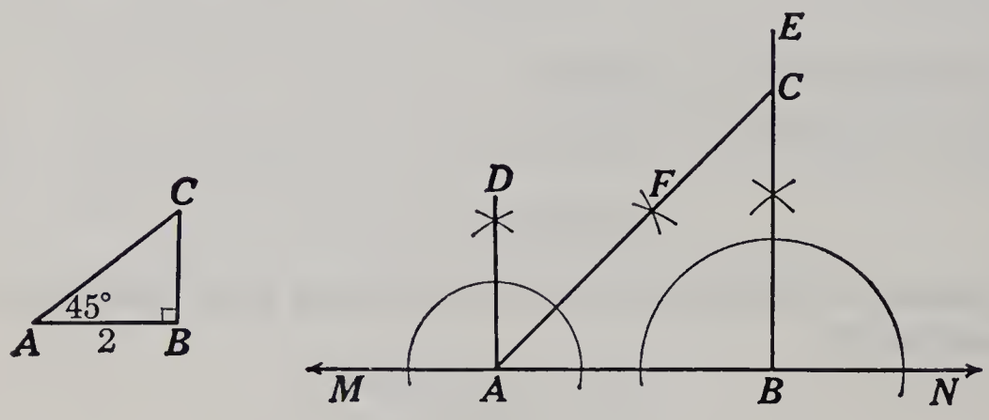


Fig. D-5

Construction: On a line MN (*Fig. D-5*) mark a point A.
With centre A and radius 2 inches, draw an arc intersecting MN in B.
At B, construct $BE \perp MN$.
At A, construct $DA \perp MN$.
Construct AF, the bisector of $\angle DAB$.
Extend AF to intersect BE on C.
ABC is the required triangle.

Example 3. Construct a right triangle having the hypotenuse and one of the sides equal to two given line segments, one of which is twice the length of the other.

PRELIMINARY SKETCH RULER-COMPASSES CONSTRUCTION

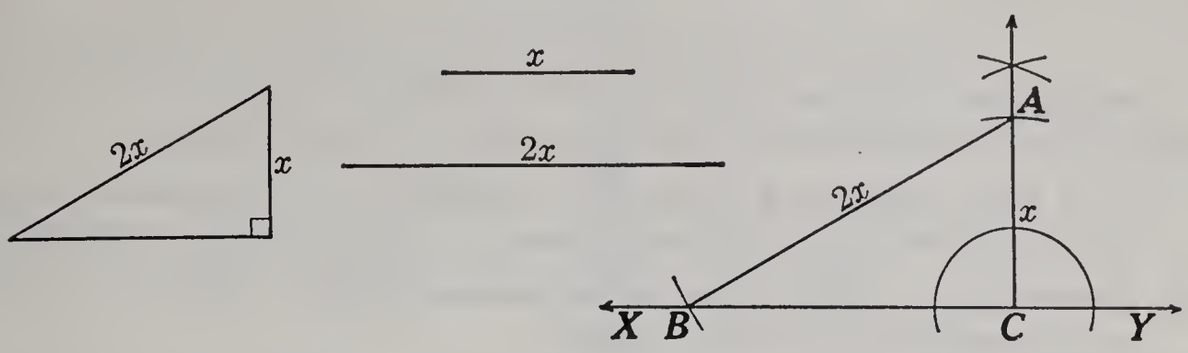


Fig. D-6

Given: Two line segments having lengths x units and $2x$ units (*Fig. D-6*).

Required: To construct a right triangle in which the hypotenuse is $2x$ units and one of the sides is x units.

Construction: Draw a line XY.
At any point C on XY, construct a right angle.
With centre C and radius x units, draw an arc to cut the perpendicular on A.
With centre A and radius $2x$ units, draw an arc to cut XY on B.
Join AB.
Then ABC is the required triangle.

Example 4. Construct $\triangle ABC$ having $AB = AC = 1\frac{1}{2}$ inches and $BC = 1$ inch.
Construct the bisector of $\angle A$ to intersect BC on D. Measure $\angle ADB$.

PRELIMINARY SKETCH RULER-COMPASSES CONSTRUCTION

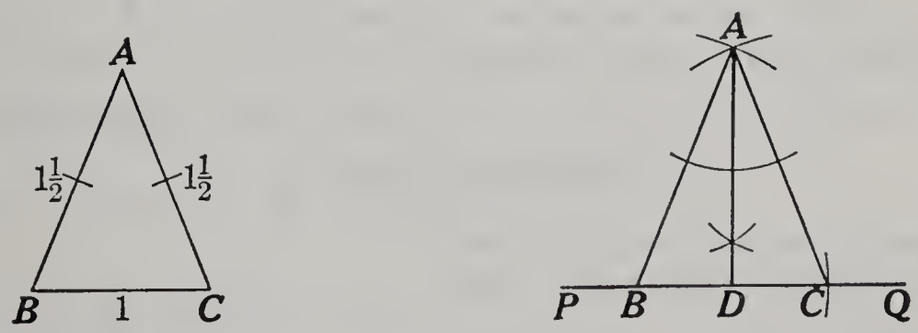


Fig. D-7

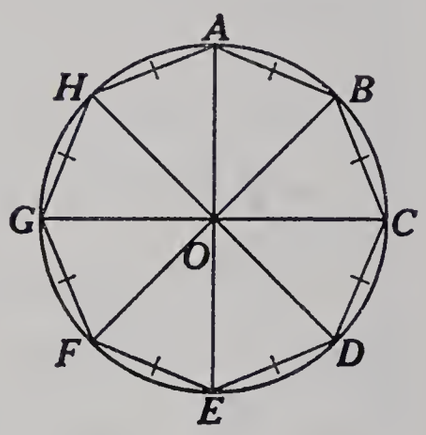
Construction: On a line PQ (*Fig. D-7*), mark a point B.
With centre B and radius 1 inch, draw an arc intersecting PQ on C.
With centres B and C and radius $1\frac{1}{2}$ inches, draw arcs intersecting on A.
Join BA and AC.
Bisect $\angle BAC$.
By measuring, $\angle ADB = 90°$.

Exercise D-1

(B)

Use ruler and compasses only to make the following constructions; show all construction lines, but do not give a description of constructions:

1. At a given point on a given line construct rays which make angles with the given line having measurements of $90°, 45°, 22\frac{1}{2}°, 135°, 112\frac{1}{2}°$.
2. Construct an angle whose measurement is the difference of the measurements of two given angles.
3. Construct $\triangle ABC$ in which the base is 3 inches in length and the two angles adjacent to the base are equal to two given acute angles.
4. Construct an angle equal to the angle sum of the angles of a given triangle; measure the constructed angle using a protractor.
5. Construct an isosceles triangle ABC in which $AB = AC = 2$ inches and $\angle BAC = 45°$.
6. Examine the equilateral octagon $ABCDEFGH$ in the diagram at the right. Using a circle as a beginning, devise a method of constructing an equilateral octagon.
7. AB and CD are two lines that intersect on O. Bisect $\angle AOD$ and $\angle AOC$. Measure the angle at O formed by the bisectors.



***D·3* Fundamental ruler-compasses constructions 4, 5, 6.** In each of the following fundamental constructions, examine the construction carefully and then describe it. Compare your description with that on page 483.

CONSTRUCTION 4

To construct a line segment perpendicular to a given line from a point not on the line.

Given: A line AB and a point C not on it (*Fig. D-8*).

Required: To construct a line segment from C perpendicular to AB.

The statement, "CG is perpendicular to AB", is abbreviated, $CG \perp AB$. The *distance* from a point to a line means the *perpendicular* distance.

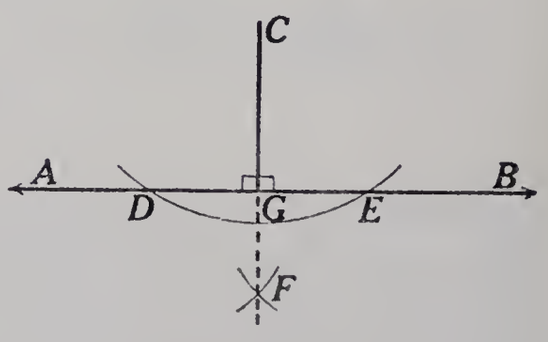


Fig. D-8

CONSTRUCTION 5

To construct the right bisector of a given line segment.

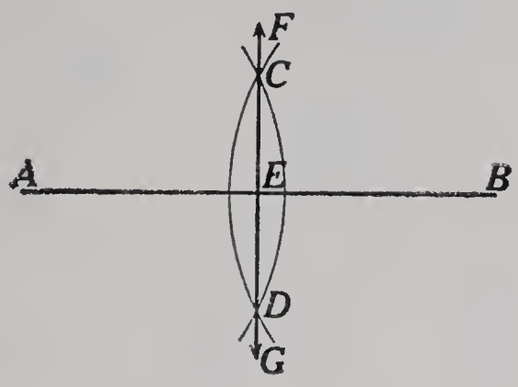


Fig. D-9

Given: A line segment AB (*Fig. D-9*).
Required: To construct the right bisector of AB.

CONSTRUCTION 6

To construct a line parallel to a given line on a given point not on the line.

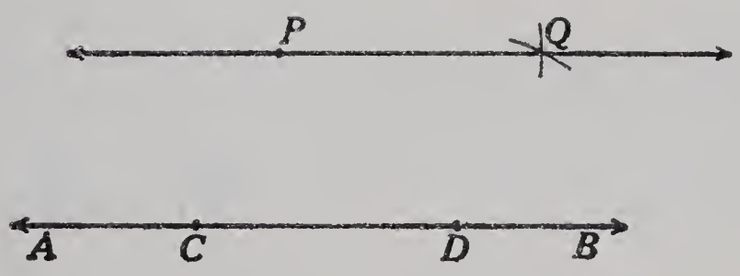


Fig. D-10

Given: A line AB and a point P not on the line (*Fig. D-10*).
Required: To construct on P a line parallel to AB.

DEFINITION: *An altitude of a triangle is the line segment drawn from a vertex perpendicular to the opposite side and terminating on the opposite side.*

Example 1. Construct the altitudes from B and C to the opposite sides of $\triangle ABC$ (*Fig. D-11*):

(i) if $\triangle ABC$ is an acute triangle;

(ii) if $\angle B$ is an obtuse angle.

Construction: Draw $BD \perp AC$, $CE \perp AB$. Then BD and CE are the required altitudes.

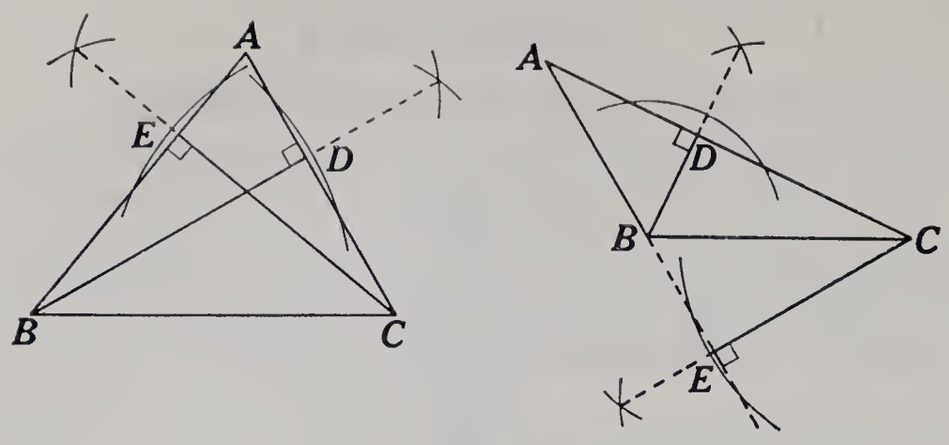
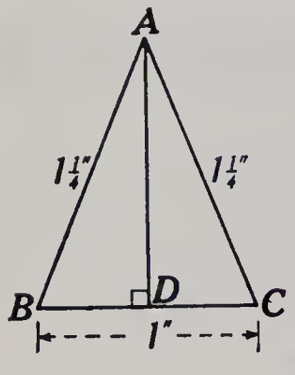


Fig. D-11

Example 2. Construct an isosceles $\triangle ABC$ in which $AB = AC = 1\frac{1}{4}$ inches and $BC = 1$ inch. Construct the altitude from A to BC.

PRELIMINARY SKETCH



RULER-COMPASSES CONSTRUCTION

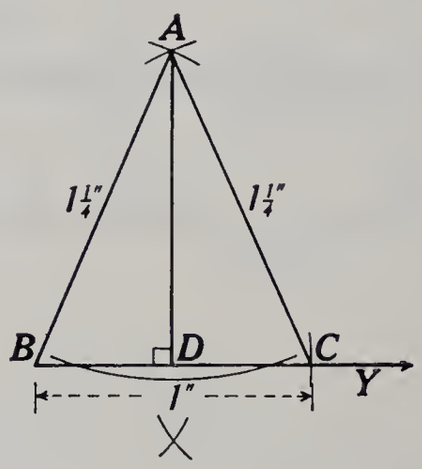


Fig. D-12

Given: $AB = AC = 1\frac{1}{4}$ inches and $BC = 1$ inch (*Fig. D-12*).

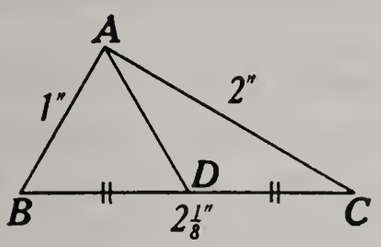
Required: To construct an isosceles triangle ABC with $AB = AC = 1\frac{1}{4}$ inches and $BC = 1$ inch; then construct the perpendicular from A to BC.

Construction: Draw a ray BY and mark off $BC = 1$ inch.
With centres B and C, and radius $1\frac{1}{4}$ inches, draw arcs to intersect on A.
Join AB and AC.
ABC is the required triangle.
From A draw a line segment perpendicular to BC.
AD is the required altitude.

DEFINITION: *The median of a triangle is the line segment joining any vertex to the midpoint of the opposite side.*

Example 3. Construct $\triangle ABC$ in which $AB = 1$ inch, $AC = 2$ inches, $BC = 2\frac{1}{8}$ inches. Construct the median AD from A to BC. Only the diagram is required.

PRELIMINARY SKETCH



RULER-COMPASSES CONSTRUCTION

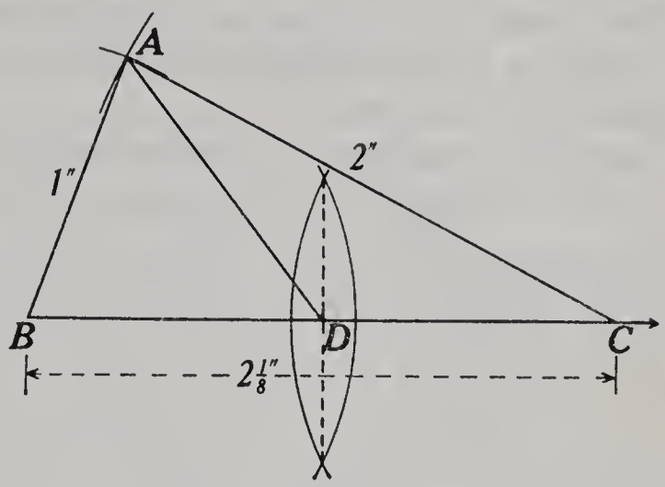


Fig. D-13

Exercise D-2

(B)

Use ruler and compasses only to make the following constructions; show all construction lines, but do not give a description of constructions;

1. Divide a given line segment into four equal parts.
2. Construct $AB = 2$ inches, $CD = 3$ inches, bisecting each other at right angles.
3. Construct any triangle ABC. Through D, the midpoint of AB, construct a line parallel to BC to intersect AC on E. Measure AE, EC, DE, and BC.
4. Construct $\triangle ABC$ in which the base BC is 3 inches, $\angle ABC$ is a given obtuse angle, and side AB is 2 inches long. Construct a line through A parallel to BC.
5. Draw an equilateral triangle whose sides are each $1\frac{1}{2}$ inches long. Through each vertex draw a line parallel to the opposite side of the triangle. Measure the lengths of the sides of the triangle formed.
6. Construct $\triangle ABC$ having $AB = 2.5$ inches, $BC = 3.9$ inches, and $CA = 4$ inches. Construct the right bisectors of AB and AC, intersecting on D. Measure AD, BD, and CD, to the nearest tenth of an inch. With centre D and radius BD, draw a circle. This circle is called the *circumscribed circle* of $\triangle ABC$. Write a definition of the circumscribed circle of a triangle.

***D·4* Inductive reasoning.** In earlier grades, many geometrical conjectures were made in the study of experimental geometry. For example, after finding by measurement and calculation the angle sum of different triangles, the *conjecture* or *probable inference* was made that the angle sum of a triangle is 180°.

The process of arriving at a conjecture by observing close agreement of evidence obtained from a relatively large number of particular cases is called *inductive reasoning.*

Inductive reasoning is important in all scientific research because it forms the basis of the scientific method. Many valuable scientific discoveries are the direct result of inductive reasoning. However, even though inductive reasoning often leads to important conjectures or probable inferences, it does not prove that any conjecture is true. In mathematics, any conclusion based on inductive reasoning is accepted only as a conjecture. The mathematician then tries to prove or disprove the conjecture by logical reasoning, not by observation or measurement. In many of the sciences, conjectures arrived at by inductive reasoning often form the basis for theories which are accepted until a contradiction is found. It should be noted that inductive reasoning proceeds from particular cases to a conjecture which is a *generalization.*

A statement is false if just one case exists for which the statement is false. This special case is called a *counterexample* of the statement. Since a single counterexample is sufficient, it is often easier to prove a statement false than to prove a statement true.

The following exercise provides practice in applying inductive reasoning to make conjectures.

Exercise D-3

(*Best results will be obtained from large diagrams.*)

1. Draw any acute triangle and any obtuse triangle. Using a set-square, draw the three altitudes of each of these triangles. Note any relation among the altitudes suggested by these diagrams. Test this suggested relation by drawing a second acute triangle and a second obtuse triangle and making ruler-compasses constructions of the three altitudes of each of these triangles. Does the suggested relation apply to a right triangle? Make a conjecture about the relation among the altitudes of any triangle.
2. Draw any right triangle, any acute triangle, and any obtuse triangle. Using a ruler, estimate the position of the midpoints of the sides of each of these triangles, and draw the three medians of each triangle. Note any relation among the medians suggested by these diagrams. Test this suggested relation by drawing a second set of three tri-

angles and making ruler-compasses constructions of the three medians of each triangle. Make a conjecture about the relation among the medians of any triangle.

3. Draw any right triangle, any acute triangle, and any obtuse triangle. Using a protractor, draw the bisectors of the three angles of each of these triangles. Note any relation among the bisectors of the angles suggested by these diagrams. Test this suggested relation by drawing the bisectors of the angles of a second set of three such triangles using ruler and compasses. Make a conjecture about the relation among the bisectors of the angles of any triangle.

4. Draw any right triangle, any acute triangle, and any obtuse triangle. Using ruler-compasses constructions, draw the right bisectors of the sides of each of the triangles. Note any relation among the right bisectors suggested by these diagrams. Test this suggested relation by drawing the right bisectors of the sides of a second set of three such triangles. Make a conjecture about the relation among the right bisectors of the sides of any triangle.

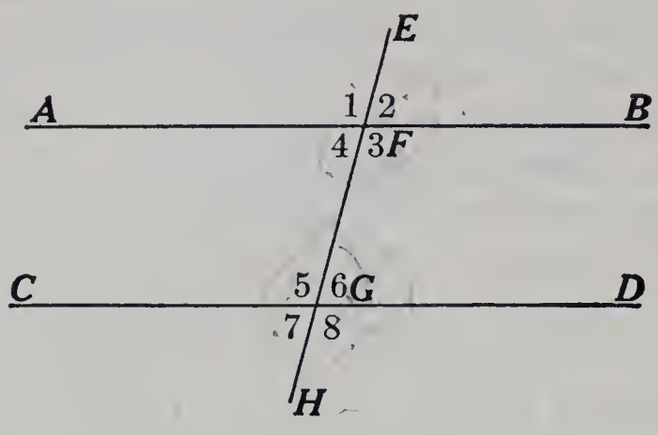
5. Draw any circle. Choose three points A, B, and C on it; join AB, BC, CA to form triangle ABC. It is said that $\triangle ABC$ is *inscribed* in the circle, or that the circle *circumscribes* $\triangle ABC$. An obvious conjecture is that, given any triangle, there exists a circumscribing circle. Make a conjecture about the method of determining the centre and radius of such a circle, so that it can be drawn in a physical model. Test your conjecture for two acute triangles and two obtuse triangles.

6. Draw any acute triangle, any obtuse triangle, and any right triangle. Using ruler-compasses constructions, determine the orthocentre, the centroid, and the circumcentre of each triangle, where the *orthocentre* is the point of intersection of the altitudes, the *centroid* is the point of intersection of the medians, and the *circumcentre* is the point of intersection of the right bisectors of the sides of a triangle. Note any relation among the orthocentre, the centroid, and the circumcentre suggested by these diagrams. Test the suggested relation for an isosceles triangle and also for an equilateral triangle. Make a conjecture about the relation among the orthocentre, the centroid, and the circumcentre of any triangle.

7. Draw any acute triangle ABC; extend BC to D. Measure $\angle ACD$, $\angle CAB$, $\angle ABC$. Angle ACD is an *exterior angle* of $\triangle ABC$, and $\angle CAB$ and $\angle ABC$ are the *interior remote angles* of $\angle ACD$. Note any relation between the exterior angle and the two interior remote angles suggested by this diagram. Test this suggested relation for an obtuse triangle, a right triangle, and another acute triangle. Make a

conjecture about the relation between the exterior angle and the two interior remote angles of any triangle.

8. Draw two parallel line segments with a ruler. Draw a line segment (*transversal*) intersecting the two lines.



Measure:

(i) pairs of *alternate angles* (4, 6), (3, 5);

(ii) pairs of *corresponding angles* (2, 6), (3, 8), (1, 5), (4, 7);

(iii) pairs of *interior angles on the same side of the transversal* (4, 5), (3, 6).

Note any relation between:

(i) pairs of alternate angles;

(ii) pairs of corresponding angles;

(iii) pairs of interior angles on the same side of the transversal suggested by the measurements.

Test this suggested relation by drawing a second pair of parallel line segments using ruler-compasses construction. Make a conjecture about pairs of alternate angles, pairs of corresponding angles, and pairs of interior angles on the same side of a transversal intersecting two parallel line segments.

Chapter VIII

INTRODUCTION TO DEDUCTIVE REASONING

8·1 Sentences, sentential connectives. In the study of inductive reasoning, we learned how to make a probable inference or conjecture from the examination of particular cases. We also learned that inductive reasoning does not provide proof that a conjecture is true. Mathematical proof demands a type of reasoning known as *deductive reasoning* or *logical deduction*. To begin with, we must become acquainted with the language of logic.

In logic, each English sentence has a form which is given an identifying name. For example, the sentence:

It is raining.

is a basic type of sentence. In logic such a sentence is called a *simple sentence* or an *atomic sentence*.

DEFINITION: *A sentence, in logic, is a statement which is either true or false, but not both.*

If we combine two simple sentences by means of a connecting word, the resulting sentence is no longer a simple sentence. It is called a *compound sentence* or a *molecular sentence*. For example, the sentence:

It is raining and it is cold.

is a compound sentence. The word *and* combines the two simple sentences:

1. *It is raining.*
2. *It is cold.*

It should be noted that the connecting word *and* is not part of either simple sentence. It merely connects the two simple sentences to produce

a compound sentence. Thus, the word *and* in logic is called a *sentential connective.*

The basic sentential connectives are *and, or, if . . . then . . ., not.*

The use of each of these connectives together with the simple sentences:

1. *It is raining.*
2. *It is cold.*

to produce compound sentences is illustrated in the following.

(i) Connective: *and*

It is raining *and* it is cold.

This sentence is called the *conjunction of the two simple sentences.*

(ii) Connective: *or*

It is raining *or* it is cold.

This sentence is called the *disjunction of the two simple sentences.*

(iii) Connective: *if . . . then . . .*

If it is raining, *then* it is cold.

This sentence is called a *conditional sentence* or an *implication.* It should be noted that the connectives *and, or, if . . . then . . .* control or modify two sentences.

(iv) Connective: *not*

It is *not* raining.

This sentence is called the *negation* of the sentence, "It is raining". It should be noted that the connective *not* controls or modifies only one sentence.

The basic forms of compound sentences are:

(i) ( ) *and* ( ).

(ii) ( ) *or* ( ).

(iii) *If* ( ) *then* ( ).

(iv) *not* ( ).

where the parenthesis may be filled by simple or compound sentences.

In logic, capital letters, such as P, Q, R, S, T, A, B, are used to represent sentences. Thus, if P represents the sentence, *It is raining,* and Q represents the sentence, *It is cold,* the compound sentence,

(*It is raining*) *and* (*it is cold*).

may be represented symbolically as:

(P) and (Q)

or simply, P and Q.

In general, if A and B represent any two sentences, the basic compound sentences may be represented symbolically as:

(i) A and B .

(ii) A or B .

(iii) If A then B; (or) A implies B; (or) $A \rightarrow B$.

(iv) Not A; (or) $\sim A$.

Exercise 8-1

(A)

Classify each of the following sentences as simple or compound:

1. Today is Monday.
2. Mathematics is the Queen of the Sciences.
3. Camping is fun.
4. Governments fold at the polls.
5. Time does not stand still.
6. John enjoys music, and he has a large record collection.
7. $4x = 3y$.
8. The talk on how to make an interesting speech was uninteresting.
9. The better team does not always win.
10. $x = y$ and $y \neq z$.
11. If $x - y = 0$, then $x = y$.
12. If two lines are parallel, then they do not have a common point.
13. 2 is rational and $\sqrt{2}$ is irrational.
14. He is intelligent.
15. Scientists are not eccentric.
16. Nero fiddled.
17. I drove to the cottage and hitch-hiked back.

(B)

In each of the following sentences make a list of connectives, if there are any:

18. Today is Monday, and tomorrow is Tuesday.
19. $2x + 3 = 9$ or $5x + 7 = 22$.
20. If $a, b \in N$, then $\frac{a}{b} \notin N$.
21. $\{0\}$ is not a null set.

In each of the following sentences make a list of connectives, if there are any:

22. The set of all even numbers greater than 1 and less than 10 is $\{2, 4, 6, 8\}$.

23. If $x + 3 = 5$, then $x = 2$.

24. If $a > b$ and $b > c$, then $a > c$.

25. Animal cells do not have real cell walls.

26. If he is uncertain, then he will not answer.

27. If $x^2 = 4$ and x represents a natural number, then $x = 2$.

Form a compound sentence by combining the simple sentences in each of the following using the connective indicated; symbolize the sentence:

28. $2x + 4 = 8$. $x = 2$. (connective: *If . . . then . . .*)

29. The rooms are warm. The food is excellent. (connective: *and*)

30. It is easy to bowl a perfect game. (connective: *not*)

31. $a > b$. $a = b$. $a < b$. (connective: *or*)

32. $x + 3 = 6$. (connective: *not*)

33. He turned around. He looked at us. (connective: *and*)

Classify each of the following compound sentences in terms of conjunction, disjunction, implication, or negation:

34. P or Q

35. M and N

36. $(\sim A)$ or $(\sim D)$

37. $(\sim B)$ and $(\sim C)$

38. $M \rightarrow X$

39. X or (P and Q)

40. Q or $(\sim Q)$

41. $\sim (X$ and $Y)$

42. $P \rightarrow (\sim Q)$

43. (M and N) $\rightarrow A$

Rewrite the following conditional statements in "if . . . then . . ." form:

44. A win today will put the team in the finals.

45. $\because\ x + 3 = 7 \quad \therefore\ x = 4$.

46. $a - b = x$ since $x + b = a$.

47. $a \div b = x$ since $x \times b = a$.

48. A cloudy day is not a warm day.

49. When it rains the crops grow.

50. When $x = 5$ the expression $2x + 4$ represents 14 .

51. A good argument is logical.

52. A logical argument is good.

53. A person who understands logic will understand geometry.

8·2 Deductive reasoning. The process of making necessary conclusions from accepted statements by applying accepted rules of logical inference is called *deductive reasoning.*

If we are given the following statements:

1. *If it is raining, then it is cold.*
2. *It is raining.*

then the inescapable consequent of these two statements is the conclusion:

It is cold.

The two given statements are the *premises* of the argument, and the conclusion is the *logical consequent* of the premises.

The rule of logical inference which permits us to make this conclusion from these statements is called the *Law of Detachment.* Thus, by this law, *if we are given any implication and precisely the* "if clause" *of this implication, we can detach the* "then clause" *and state it as a logical consequent.*

In the above example, we say we have proved or deduced the statement: *It is cold.* That is, we have proved that the derived sentence follows logically from the given sentences.

The complete argument may be arranged as follows:

Premises: 1. If it is raining, then it is cold.
2. It is raining.

Logical consequent: It is cold. (Law of Detachment)

In symbols the argument may be represented as follows:

Premises: 1. $P \rightarrow Q$
2. P

Logical consequent: Q (Law of Detachment)

It should be noted that the Law of Detachment ensures that if P and $P \rightarrow Q$ are true (assumed or previously proved statements), then Q is a true statement.

The implication in the above is sometimes referred to as the *major premise* of the argument; the second statement is referred to as the *minor premise;* together the two premises form the *hypothesis* of the argument.

Example. State the logical consequent, if there is one, for each of the following sets of premises:

(i) 1. If logic is easy, then he will master it.
2. Logic is easy.

(ii) 1. If logic is easy, then he will master it.
2. He will master it.

(iii) 1. If $x - 3 = 0$ or $x + 2 = 0$, then $x = 3$ or $x = -2$.
2. $x - 3 = 0$ or $x + 2 = 0$.

(iv) 1. If $x = -2$, then $x^2 = 4$.
2. $x^2 = 4$.

Solution. (i) Logical consequent: He will master it. (Law of Detachment)

(ii) No logical consequent, since we are not given precisely the "if clause" of the implication.

(iii) Logical consequent: $x = 3$ or $x = -2$ (Law of Detachment)

(iv) No logical consequent, since we are not given precisely the "if clause" of the implication.

Exercise 8-2

(A)

In each of the following you are given a set of premises; deduce a logical consequent where possible:

1. (i) If we win the game, then we are happy.
(ii) We win the game.

2. (i) If he likes football, then he goes to the games.
(ii) He goes to the games.

3. (i) When it rains, the river rises.
(ii) It rains.

4. (i) If P, Q represent simple sentences, then P or Q is a compound sentence.
(ii) P, Q represent simple sentences.

5. (i) If $x, y \in Q$, then $x + y = 6$.
(ii) $x + y = 6$.

6. (i) If I pass my examinations, I am very happy.
(ii) I am very happy.

7. (i) $\because 3x + 7 = 28 \quad \therefore x = 7$.
(ii) $3x + 7 = 28$.

8. (i) If $x \in N$, then $(x + 9) \in N$.
(ii) $x \in N$.

9. (i) $M \rightarrow N$
(ii) M

10. (i) $M \rightarrow N$
(ii) N

11. (i) A
(ii) $A \rightarrow B$

12. (i) $\sim A \rightarrow B$
(ii) $\sim A$

13. (i) (A and B) $\rightarrow C$
(ii) A and B

14. (i) $S \rightarrow (R$ or $T)$
(ii) R or T

15. (i) (M or N) $\rightarrow O$
(ii) M or N

16. (i) $(x \neq 4) \rightarrow (x + y = 7)$.
(ii) $x \neq 4$.

17. (i) $(2x + 4 = 8) \rightarrow (x = 2)$.
(ii) $2x + 4 = 8$.

18. (i) $(x + y = 3) \rightarrow ((x + y)^2 = 9)$.
(ii) $(x + y)^2 = 9$.

19. (i) $(\angle ADB = \angle ADC = 90°) \rightarrow (AD \perp BC)$.
(ii) $\angle ADB = \angle ADC = 90°$.

20. (i) $(x > y$ and $y > z) \rightarrow (x > z)$.
(ii) $(x > y$ and $y > z)$.

21. (i) $ABCD$ is a parallelogram.
(ii) ($ABCD$ is a $\|^{gm}$) $\rightarrow$ ($AB \| DC$ and $AD \| BC$).

22. (i) P is on the right bisector of AB.
(ii) P is on the right bisector of $AB \rightarrow PA = PB$.

23. (i) If $\sim A$, then $\sim B$.
(ii) $\sim A$

24. (i) If (A and B), then Q.
(ii) A and B

25. (i) If $\sim A$, then (A and B).
(ii) A and B

26. (i) $B \rightarrow \sim A$
(ii) B

27. *Hypothesis:* $(\angle ADB = \angle ADC) \rightarrow (AB = AC)$
$AB = AC$

(B)

Write the logical consequent, if there is one, in each of the following by applying the Law of Detachment:

28. *Hypothesis:* $S \rightarrow R$
R

29. *Hypothesis:* $\sim A \rightarrow \sim B$
$\sim A$

30. *Hypothesis:* (M and N) $\rightarrow T$
M and N

31. *Hypothesis:* X
$Y \rightarrow X$

32. *Hypothesis:* $(AD \perp BC) \rightarrow (\angle ADC = 90°)$
$AD \perp BC$

33. *Hypothesis:* $(a + b = c + d$ and $a = c) \rightarrow (b = d)$
$b = d$

34. *Hypothesis:* ($ABCD$ is a parallelogram) $\rightarrow$ ($AB \| DC$ and $AD \| BC$).
$ABCD$ is a parallelogram.

35. *Hypothesis:* (l_1 intersects l_2) $\rightarrow$ ($\angle 1 = \angle 2$)
l_1 intersects l_2.

36. *Hypothesis:* ($AD = DB$ and $DE \| BC$) $\rightarrow$ ($AE = EC$)
$AE = EC$

37. *Hypothesis:* ($AD = BC$ and $AD \| BC$) $\rightarrow$ ($AB = DC$)
$AD = BC$ and $AD \| BC$

38. *Hypothesis:* $(\angle ADB = \angle ADC = 90°) \rightarrow (\angle BCD = 180°)$
$\angle ADB = \angle ADC$

39. *Hypothesis:* ($\angle 1 = \angle 2$ and l_1 is a line) $\rightarrow$ (l_2 is a line).
l_2 is a line.

40. *Hypothesis:* ($l_1 \parallel l_2$ and $l_2 \parallel l_3$) $\rightarrow$ ($l_1 \parallel l_3$)
$l_1 \parallel l_2$ and $l_2 \parallel l_3$

41. *Hypothesis:* ($\angle 1 = \angle 2$) $\rightarrow$ ($l_1 \parallel l_2$)
$\angle 1 = \angle 2$

42. *Hypothesis:* ($ABCD$ is a quadrilateral inscribed in a circle)
$\rightarrow$ ($\angle A + \angle C = 180°$).
$\angle A + \angle B = 180°$

43. *Hypothesis:* ($AB = r_1 + r_2$) $\rightarrow$ (two circles touch externally).
$AB = r_1 + r_2$

44. *Hypothesis:* $\left(\frac{AD}{DB} = \frac{AE}{EC}\right) \rightarrow (DE \parallel BC)$

$$\frac{AE}{EC} = \frac{AD}{DB}$$

***8·3* Deductive proofs involving several steps.** Most deductive arguments consist of more than one application of the Law of Detachment. This is illustrated in the following example.

Premises: 1. If it is snowing, then it is cold.
2. It is snowing.
3. If it is cold, then I will stay at home.

Conclusion: I will stay at home.

This conclusion may be shown to be a logical consequent of the premises by applying the Law of Detachment twice. Thus, from statements 1 and 2 we obtain the conclusion, *It is cold*, by a first application of the law. Then by combining this conclusion with statement 3, the Law of Detachment brings us to the logical consequent, *I will stay at home.*

A formal deductive proof involving more than one application of the Law of Detachment follows:

Hypothesis: 1. $P \rightarrow Q$
2. P
3. $Q \rightarrow R$

Conclusion: R

Proof:

STATEMENTS	AUTHORITIES
1. $P \rightarrow Q$	1. Hypothesis
2. P	2. Hypothesis
3. Q	3. Law of Detachment 1, 2
4. $Q \rightarrow R$	4. Hypothesis
5. R	5. Law of Detachment 3, 4

A formal deductive proof is a succession of statements which lead to the desired conclusion. Each statement is either a premise or a statement derived directly by a rule of inference. The proof is arranged in two columns: *statements* in the first column and *authorities* for these statements in the second column. The process of building a formal deductive proof is called *synthesis.*

Example 1. Complete the following formal deductive proof; compare your proof with the complete proof on page 483.

Hypothesis: $A \to B$, $B \to C$, $C \to D$, A

Conclusion: D

Proof:

STATEMENTS	AUTHORITIES
1. $A \to B$	1.
2. A	2.
3. B	3.
4. $B \to C$	4.
5. C	5.
6. $C \to D$	6.
7. D	7.

Exercise 8-3

(A)

Complete the authorities column of each of the following formal deductive proofs:

1. *Hypothesis:* A, $A \to (\sim B)$, $(\sim B) \to C$

Conclusion: C

Proof:

STATEMENTS	AUTHORITIES
1. A	1.
2. $A \to (\sim B)$	2.
3. $(\sim B)$	3.
4. $(\sim B) \to C$	4.
5. C	5.

2. *Hypothesis:* $AB = AC \rightarrow \angle B = \angle C$, $(\angle B = \angle C) \rightarrow (\angle B = 72°)$
$(\angle B = 72°) \rightarrow (\angle A = 36°)$, $AB = AC$

Conclusion: $\angle A = 36°$

Proof:

STATEMENTS	AUTHORITIES
1. $AB = AC$	1.
2. $(AB = AC) \rightarrow (\angle B = \angle C)$	2.
3. $\angle B = \angle C$	3.
4. $(\angle B = \angle C) \rightarrow (\angle B = 72°)$	4.
5. $\angle B = 72°$	5.
6. $\angle B = 72° \rightarrow \angle A = 36°$	6.
7. $\angle A = 36°$	7.

(B)

Write a formal proof for each of the following:

3. *Hypothesis:* $A \rightarrow B$, $B \rightarrow C$, A, $C \rightarrow D$, $D \rightarrow E$
Conclusion: E

4. *Hypothesis:* $P \rightarrow (Q \text{ and } R)$, $(Q \text{ and } R) \rightarrow S$, P
Conclusion: S

5. *Hypothesis:* $(A \text{ and } B)$, $(A \text{ and } B) \rightarrow C$, $C \rightarrow \sim D$
Conclusion: $\sim D$

6. *Hypothesis:* $S \rightarrow \sim T$, $\sim T \rightarrow R$, S
Conclusion: R

7. *Hypothesis:* $(\sim P) \rightarrow Q$, $Q \rightarrow (\sim R)$, $(\sim P)$
Conclusion: $\sim R$

8. *Hypothesis:* $\sim A \rightarrow \sim B$, $C \rightarrow \sim A$, C
Conclusion: $\sim B$

9. *Hypothesis:* $(a = b \text{ and } b = c) \rightarrow (a = c)$
$(a = c) \rightarrow (c = a)$, $(a = b \text{ and } b = c)$
Conclusion: $c = a$

10. Symbolize the following sentences and write a proof to justify the conclusion.
Hypothesis: 1. If humid air rises, it cools.
2. When it cools, clouds form.
3. Humid air rises.
Conclusion: Clouds form.

8·4 Analysis of a deduction. *Analysis* is the method used to discover the plan for a formal proof. It works in *reverse* order to synthesis. We begin with what we are trying to prove and work back to the given conditions, as illustrated in the following example.

Hypothesis: $P \rightarrow Q,\ Q \rightarrow R,\ P$

Conclusion: R

Analysis:

I CAN PROVE	IF I CAN PROVE
1. R	1. Q ($\because\ Q \rightarrow R$) .
2. Q	2. P ($\because\ P \rightarrow Q$) .
3. P by hypothesis.	

The statements contained in the analysis taken in reverse order are the statements of the formal proof; thus,

Proof:

STATEMENTS	AUTHORITIES
1. P	1. Hypothesis
2. $P \rightarrow Q$	2. Hypothesis
3. Q	3. Law of Detachment 1, 2
4. $Q \rightarrow R$	4. Hypothesis
5. R	5. Law of Detachment 3, 4

The process of analysis becomes increasingly valuable as the number of the steps in the proof and the number of premises increase. When a given hypothesis contains many statements, analysis enables us to recognize those which are relevant to the proof.

Example 1. Read the analysis of the following deduction; write a complete formal proof and compare it with the proof on page 484.

Hypothesis: $(A \text{ and } B) \rightarrow C,\ C \rightarrow D,\ C \rightarrow (D \text{ or } E)$
$C \rightarrow E,\ A,\ B,\ (A \text{ and } B)$

Conclusion: D or E

Analysis:

I CAN PROVE	IF I CAN PROVE
1. D or E	1. C ($\because\ C \rightarrow (D \text{ or } E)$) .
2. C	2. A and B ($\because\ (A \text{ and } B) \rightarrow C$) .
3. $(A \text{ and } B)$ by hypothesis .	

Example 2. Write a complete analysis and proof of the following deduction; compare your solution with that on page 484.

Hypothesis: $\sim C \rightarrow D,\ C \rightarrow \sim D,\ \sim E \rightarrow \sim C,$
$D \rightarrow B,\ E \rightarrow C,\ \sim E$

Conclusion: B

Exercise 8-4

(B)

Write a complete analysis and proof of each of the following deductions:

1. *Hypothesis:* $C \rightarrow \sim D$, $\sim D \rightarrow B$, $B \rightarrow A$, C
   Conclusion: A
2. *Hypothesis:* $B \rightarrow \sim C$, B, $\sim C \rightarrow D$, $D \rightarrow \sim A$
   Conclusion: $\sim A$
3. *Hypothesis:* (A or B), (A or B) $\rightarrow \sim D$,
   (E and $\sim F$) $\rightarrow$ (G or H), $\sim D \rightarrow$ (E and $\sim F$)
   Conclusion: G or H
4. *Hypothesis:* $\sim Q \rightarrow R$, $R \rightarrow S$, $\sim Q$, $S \rightarrow \sim T$, $U \rightarrow P$, $\sim T \rightarrow U$
   Conclusion: P

***8·5* The syllogism.** Examination of the following statements:

1. *All persons who live in Toronto live in Ontario.*
2. *John lives in Toronto.*

leads to the conclusion: *John lives in Ontario.*

This conclusion appears to be a necessary consequent of statements 1 and 2. However, the Law of Detachment cannot be used to justify a conclusion unless we are given an implication and precisely its "if clause".

In logic, the statement, *All persons who live in Toronto live in Ontario,* has the meaning, *For all persons, if a person lives in Toronto, then the person lives in Ontario,* which is an implication. The statement, *John lives in Toronto,* is not precisely the "if clause" of the implication; it is a particular case of the "if clause". Thus, it is not possible to go directly to the conclusion, *John lives in Ontario,* by simply applying the Law of Detachment. To reach the conclusion logically, we must first agree that if an implication applies to all members of a set, it applies to any specified member.

An argument of this form, called a *syllogism,* involves three steps:

(a) *Major premise:* a general implication
(b) *Minor premise:* a particular case of (a)
(c) *Conclusion:* the logical consequent of the general and particular statements.

It should be noted that if the major and minor premises are true (accepted) statements, the logical consequent is a true statement.

Example. State the logical consequent, if there is one, in each of the following; compare your solutions with those on page 484.

(i) **Hypothesis:** 1. All birds fly.
2. Polly is a bird.

(ii) **Hypothesis:** 1. All right angles are equal angles.
2. A and B are right angles.

(iii) **Hypothesis:** 1. All right angles are equal angles.
2. A and B are equal angles.

(iv) **Hypothesis:** 1. Any acute angle is less than 90°.
2. $\angle P$ is an acute angle.

(v) **Hypothesis:** 1. If a line joins the midpoints of two sides of a triangle, then it is parallel to the third side and equal to one-half its length.

2.

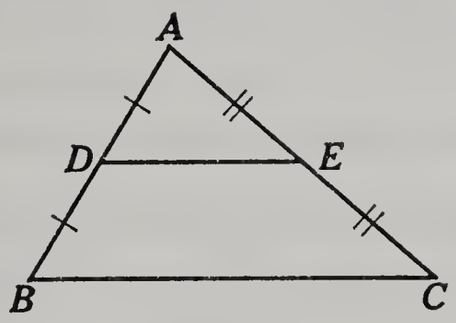


Fig. 8-1

(vi) **Hypothesis:** 1. If two angles have an angle sum of 90°, then the two angles are complementary angles.

2.

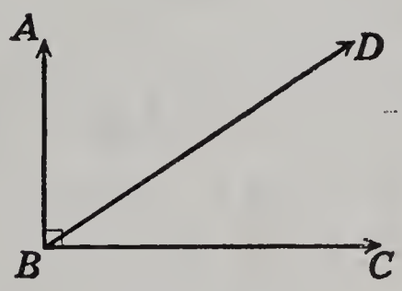


Fig. 8-2

(vii) **Hypothesis:** 1. If two angles have an angle sum of 180°, then the two angles are supplementary angles.

2.

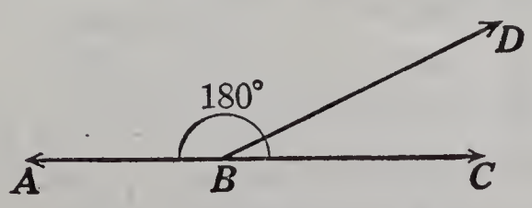


Fig. 8-3

Exercise 8-5

(A)

State each of the following conditional sentences in "if . . . then. . . " form:

1. A statement which is accepted as true without proof is an assumption.
2. The logical consequent of two accepted premises is called a deduction.
3. When we say a statement is true, we mean it is an assumption or it has been proved.
4. A statement is proved when it is the logical consequent of two accepted premises.
5. A statement is a logical consequent of two statements if it is obtained from the two statements by applying a rule of inference.
6. Deductive reasoning is the process of applying a rule of inference.
7. A logical consequent is deduced from two accepted premises.

State the logical consequent, if there is one, in each of the following:

8. *Hypothesis:* 1. If John lives in Toronto, then he lives in Ontario.
   2. John lives in Toronto.
9. *Hypothesis:* 1. If John lives in Toronto, then he lives in Ontario.
   2. John lives in Ontario.
10. *Hypothesis:* 1. If $a, b, c \in N$, then $(a+b) + c = a + (b + c)$.
    2. $a, b, c \in N$
11. *Hypothesis:* 1. If $2x + 3 = 7$, then $x = 2$.
    2. $x = 2$
12. *Hypothesis:* 1. If $\frac{a}{b}, \frac{c}{d} \in Q$, then $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.
    2. $\frac{a}{b} = \frac{3}{5}, \quad \frac{c}{d} = \frac{4}{3}$

Assess the conclusion in each of the following as valid if it follows as a logical consequent of the given statements, or not valid if it does not necessarily follow:

13. *Hypothesis:* (i) If a church is white, then it is painted.
    (ii) St. Paul's Church is white.
    Conclusion: St. Paul's Church is painted.
14. *Hypothesis:* (i) If a church is white, then it is painted.
    (ii) St. Andrew's Church is painted.
    Conclusion: St. Andrew's Church is white.

15. *Hypothesis:* (i) If a church is white, then it is painted.
(ii) St. Mark's is a church.
Conclusion: St. Mark's is painted white.

16. *Hypothesis:* (i) All good singers give a recital.
(ii) Mary is giving a recital.
Conclusion: Mary is a good singer.

17. *Hypothesis:* (i) All children are admitted free to the circus.
(ii) John was admitted free to the circus.
Conclusion: John is a child.

(B)

Rate the argument in each of the following as valid if the conclusion is a logical consequent, or not valid if it is not a logical consequent of the given premises (i) *and* (ii):

18. *Hypothesis:* (i) All real numbers x, y satisfy the equation $x+y=y+x$.
(ii) $-\frac{4}{6}\frac{7}{2}, \sqrt{2}$ are real numbers.
Conclusion: $-\frac{4}{6}\frac{7}{2}, \sqrt{2}$ satisfy the equation.

19. *Hypothesis:* (i) All irrational numbers are real numbers.
(ii) $\sqrt{2}$ is a real number.
Conclusion: $\sqrt{2}$ is an irrational number.

20. *Hypothesis:* (i) All policemen must be at least 5 ft. 10 in. tall.
(ii) John Law is a policeman.
Conclusion: John Law is 5 ft. 10 in. tall.

21. *Hypothesis:* (i) All dogs disturb their owners at night.
(ii) Rex disturbs his owner at night.
Conclusion: Rex is a dog.

22. *Hypothesis:* (i) All angles can be trisected.
(ii) ABC is an angle.
Conclusion: $\angle ABC$ can be trisected.

In each of the following, state the logical consequent, if there is one:

23. *Hypothesis:* (i) All animals have four legs.
Conclusion: (ii) A monkey is an animal.

24. *Hypothesis:* (i) If an angle has a measurement of 180°, then it is a straight angle.
Conclusion: (ii) $\angle ABC$ has a measurement of 180°.

25. *Hypothesis:* (i) All acute angles have measurements less than 90°.
Conclusion: (ii) $\angle DEF$ is an acute angle.

26. *Hypothesis:* (i) All vertical angles are equal.
Conclusion: (ii) $\angle ABC = \angle DEF$

27. *Hypothesis:* (i) All vertical angles are equal.
Conclusion: (ii) $\angle ABC$ and $\angle DEF$ are vertical angles.

28. *Hypothesis:* (i) All sets of points are geometric figures.
Conclusion: (ii) A single point is a set of points.

29. *Hypothesis:* (i) If a figure has four equal sides, then it is a rhombus.
Conclusion: (ii) A square has four equal sides.

30. *Hypothesis:* (i) If $\frac{a}{b}, \frac{c}{d} \in Q$, then $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.

Conclusion: (ii) $\frac{-2}{-3}, \frac{-4}{+5} \in Q$

***8·6* Logical consequents deduced from geometric definitions, postulates, and number axioms.** In geometry, the definitions, postulates, and number axioms are the basic implications from which geometric relationships are deduced by logical reasoning.

The following is a list of postulates and number axioms used frequently as premises for geometric deductions.

a. *Existence Postulates.*

(i) There exists one and only one midpoint of a line segment.

(ii) There exists one and only one bisector of an angle.

(iii) There exists an unlimited number of line segments equal to a given line segment.

(iv) There exists an unlimited number of angles equal to a given angle.

(v) There exists a real number for each line segment which represents, in some unit, the length of the line segment.

(vi) There exists a unique real number x for each angle which represents the number of degrees in the angle where $0 < x \leqq 180$.

b. *Completion Postulates.*

(i) If B is between A and C, then

$AB + BC = AC$
$AC - BC = AB$
$AC - AB = BC$
$AC > AB$
$AC > BC$.

Fig. 8-4

(ii) If ray AC is between rays AB and AD, then

$\angle BAD = \angle BAC + \angle CAD$

$\angle BAD - \angle CAD = \angle BAC$

$\angle BAD - \angle BAC = \angle CAD$

$\angle BAD > \angle BAC$

$\angle BAD > \angle CAD$.

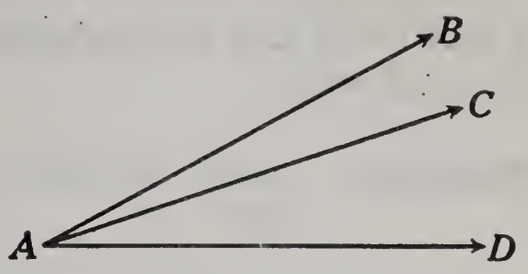


Fig. 8-5

c. *Number Axioms.*

(i) Properties of the real number system.

If $a, b, c \in R$, then

	ADDITION	MULTIPLICATION
Closure Law	$a + b$ *represents one and only one real number* (*ClA*)	ab *represents one and only one real number* (*ClM*)
Commutative Law	$a + b = b + a$ (*CA*)	$ab = ba$ (*CM*)
Associative Law	$(a+b)+c = a+(b+c)$ (*AA*)	$(ab)c = a(bc)$ (*AM*)
Distributive Law	$a(b + c) = ab + ac$ (*D*)	
Identity Elements	*Zero*	*Unity*
Inverse Elements	$-a$	$\frac{1}{a}$, $a \neq 0$

(ii) Properties of the equality relation.

$a = a$.	(Reflexive property)
If $a = b$, then $b = a$.	(Symmetric property)
If $a = b$ and $b = c$, then $a = c$.	(Transitive property)

(iii) Properties resulting from the Closure Law.

If $a = b$, then $a + c = b + c$.	(Addition property)
If $a = b$, then $a - c = b - c$.	(Subtraction property)
If $a = b$, then $ac = bc$.	(Multiplication property)
If $a = b$, then $\frac{a}{c} = \frac{b}{c}$, $c \neq 0$.	(Division property)
If $a = b$, then a may be replaced by b.	(Replacement property)

(iv) Order properties.

Either $a = b$ or $a \neq b$.	(Dichotomy property)
Either $a < b$ or $a = b$ or $a > b$.	(Trichotomy property)
If $a > b$ and $b > c$, then $a > c$.	(Transitive property)

Example 1. Examine the proof of the following geometric deduction; complete the second column; compare it with that on page 484.

If in $\triangle ABC$ (*Fig. 8-6*), D is in AB, and E is in AC such that $AD = AE$ and $DB = EC$, then $AB = AC$.

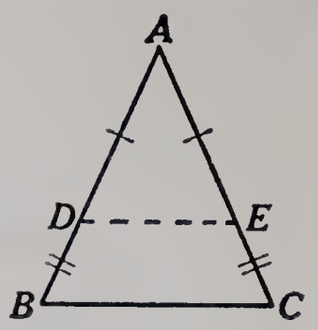


Fig. 8-6

Hypothesis: $\triangle ABC$ in which $AD = AE$, $DB = EC$

Conclusion: $AB = AC$

Proof:

STATEMENTS	AUTHORITIES
1. $AD = AE$	1. HYP
2. $DB = EC$	2. HYP
3. $AD + DB = AE + EC$	3. Addition
4. $AB = AC$	4. Completion

It should be noted that in line 3 of the proof, the conclusion, $AD + DB = AE + EC$, is obtained by recognizing that statements 1 and 2 provide a particular case to which the addition property may be applied.

The process of building a proof of a deduction is called *synthesis.*

Examine the proof of the geometric deductions in each of the following examples; complete the second column in each case and compare your solutions with those on page 484.

Example 2. In *Fig. 8-7*, if $AD = BC$ and $BC = EF$, then $AD = EF$.

Hypothesis: $AD = BC$ and $BC = EF$

Conclusion: $AD = EF$

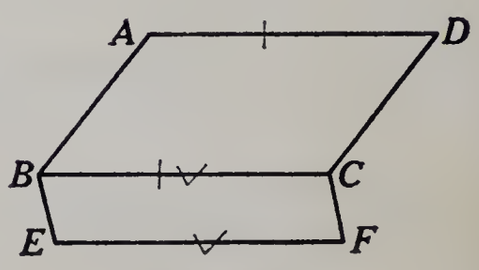


Fig. 8-7

Proof:

STATEMENTS	AUTHORITIES
1. $AD = BC$	1.
2. $BC = EF$	2.
3. $AD = EF$	3. TRANSITIVE REPLACEMENT

Example 3. In *Fig. 8-8*, if $AB = AC$ and $DB = EC$, then $AD = AE$.

Hypothesis: In *Fig. 8-8*, $AB = AC$ and $DB = EC$.

Conclusion: $AD = AE$

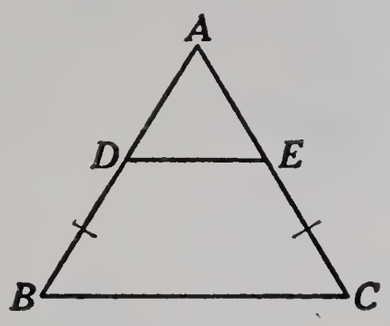


Fig. 8-8

Proof:

STATEMENTS	AUTHORITIES
1. $AB = AC$	1.
2. $DB = EC$	2.
3. $AB - DB = AC - EC$	3.
4. $AD = AE$	4.

Example 4. In *Fig. 8-9*, if $\angle 1 = \angle 2 = \angle 3 = 30°$, then $\angle ABC = 90°$.

Hypothesis: *Fig. 8-9*, in which $\angle 1 = \angle 2 = \angle 3 = 30°$

Conclusion: $\angle ABC = 90°$

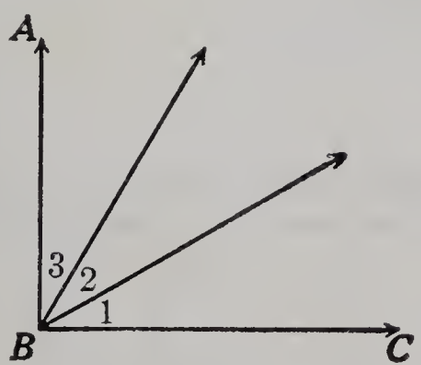


Fig. 8-9

Proof:

STATEMENTS	AUTHORITIES
1. $\angle ABC = \angle 1 + \angle 2 + \angle 3$	1.
2. $\angle 1 = \angle 2 = \angle 3 = 30°$	2.
3. $\angle ABC = (30 + 30 + 30)°$	3.
4. $\angle ABC = 90°$	4.

Example 5. In *Fig. 8-10*, if $\angle 1 + \angle 3 = 180°$ and $\angle 1 = \angle 2$, then $\angle 2 + \angle 3 = 180°$.

Hypothesis: In *Fig. 8-10*, $\angle 1 + \angle 3 = 180°$ and $\angle 1 = \angle 2$.

Conclusion: $\angle 2 + \angle 3 = 180°$

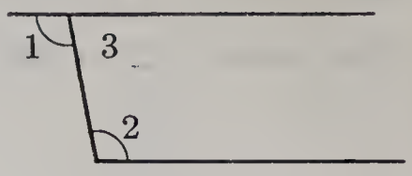


Fig. 8-10

Proof:

STATEMENTS	AUTHORITIES
1. $\angle 1 + \angle 3 = 180°$	1.
2. $\angle 1 = \angle 2$	2.
3. $\angle 2 + \angle 3 = 180°$	3.

Exercise 8-6

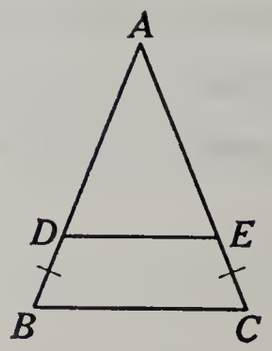
(A)

In each of the following proofs, state the authority required to justify each statement:

1. *Hypothesis:* $\triangle ABC$ in which $AB = AC$, $BD = CE$

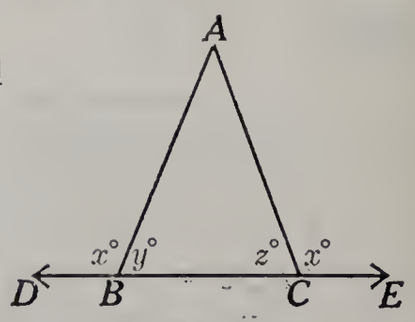
Conclusion: $AD = AE$

Proof:



STATEMENTS	AUTHORITIES
1. $AB = AC$	1.
2. $BD = CE$	2.
3. $AB - BD = AC - CE$	3.
4. $AD = AE$	4.

2. *Hypothesis:* $\triangle ABC$ in which BC is extended to E, CB is extended to D, $\angle ABD = \angle ACE = x°$, $\angle ABC = y°$, and $\angle ACE = z°$



Conclusion: $y = z$

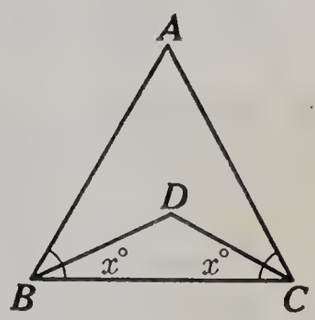
Proof:

STATEMENTS	AUTHORITIES
1. $\angle DBC = 180°$	1.
2. $x + y = 180$	2.
3. $\angle BCE = 180°$	3.
4. $z + x = 180$	4.
5. $x + y = z + x$	5.
6. $y = z$	6.

In each of the following, give the statements which should appear in the statement column opposite each of the given authorities:

3. *Hypothesis:* $\triangle ABC$ in which $\angle ABC = \angle ACB$ and $\angle DBC = \angle DCB$

Conclusion: $\angle ABD = \angle ACD$

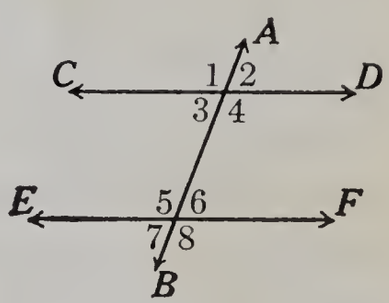


Proof:

STATEMENTS	AUTHORITIES
1. $\angle ABC = \angle ACB$	1. Hypothesis
2. $\angle DBC = \angle DCB$	2. Hypothesis
3. $\angle ABD = \angle ACD$	3. Subtraction

4. *Hypothesis:* Line AB intersecting lines CD and EF so that $\angle 4 + \angle 6 = 180°$

Conclusion: $\angle 3 = \angle 6$



Proof:

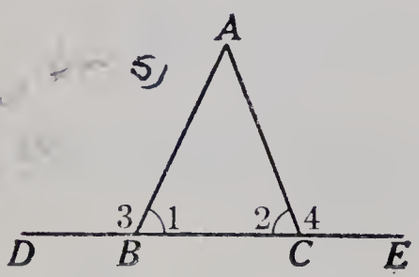
STATEMENTS	AUTHORITIES
1. $\angle 3 + \angle 4 = 180°$	1. Definition (straight angle)
2. $\angle 4 + \angle 6 = 180°$	2. Hypothesis
3. $\angle 3 + \angle 4 = \angle 4 + \angle 6$	3. Replacement
4. $\angle 3 = \angle 6$	4. Subtraction

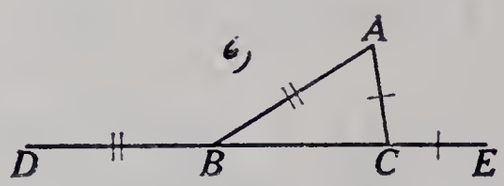
(B)

Write a complete proof of the following deductions:

5. *Hypothesis:* $\triangle ABC$ having $\angle 1 = \angle 2$, BC extended to E forming straight angle BCE, CB extended to D forming straight angle CBD (left below)

   Conclusion: $\angle 3 = \angle 4$





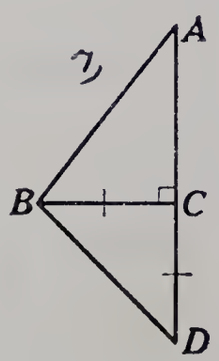
6. *Hypothesis:* $\triangle ABC$, at the right above, in which BC is extended to E so that $AC = CE$, CB is extended to D so that $BD = BA$

   Conclusion: The perimeter of $\triangle ABC$ is equal to DE.

7. *Hypothesis:* Right triangle ABC, at the right, in which AC is extended to D so that $CD = CB$

   Conclusion: (i) $\angle BCD = 90°$

   (ii) $AC + CB = AD$



***8·7* Analysis of geometric deductions.** In section 8·4 we discussed analysis as the method used to discover the plan for a formal proof. Analysis works in reverse order to synthesis. We begin with what we are trying to prove and work back to the given conditions, as illustrated in the following examples.

Examine the analysis of each of the following deductions; write complete two-column proofs and compare them with those on page 485.

Example 1. From point B in straight angle ABC (*Fig. 8-11*), rays BE, BD, and BF are drawn so that $\angle ABE = \angle EBD = x°$, $\angle DBF = \angle FBC = y°$, then $\angle EBF = 90°$.

Hypothesis: $\angle ABC = 180°$
$\angle ABE = \angle EBD = x°$
$\angle DBF = \angle FBC = y°$

Conclusion: $\angle EBF = 90°$

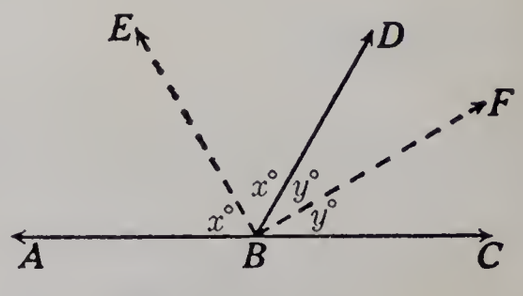


Fig. 8-11

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle EBF = 90°$	1. $x + y = 90$.
2. $x + y = 90$	2. $2x + 2y = 180$.
3. $2x + 2y = 180$	3. $\angle ABC = 180°$.
4. $\angle ABC = 180°$ by hypothesis.	

Example 2.

Hypothesis: $\triangle ABC$, *Fig. 8-12*, with $\angle 1 = \angle 2$. AB is extended to D, forming straight angle ABD; AC is extended to E, forming straight angle ACE.

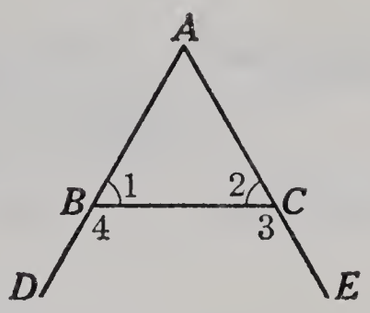


Fig. 8-12

Conclusion: $\angle 4 = \angle 3$

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle 4 = \angle 3$	1. $\angle 4 + \angle 1 = \angle 3 + \angle 2$.
2. $\angle 4 + \angle 1 = \angle 3 + \angle 2$	2. $\angle ABD = \angle ACE$.
3. $\angle ABD = \angle ACE$ by hypothesis.	

Write a two-column analysis and a two-column proof of the following deduction; compare your solution with that on page 485.

Example 3. If O is the centre of two concentric circles (circles with the same centre and different radii), and OAB and OCD are line segments with A, C, B, D on the circles, as shown in *Fig. 8-13*, then $AB = CD$.

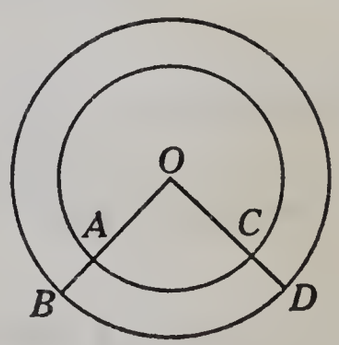


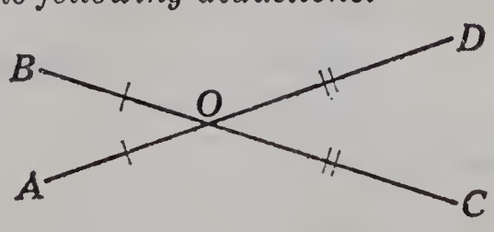
Fig. 8-13

Exercise 8-7

(A)

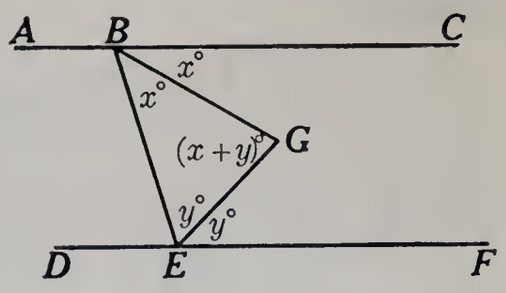
Give an oral analysis for each of the following deductions:

1. *Hypothesis:* $OA = OB$
   $OC = OD$
   Conclusion: $AD = BC$



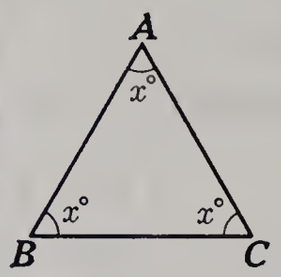
2. *Hypothesis:* $\angle CBE + \angle BEF = 180°$
$\angle CBG = \angle GBE = x°$
$\angle BEG = \angle GEF = y°$
$\angle BGE = (x + y)°$

Conclusion: $\angle BGE = 90°$



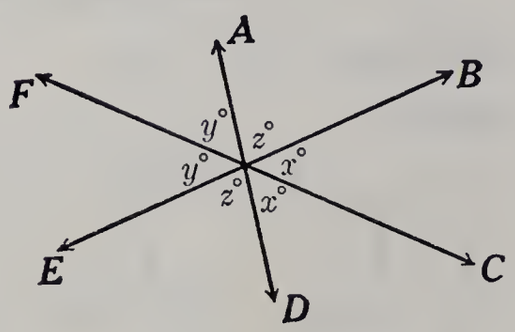
3. *Hypothesis:* $\triangle ABC$ in which
$\angle A = \angle B = \angle C = x°$ and
$\angle A + \angle B + \angle C = 180°$

Conclusion: $\angle A = 60°$



4. *Hypothesis:* Rays forming angles with the measurements indicated

Conclusion: $\angle FOC = 180°$

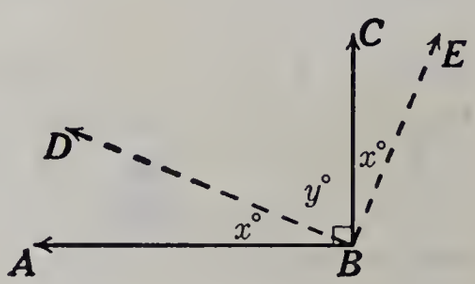


(B)

Write a proof for each of the following deductions:

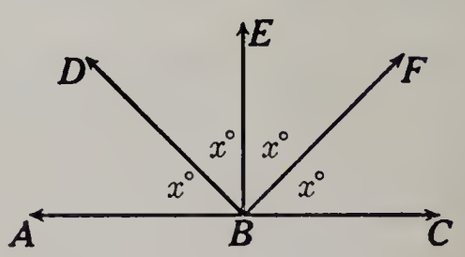
5. *Hypothesis:* $\angle ABC = 90°$
$\angle ABD = \angle CBE = x°$

Conclusion: $\angle DBE = 90°$



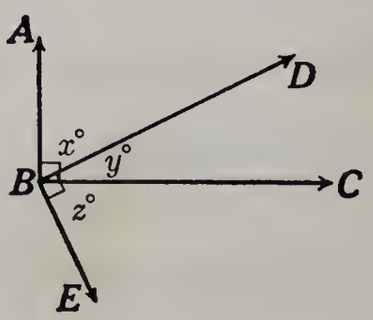
6. *Hypothesis:* $\angle ABC$ is a straight angle.
$\angle ABD = \angle DBE$
$= \angle EBF = \angle FBC = x°$.

Conclusion: $x = 45$



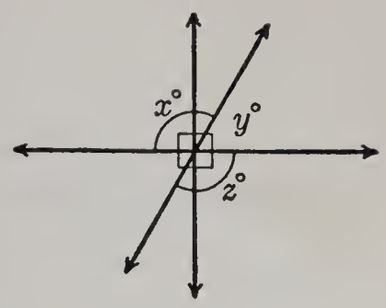
7. *Hypothesis:* Rays from B forming angles with the measurements indicated

Conclusion: $x = z$



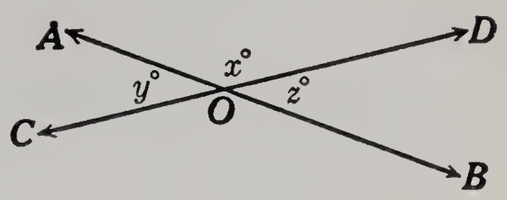
8. *Hypothesis:* Rays forming angles with the measurements indicated

Conclusion: $x = z$



9. *Hypothesis:* Rays from O such that $\angle AOB = \angle COD$

Conclusion: $y = z$



10. *Hypothesis:* $A \rightarrow B,\ B \rightarrow \sim C,\ A, \sim C \rightarrow D$

Conclusion: D

Determine by analysis if the conclusion in each of the following is a logical consequent of the premises:

11. *Hypothesis:* $P,\ Q \rightarrow \sim P,\ S \rightarrow \sim R, \sim R \rightarrow Q$

Conclusion: S

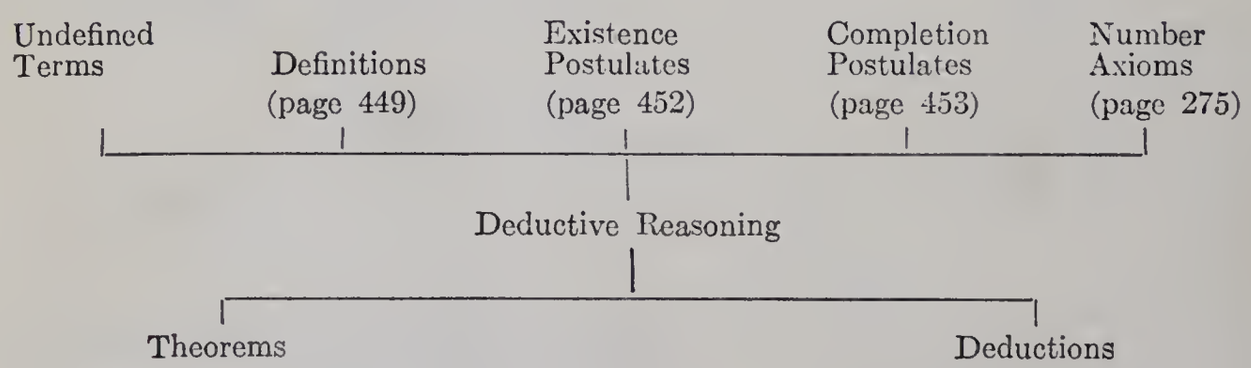
12. *Hypothesis:* $\sim A,\ B \rightarrow \sim D, \sim C \rightarrow B, \sim D \rightarrow \sim A$

Conclusion: B

Chapter IX

FUNDAMENTAL ANGLE PAIR THEOREMS

***9·1* A logical system of geometry.** Now that the basic principles of deductive reasoning have been studied, it is possible to begin to organize our geometric knowledge into a logical system in which each statement is derived from preceding statements by deductive reasoning. The beginning ideas and statements in our system are undefined terms, definitions, postulates, and number axioms. Conclusions deduced by deductive reasoning from these beginning statements are called *deductions*. The generalizations expressed in these deductions may be used in turn as the major premise for further logical deductions. However, the conclusions of some deductions are used to deduce further conclusions more frequently than others; such deductions are called *theorems*. The following diagram illustrates the structure of a logical system.



(General conclusions which are particularly useful in deducing further conclusions)

9·2 Complementary Angle Theorem.

THEOREM 1

Complementary Angle Theorem

If angles are complements of equal angles, then they are equal.

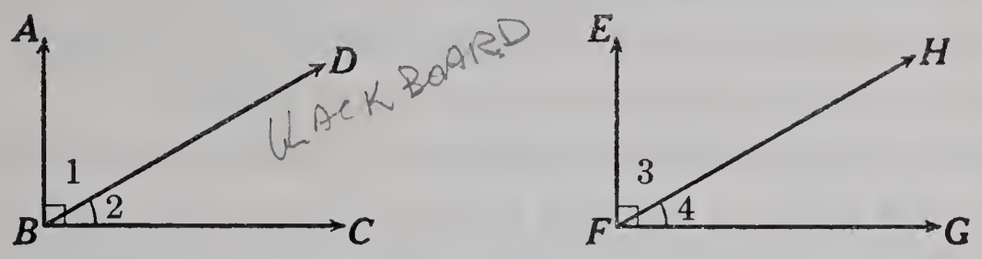


Fig. 9-1

Hypothesis: $\angle ABC = \angle EFG = 90°$, $\angle 2 = \angle 4$, $\angle 1$ is the complement of $\angle 2$, $\angle 3$ is the complement of $\angle 4$ (*Fig. 9-1*).

Conclusion: $\angle 1 = \angle 3$

Proof:

STATEMENTS	AUTHORITIES
1. $\angle ABC = \angle EFG$	1. Hypothesis
2. $\angle 2 = \angle 4$	2. Hypothesis
3. $\angle ABC - \angle 2 = \angle EFG - \angle 4$	3. Subtraction
4. $\angle 1 = \angle 3$	4. Completion

DEFINITION: *A corollary of a theorem is an implication which follows as a direct consequent of the theorem.*

Corollary: If angles are complements of the same angle, then they are equal.

Hypothesis: *Fig. 9-2*, in which $\angle 2$ is the complement of $\angle 1$, $\angle 3$ is the complement of $\angle 1$

Conclusion: $\angle 2 = \angle 3$

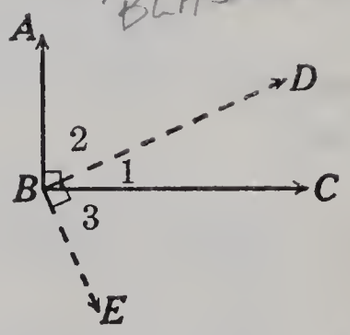


Fig. 9-2

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle 2 = \angle 3$	1. $\angle 2$ and $\angle 3$ are complements of equal angles.
2. $\angle 2$ and $\angle 3$ are complements of $\angle 1$ by hypothesis.	

Proof: Write a complete proof; compare your proof with that on page 486.

9·3 The parts of a demonstration of proof of a theorem or deduction. A geometric demonstration of proof of a theorem or deduction consists of the following five parts:

1. The conditional statement of the theorem or deduction. This statement is sometimes referred to as the *general enunciation* of the theorem.
2. A diagram which illustrates the ideas involved in the theorem.
3. Statement of the hypothesis in terms of the diagram.
4. Statement of the conclusion to be proved in terms of the diagram.
5. The proof which is arranged in two columns. The statements are listed in order in the left column. The authorities are given in the right column. Each authority is numbered to correspond to the statement it justifies.

9·4 Supplementary Angle Theorem.

THEOREM 2

Supplementary Angle Theorem

If angles are supplements of equal angles, then they are equal.

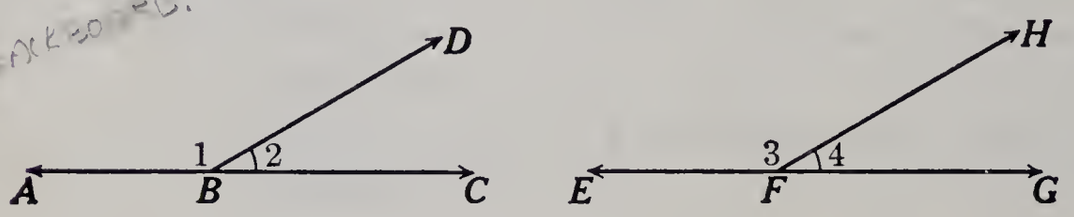


Fig. 9-3

Hypothesis: $\angle ABC = \angle EFG = 180°$, $\angle 2 = \angle 4$, $\angle 1$ is the supplement of $\angle 2$, $\angle 3$ is the supplement of $\angle 4$ (*Fig. 9-3*).

Conclusion: $\angle 1 = \angle 3$

Proof:

STATEMENTS	AUTHORITIES
1. $\angle ABC = \angle EFG$	1. Hypothesis
2. $\angle 2 = \angle 4$	2. Hypothesis
3. $\angle ABC - \angle 2 = \angle EFG - \angle 4$	3. Subtraction
4. $\angle 1 = \angle 3$	4. Completion

Corollary: If angles are supplements of the same angle, then they are equal.

Hypothesis: $\angle 2$ is the supplement of $\angle 1$, $\angle 3$ is the supplement of $\angle 1$ (*Fig. 9-4*).

Conclusion: $\angle 2 = \angle 3$

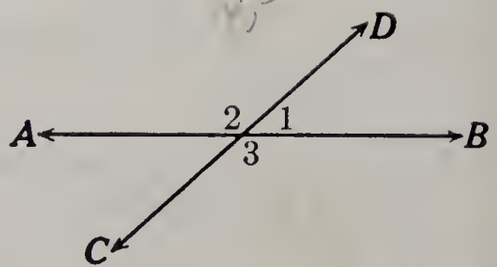


Fig. 9-4

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle 2 = \angle 3$	1. $\angle 2$ and $\angle 3$ are supplements of equal angles.
2. $\angle 2$ and $\angle 3$ are supplements of $\angle 1$ by hypothesis.	

Proof: Write a complete proof; compare your proof with that on page 486.

Example. Examine the analysis of the following deduction; write a complete proof and compare your proof with that on page 486.

Hypothesis: $\angle ABC = 180°$
$\angle ABD = \angle CBD = 90°$
$\angle ABE = \angle CBF = y°$
$\angle EBD = (2x - 60)°$
$\angle FBD = (x + 10)°$

Conclusion: $\angle ABE = 10°$

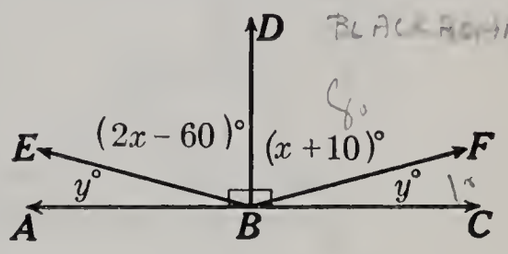


Fig. 9-5

Analysis:

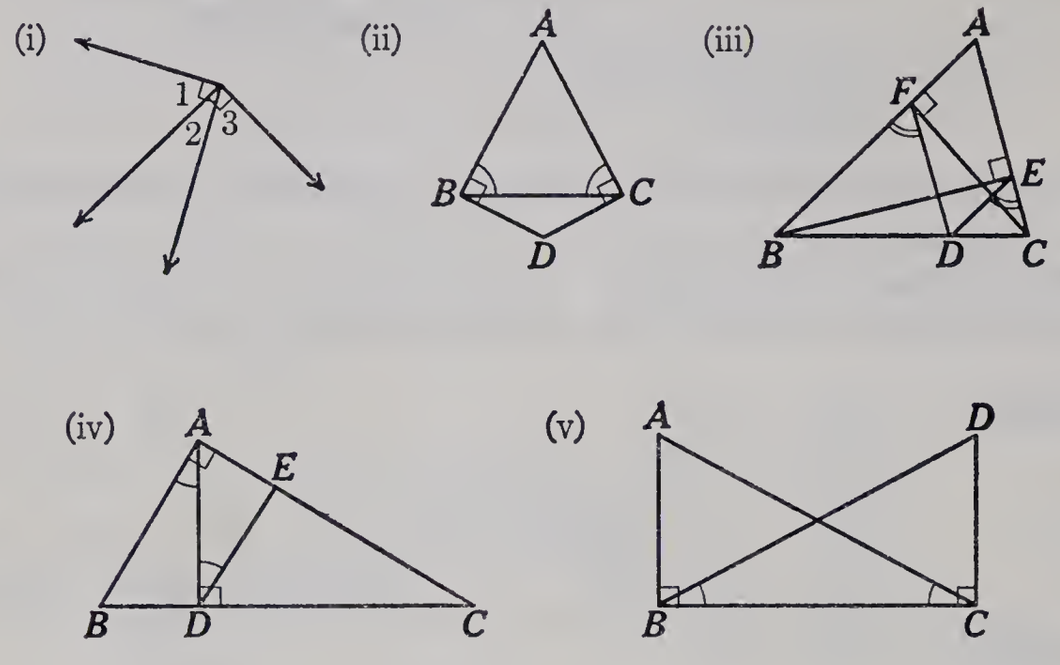
I CAN PROVE	IF I CAN PROVE
1. $\angle ABE = 10°$	1. $y = 10$.
2. $y = 10$	2. $x + 10 = 80$.
3. $x + 10 = 80$	3. $x = 70$.
4. $x = 70$	4. $2x - 60 = x + 10$.
5. $2x - 60 = x + 10$	5. $\angle ABE = \angle CBF$.
6. $\angle ABE = \angle CBF$ by hypothesis.	

Exercise 9-1

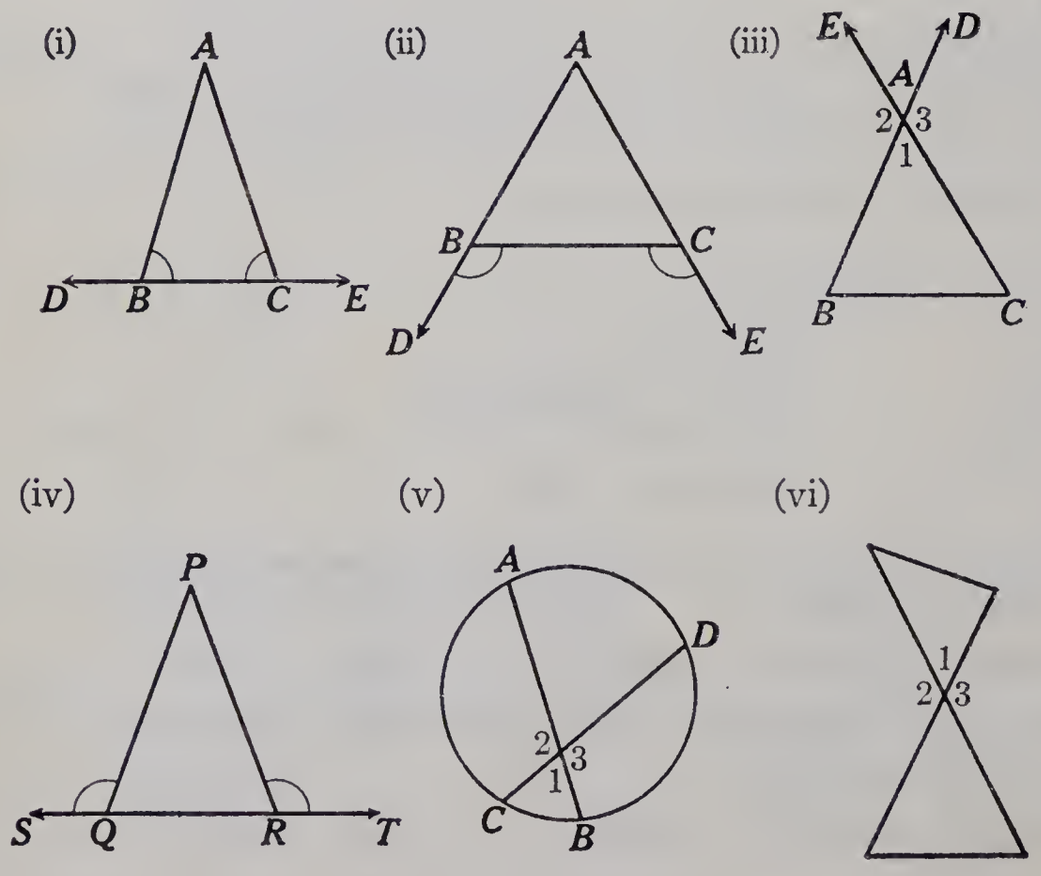
(A)

1. State the measurement of the angle which is the complement of each of the following:
   (i) 60° (ii) 30° (iii) 45° (iv) 10° (v) 89°
2. State the measurement of the angle which is the supplement of each of the following:
   (i) 60° (ii) 30° (iii) 45° (iv) 10° (v) 89°

3. Using the Complementary Angle Theorem as an authority, name pairs of equal angles in each of the following geometric figures:



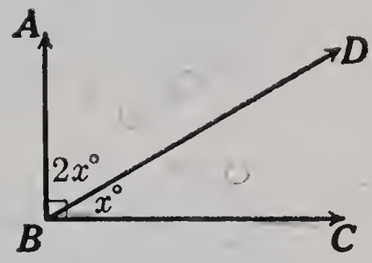
4. Applying the Supplementary Angle Theorem, name pairs of equal angles in each of the following geometric figures:



(B)

In each of the following, copy and complete the proofs by writing the authority that justifies each statement:

5. *Hypothesis:* In the diagram at the right,
$AB \perp BC$, $\angle DBC = x°$,
$\angle ABD = 2x° = 2\ \angle DBC$.

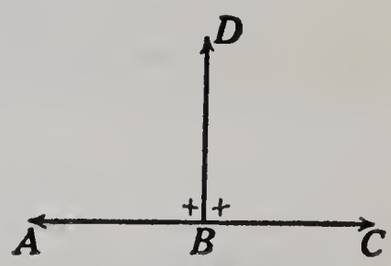


Conclusion: $\angle DBC = 30°$, $\angle ABD = 60°$

Proof:

STATEMENTS	AUTHORITIES
1. $\angle ABC = 90°$	1.
2. $2x + x = 90$	2.
3. $3x = 90$	3.
4. $x = 30$	4.
5. $\angle DBC = 30°$	5.
6. $\angle ABD = 60°$	6.

6. *Hypothesis:* In the diagram at the right,
$\angle ABC = 180°$,
$\angle ABD = \angle DBC$.

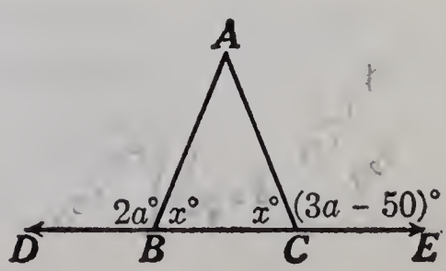


Conclusion: $DB \perp AC$

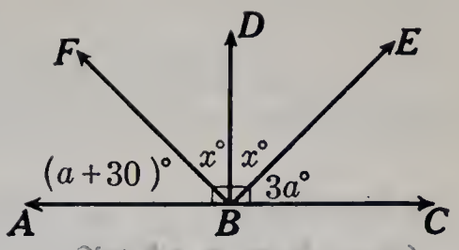
Proof:

STATEMENTS	AUTHORITIES
1. $\angle ABC = 180°$	1.
2. $\angle ABD + \angle DBC = 180°$	2.
3. $\angle ABD + \angle ABD = 180°$	3.
4. $2\ \angle ABD = 180°$	4.
5. $\angle ABD = 90°$	5.
6. $DB \perp AC$	6.

7. In the accompanying diagram, the base of $\triangle ABC$ is extended as indicated. Deduce the measurement of each of the four angles that are labelled.



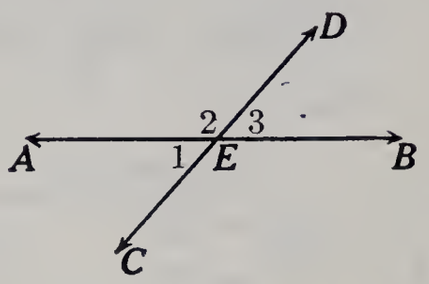
8. Write a complete analysis and proof of the following deduction: In the diagram at the right, if $\angle EBC = 3a°$, $\angle FBA = (a + 30)°$, $DB \perp AC$, $\angle FBD = \angle DBE = x°$, then $\angle FBA = 45°$.



9. Rewrite and complete the following demonstration of proof:

Hypothesis: AB and CD are two lines intersecting at E.

Conclusion: $\angle 1 = \angle 3$



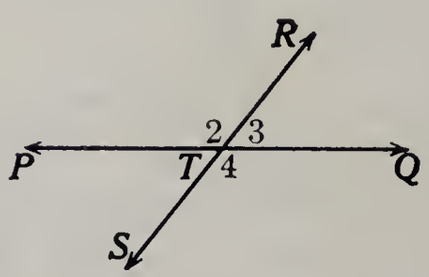
Proof:

STATEMENTS	AUTHORITIES
1. $\angle CED$ is a straight angle.	1.
2. $\angle 1$ is the supplement of $\angle 2$.	2.
3. $\angle AEB$ is a straight angle.	3.
4. $\angle 3$ is the supplement of $\angle 2$.	4.
5. $\angle 1 = \angle 3$	5.

10. Using the proof of question 9 as a guide, write a complete proof of the following deduction:

Hypothesis: PQ and RS are two lines intersecting at T.

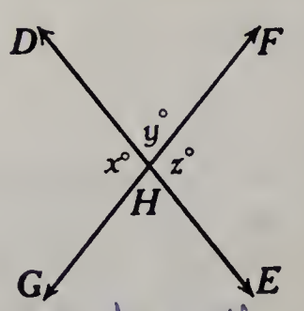
Conclusion: $\angle 2 = \angle 4$



11. Write a complete proof of the following deduction:

Hypothesis: In the diagram at the right, $\angle DHE$ and $\angle GHF$ are straight angles, $\angle GHD = x°$, $\angle DHF = y°$, $\angle FHE = z°$.

Conclusion: $x = z$.



9·5 Vertical Angle Theorem.

DEFINITION: *Two angles are vertical angles if their sides form two pairs of opposite rays.*

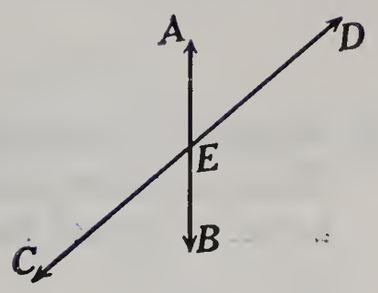


Fig. 9-6

In *Fig. 9-6*:

(i) $\angle AED$ and $\angle CEB$ are vertical angles;

(ii) $\angle DEB$ and $\angle AEC$ are vertical angles.

THEOREM 3

Vertical Angle Theorem

If two lines intersect, then the vertical angles are equal.

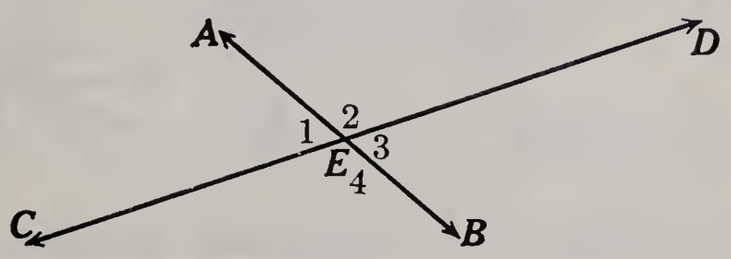


Fig. 9-7

Hypothesis: Lines AB and CD intersect at E forming vertical angle pairs $\angle 1$, $\angle 3$; $\angle 2$, $\angle 4$ (*Fig. 9-7*).

Conclusion: $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$

Proof:

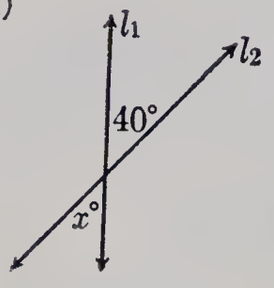
STATEMENTS	AUTHORITIES
1. $\angle CED$ is a straight angle.	1. Hypothesis
2. $\angle 1$ is the supplement of $\angle 2$.	2. Definition
3. $\angle AEB$ is a straight angle.	3. Hypothesis
4. $\angle 3$ is the supplement of $\angle 2$.	4. Definition
5. $\angle 1 = \angle 3$	5. Supplementary Angle Theorem
6. Similarly, $\angle 2$ and $\angle 4$ are each supplements of $\angle 1$.	6. As in 1, 2, 3, 4 above
7. $\angle 2 = \angle 4$	7. Supplementary Angle Theorem

Exercise 9-2

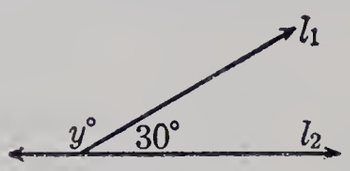
(A)

1. Using the Vertical Angle Theorem and the Supplementary Angle Theorem where applicable, state the measurements of the labelled angles in each of the following:

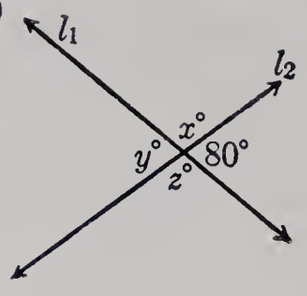
(i)



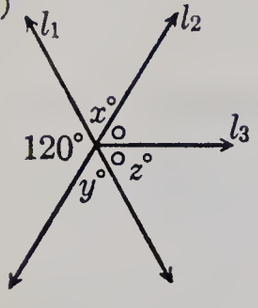
(ii)



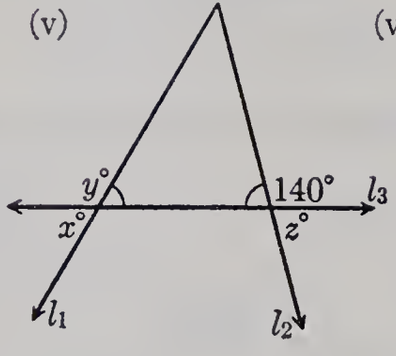
(iii)



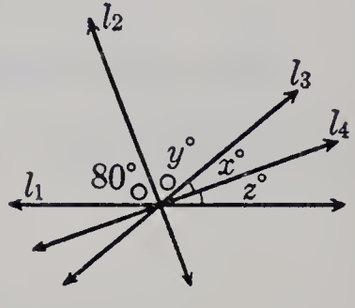
(iv)



(v)



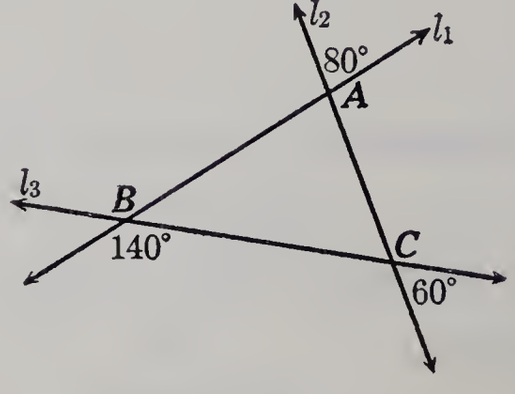
(vi)



2. State the authorities which are necessary to complete the proof of the following deduction:

Hypothesis: Lines l_1, l_2, l_3 intersecting at A, B, C, as shown, so that the labelled angles have the measurements indicated

Conclusion: $\angle BAC + \angle ABC + \angle ACB = 180°$



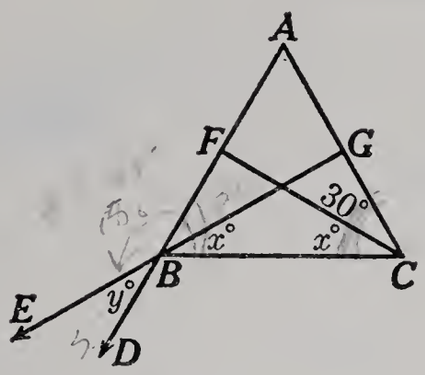
Proof:

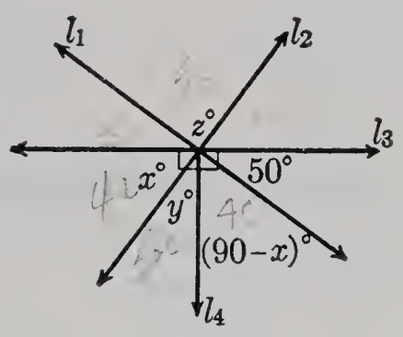
STATEMENTS	AUTHORITIES
1. $\angle BAC = 80°$	1.
2. $\angle ABC = 40°$	2.
3. $\angle ACB = 60°$	3.
4. $\angle BAC + \angle ABC + \angle ACB = 80° + 40° + 60°$	4.
5. $\angle BAC + \angle ABC + \angle ACB = 180°$	5.

(B)

Write complete proofs of each of the following deductions:

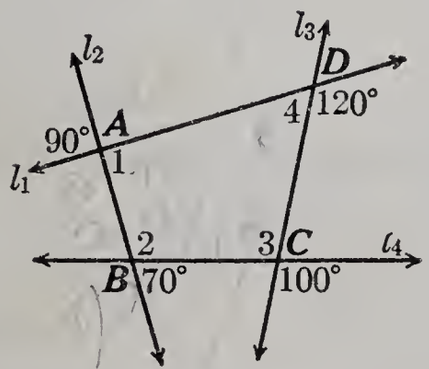
3. In the diagram at the left below, the angles have the measurements indicated.
Prove $y = 30$, if $\angle ABC = \angle ACB$.

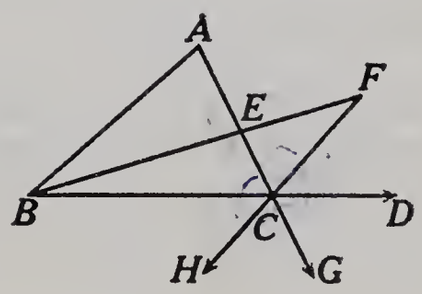




4. In the diagram at the right above, l_1, l_2, l_3, l_4 intersect as shown. The angles have the measurements indicated. Find the number of degrees in each angle and give a complete proof.

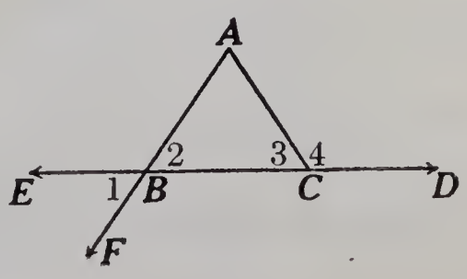
5. In the diagram at the left below, l_1, l_2, l_3, l_4 intersect as shown. The angles have the measurements indicated. Calculate $\angle 1 + \angle 2 + \angle 3 + \angle 4$; give a complete proof to justify your calculation.



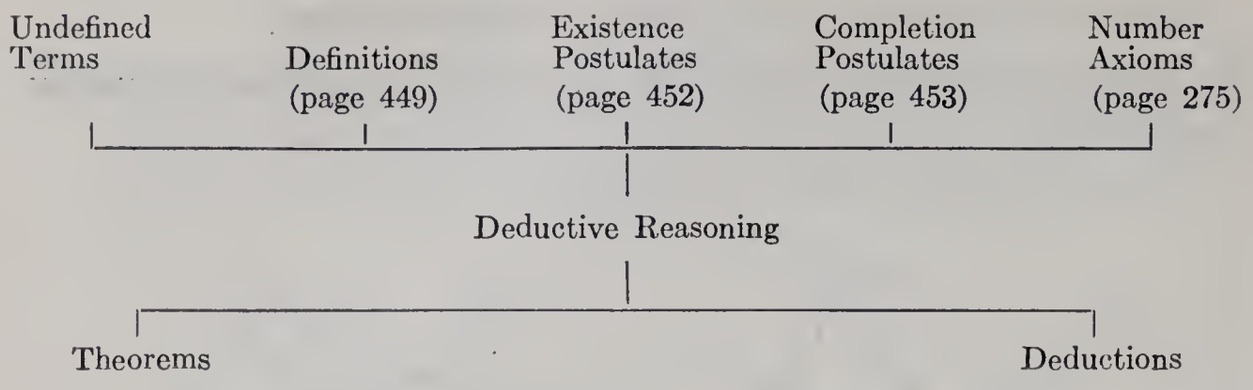


6. In the figure at the right above, $\triangle ABC$ has side AC produced to G, BC produced to D. Line segments AC and BF intersect at E. Rays AG, BD, FH intersect at C. Make a list of 8 pairs of equal vertical angles.

7. *Hypothesis:* Line DE and ray AF intersect at B, CA intersects line ED at C, $\angle 2 = \angle 3$.
Conclusion: $\angle 1 + \angle 4 = 180°$



9·6 Summary.



1. Complementary Angle Theorem
2. Supplementary Angle Theorem
3. Vertical Angle Theorem

Review Exercise 9-3

(A)

1. Give the general statement of the following theorems:
   (i) Complementary Angle Theorem
   (ii) Supplementary Angle Theorem
   (iii) Vertical Angle Theorem

2. Define vertical angles.

3. In each of the following, state the measurement of the complement of the angle with the measurement given:

   (i) $1°$ (ii) $85°$ (iii) $46°$ (iv) $27°$
   (v) $60°$ (vi) $x°$ (vii) $(90 - x)°$ (viii) $(x + 30)°$
   (ix) $90°-10°$ (x) $60° - x°$ (xi) $\left(\frac{a}{2}\right)°$ (xii) $52°$

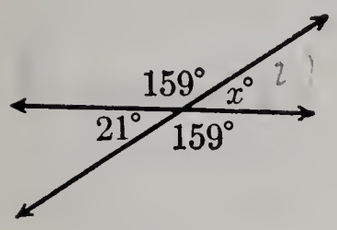
4. In each of the following, state the measurement of the supplement of the angle with the measurement given:

   (i) $1°$ (ii) $100°$ (iii) $135°$ (iv) $60°$
   (v) $179°$ (vi) $30°$ (vii) $120°$ (viii) $150°$
   (ix) $(90-a)°$ (x) $90°-20°$ (xi) $180°-y°$ (xii) $180°-30°$
   (xiii) $(m+n)°$ (xiv) $\left(\frac{180}{2}\right)°$ (xv) $180°-(a+b)°$ (xvi) $45°-a°$

(B)

In each of the following, determine the measurement of each of the labelled angles; justify your answer by quoting the authority:

5.



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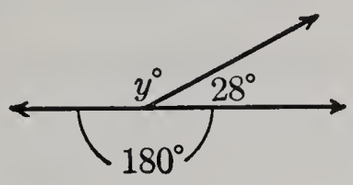

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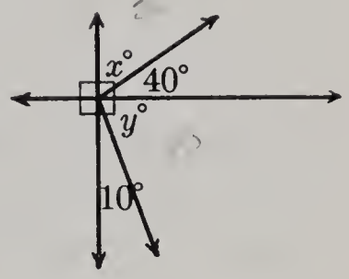

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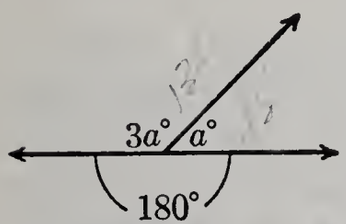

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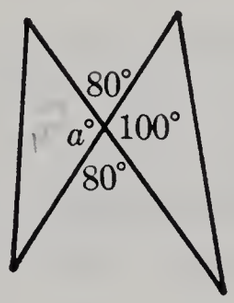

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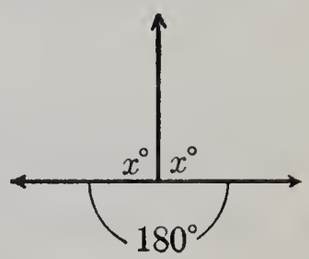

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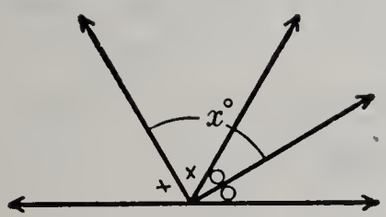

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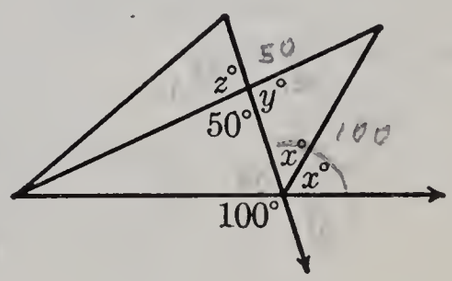


In each of the following, use only ruler and compasses; show all construction lines, but do not give descriptions of constructions:

13. Construct an angle which is the complement of an angle whose measurement is $22\frac{1}{2}°$.

14. Construct an angle which is the supplement of an angle whose measurement is 45°.

15. Construct an angle which is the supplement of an angle whose measurement is $67\frac{1}{2}°$.

16. Construct an angle which is the complement of the supplement of an angle whose measurement is $112\frac{1}{2}°$.

Chapter X

CONGRUENCE

10·1 **Fundamental congruence postulates.**

a. *Line segments.* The statement, "$AB = CD$", implies that the length of line segment AB is equal to the length of line segment CD.

One way to compare the two point sets AB and CD is to match point A with point C, *Fig. 10-1*, and each point P in AB with a corresponding point P' in CD such that $AP = CP'$. Since $AB = CD$, then point B is matched with point D.

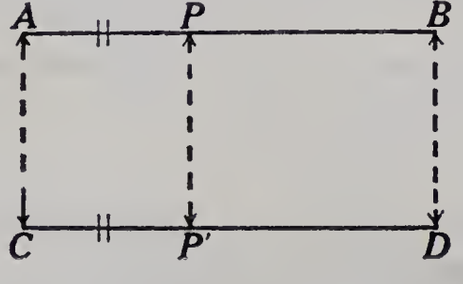


Fig. 10-1

This method of matching ensures that the point sets are placed in a one-to-one correspondence, such that the distance between any two points in AB is equal to the distance between the corresponding two points in CD. A matching of point sets in this way is called a *congruence matching*, and AB is said to be *congruent to* CD. The symbol $\cong$ is used to represent the words *is congruent to*. Thus, $AB \cong CD$ is read, "Line segment AB is congruent to line segment CD".

The idea of a congruence matching of line segments suggests the following postulate.

CONGRUENCE POSTULATE FOR LINE SEGMENTS

Equal line segments are congruent; congruent line segments are equal.

The statement $\overset{\downarrow\quad\downarrow}{AB} \cong \underset{\uparrow\quad\uparrow}{CD}$ tells us that:

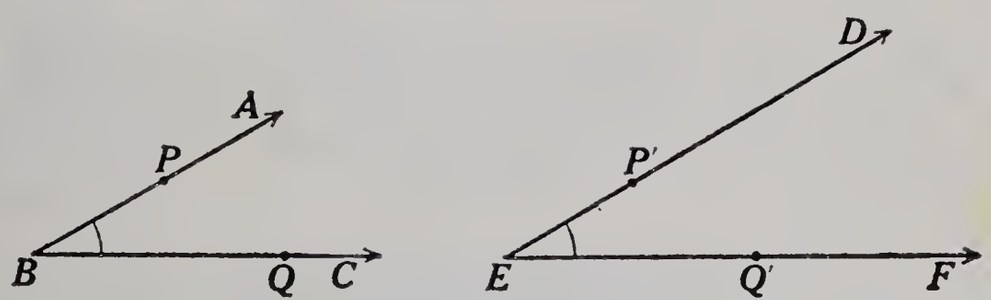
(i) A corresponds to C, and C corresponds to A, which may be expressed in symbols as $A \leftrightarrow C$;

(ii) B corresponds to D, and D corresponds to B, which may be written as $B \leftrightarrow D$;

(iii) $AB = CD$.

The statement $BA \cong DC$ provides the same information.

b. *Angles.* The idea of a congruence matching of point sets suggests that equal angles ABC and DEF, *Fig. 10-2*, are congruent. That is, point B can be matched with point E, the points on rays BA and BC with the points on rays ED and EF, respectively, such that for all pairs of corresponding points P, P' and Q, Q', then $BP = EP'$, $BQ = EQ'$, $PQ = P'Q'$.



(*a*) (*b*)

Fig. 10-2

This illustration suggests the following postulate.

CONGRUENCE OF ANGLES POSTULATE

Equal angles are congruent; congruent angles are equal.

The statement $\angle ABC \cong \angle DEF$ tells us that:

(i) ray $BA \leftrightarrow$ ray ED, and ray $BC \leftrightarrow$ ray EF;

(ii) $\angle ABC = \angle DEF$.

c. *Triangles.* The idea of a congruence matching (the line segment congruence postulate and the angle congruence postulate) suggests the required conditions for congruence of two triangles, ABC and DEF, *Fig. 10-3*.

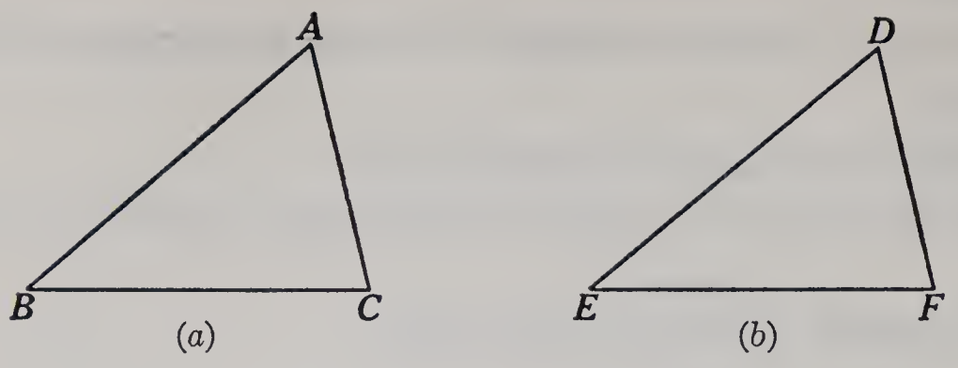


Fig. 10-3

In order to have a congruence matching of the point sets of $\triangle ABC$ and $\triangle DEF$ so that:

$$A \leftrightarrow D$$
$$B \leftrightarrow E$$
$$C \leftrightarrow F$$

the following are the required conditions:

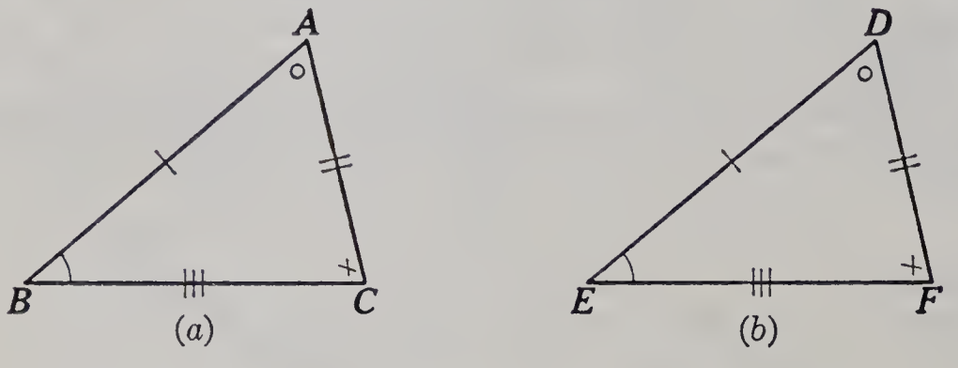


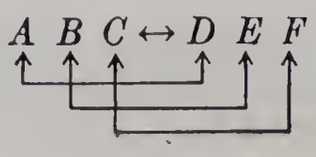
Fig. 10-4

$$AB \cong DE, \text{ that is } AB = DE .$$
$$AC \cong DF, \text{ that is } AC = DF .$$
$$BC \cong EF, \text{ that is } BC = EF .$$
$$\angle ABC \cong \angle DEF, \text{ that is } \angle ABC = \angle DEF .$$
$$\angle ACB \cong \angle DFE, \text{ that is } \angle ACB = \angle DFE .$$
$$\angle BAC \cong \angle EDF, \text{ that is } \angle BAC = \angle EDF .$$

All of the above information is summed up in the statement:

$$\triangle ABC \cong \triangle DEF$$

This statement implies that the vertex named by the first letter on the left is matched with the vertex named by the first letter on the right, and so on. This implied correspondence is illustrated below:

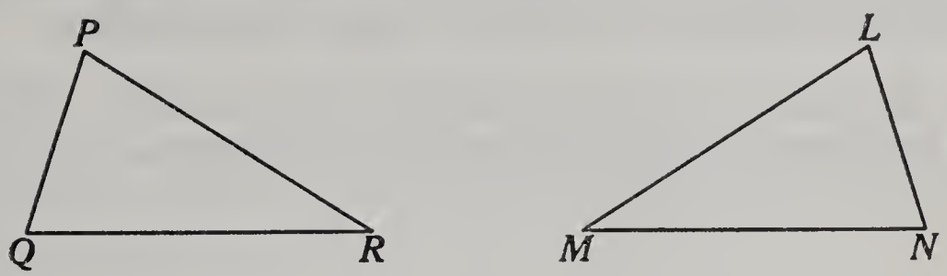


DEFINITION: *If a correspondence $ABC \leftrightarrow DEF$ between the vertices of two triangles is such that each pair of corresponding sides is congruent and each pair of corresponding angles is congruent, then this correspondence is a congruence matching, and $\triangle ABC \cong \triangle DEF$.*

It should be noted that the definition of congruent triangles includes the fact that if two triangles are congruent, then each pair of corresponding line segments is congruent (or equal) and each pair of corresponding angles is congruent (or equal).

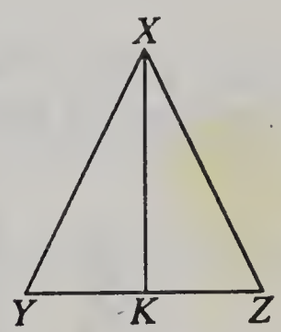
Write solutions to each of the following problems and compare your solutions with those on page 486.

1.



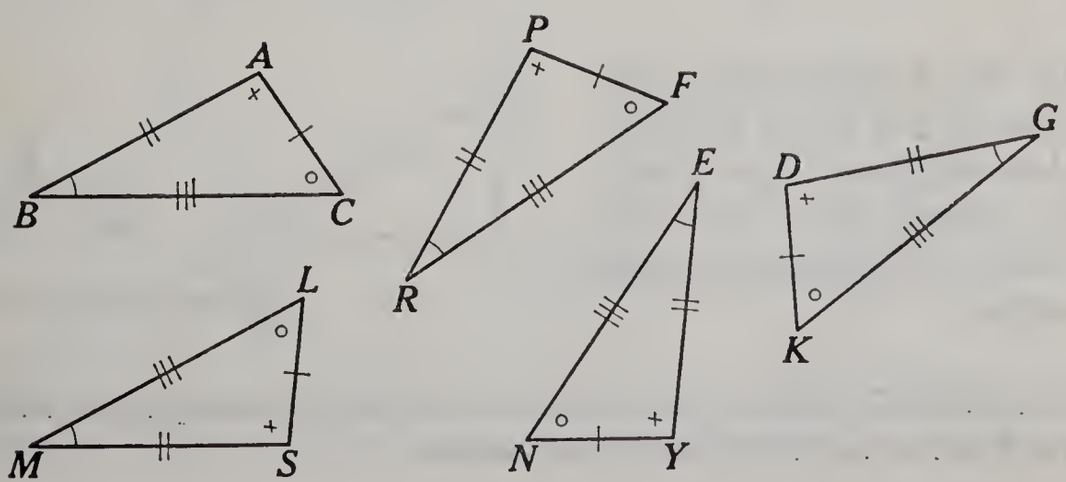
If $\triangle PQR \cong \triangle LNM$ in the diagrams above, list corresponding vertices, pairs of equal line segments, pairs of equal angles.

2. If $\triangle XYK \cong \triangle XZK$ in the diagram at the right, list corresponding vertices, pairs of equal line segments, pairs of equal angles.

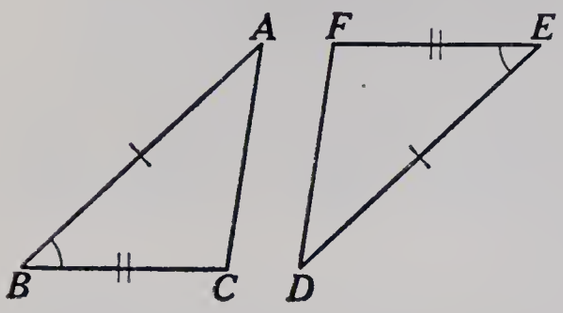


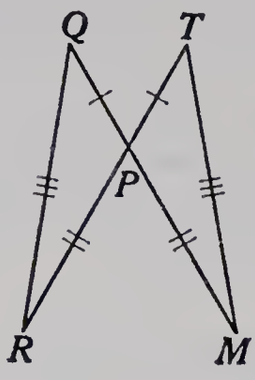
3. If $\triangle KGR \cong \triangle EGS$, list corresponding vertices, pairs of equal line segments, pairs of equal angles.

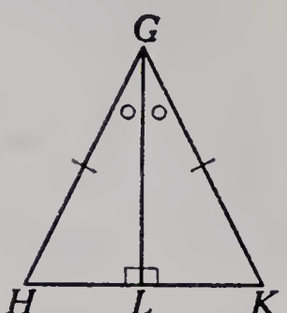
4. Name pairs of congruent triangles in the following:



5. In the diagram at the left below $\triangle ABC \cong \triangle DEF$. Name all pairs of equal parts not indicated equal in the diagram.







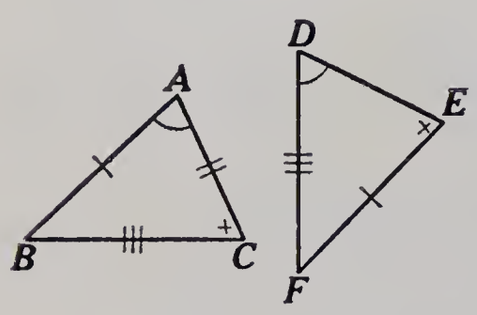
6. In the diagram in the centre above $\triangle PQR \cong \triangle PTM$. Name all pairs of equal parts not indicated equal in the diagram.

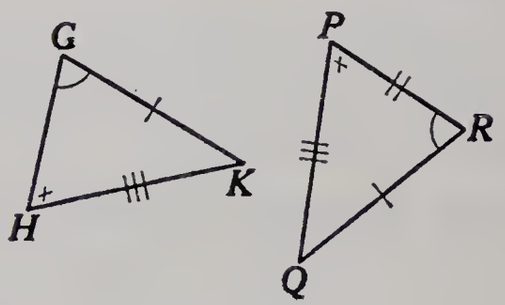
7. In the diagram at the right above $\triangle GHL \cong \triangle GKL$. Name all pairs of equal parts not indicated equal in the diagram.

Exercise 10-1

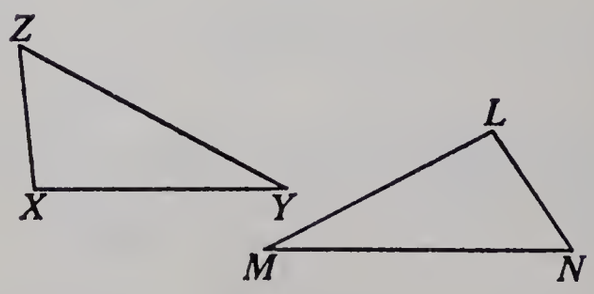
(A)

1. Select pairs of congruent triangles from the following and name corresponding parts:



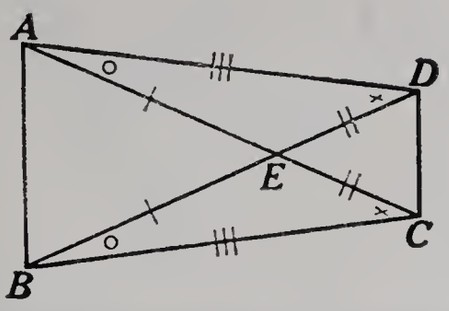


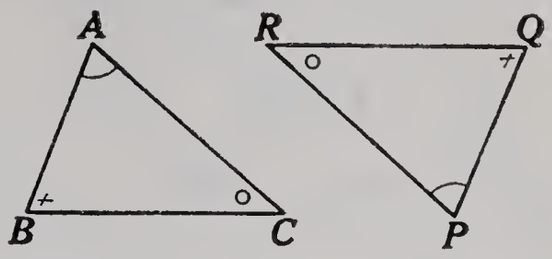
2. In the accompanying diagram, if $\triangle XYZ \cong \triangle LMN$, state the pairs of corresponding equal sides and the pairs of corresponding equal angles.



3. If $\triangle MQT \cong \triangle SPN$, state the pairs of corresponding equal sides and the pairs of corresponding equal angles.

4. Name a pair of congruent triangles in the diagram at the left below. Justify your selection by naming all pairs of corresponding equal parts.

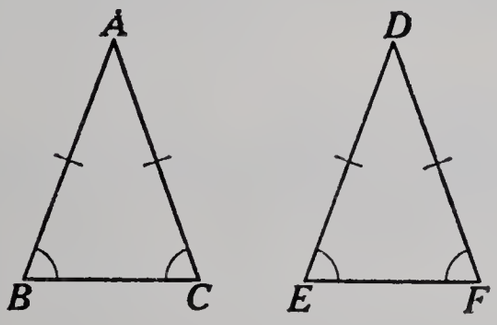


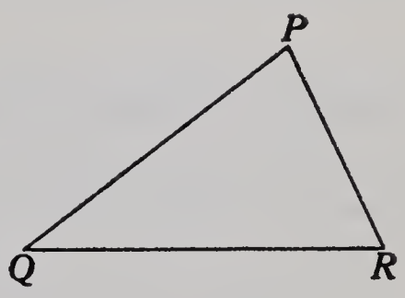


5. In the diagram at the right above, if $\triangle ABC \cong \triangle PQR$, name the corresponding equal parts not indicated equal in the diagram.

(B)

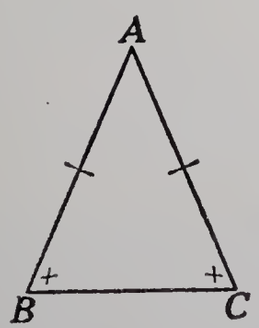
6. In the diagram at the left below, note that (i) $\triangle ABC \cong \triangle DEF$ (ii) $\triangle ABC \cong \triangle DFE$. For each of these congruencies write down the corresponding angles and sides.

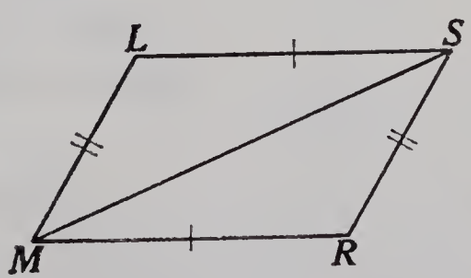




7. In the diagram at the right above, $\triangle PQR \cong \triangle PQR$; list the corresponding angles and sides for this congruency.

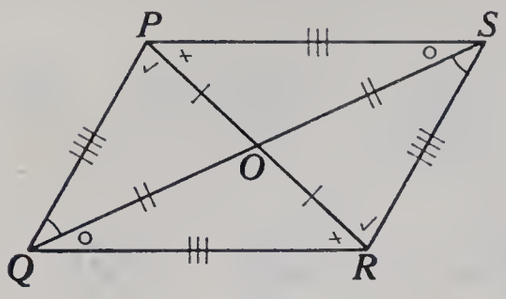
8. In the diagram at the left below, note that (i) $\triangle ABC \cong \triangle ABC$ (ii) $\triangle ABC \cong \triangle ACB$. Name the corresponding equal parts not indicated equal in the diagram.

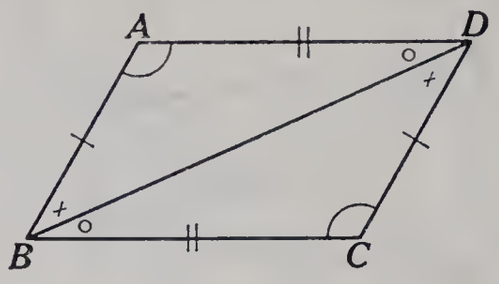




9. In the diagram at the right above, if $\triangle LMS \cong \triangle RSM$, list all pairs of equal parts not indicated equal in the diagram.

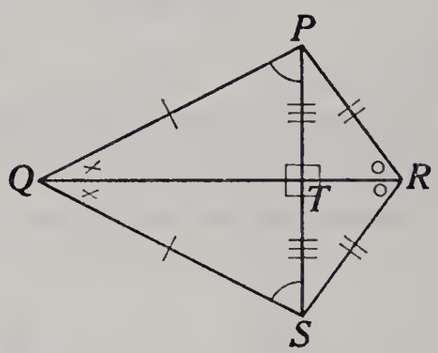
10. In the diagram at the left below, name four pairs of congruent triangles. Justify each congruency by listing all pairs of equal parts.





11. In the diagram at the right above, name a pair of congruent triangles which have AB and CD as corresponding sides; list all pairs of equal parts of the triangles named.

12. In the accompanying diagram, name two pairs of congruent triangles which have $\angle PQR$ and $\angle SQR$ as corresponding angles; for each pair of triangles list all pairs of equal parts.



10·2 **Triangle Congruence Postulate (sas).** Suppose two sets of noncollinear points are represented by $\{A, B, C\}$ and $\{D, E, F\}$. Under what conditions are the triangles determined by these points congruent? By the definition of congruent triangles, it is required that the three sides and three angles of one triangle be respectively equal to the three sides and the corresponding three angles of the other triangle. However, we are now interested in determining which of these six conditions are *sufficient* to determine the others and, hence, sufficient to determine congruent triangles.

Consider that the selected points are such that

$$AB = DE$$
$$AC = DF$$
$$\angle BAC = \angle EDF$$

as shown in *Fig. 10-5.*

Since $AB = DE$, there is a congruence matching of point set AB with point set DE, such that $A \leftrightarrow D$ and $B \leftrightarrow E$. Since $\angle BAC = \angle EDF$, there is a congruence matching of these angles, and since $AC = DF$, there is a congruence matching of these point sets such that $A \leftrightarrow D$ and $C \leftrightarrow F$. Since line segment BC is determined by points B and C, and line segment

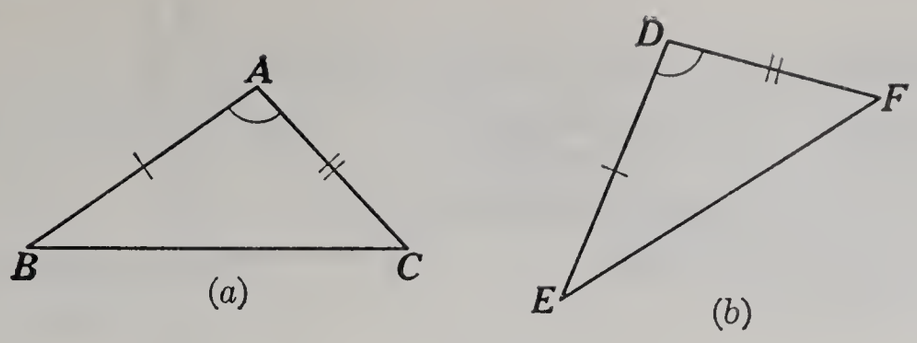


Fig. 10-5

EF by points E and F, there is a congruence matching of BC with EF. Further, $\angle ABC \cong \angle DEF$ and $\angle ACB \cong \angle DFE$.

Thus, the given conditions are sufficient to provide a congruence matching of $\triangle ABC$ with $\triangle DEF$.

This illustration suggests the following postulate.

TRIANGLE CONGRUENCE POSTULATE (sas)

If two sides and the contained angle of one triangle are respectively equal to two sides and the contained angle of another triangle, the two triangles are congruent.

This assumption may be referred to as the "sas (side, angle, side) postulate" for congruence of triangles.

Examples 1 and 2, which follow, emphasize the conditions contained in the sas postulate.

Example 1. In each triangle (*Fig. 10-6*), name the angle *contained* by the marked sides.

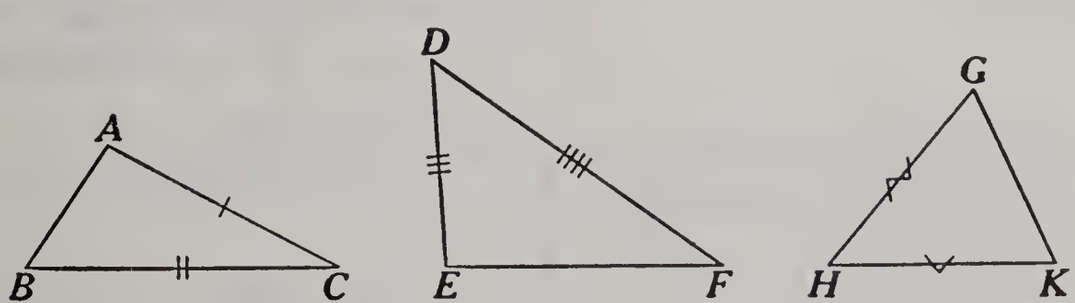


Fig. 10-6

In $\triangle ABC$, the contained angle is $\angle C$.
In $\triangle DEF$, the contained angle is $\angle D$.
In $\triangle GHK$, the contained angle is $\angle H$.

Example 2. In *Fig. 10-7*, using AC and DF as one pair of corresponding equal sides, give the additional conditions sufficient to prove congruence of the triangles by the sas postulate.

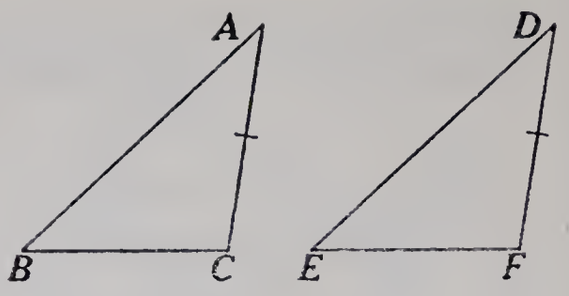


Fig. 10-7

The conditions are either: (i) $AB = DE$ and $\angle A = \angle D$ or
(ii) $BC = EF$ and $\angle C = \angle F$.

By using the sas postulate as the major premise for our reasoning, it is possible to prove logically many conditional statements and thus deduce many fundamental geometric relations. This is illustrated in Examples 3 and 4.

Example 3. If, in *Fig. 10-8*, AD represents the altitude of $\triangle ABC$ and $BD = DC$, then $AB = AC$.

Hypothesis: AD is the altitude of $\triangle ABC$ and $BD = CD$.

Conclusion: $AB = AC$

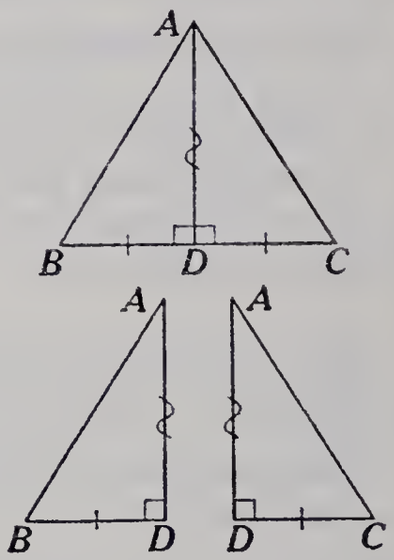


Fig. 10-8

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $AB = AC$	1. Two triangles congruent, with AB and AC as corresponding sides.
2. Two triangles congruent with AB and AC as corresponding sides	2. $\triangle ABD \cong ACD$.
3. $\triangle ABD \cong \triangle ACD$	3. $AD = AD$, $BD = CD$, $\angle ADB = \angle ADC$.
4. $AD = AD$ by the reflexive property.	
5. $BD = CD$ by hypothesis.	
6. $\angle ADB = \angle ADC$ by hypothesis.	

Proof:

STATEMENTS		AUTHORITIES
1. In $\triangle$'s ABD and ACD	$AD = AD$	1. Reflexive
2.	$BD = CD$	2. Hypothesis
3.	$\angle ADB = \angle ADC$	3. Hypothesis
4. $\triangle ABD \cong \triangle ACD$		4. sas
5. $AB = AC$		5. Definition

Example 4. If in quadrilateral $ABCD$ (*Fig. 10-9*), $AD = CB$ and $\angle ADB = \angle CBD$, then $AB = CD$.

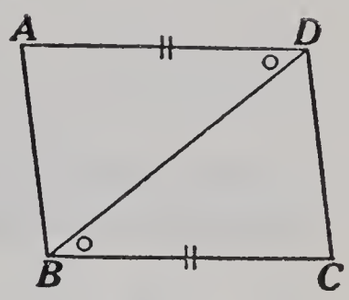


Fig. 10-9

Hypothesis: $ABCD$ is a quadrilateral with $AD = CB$ and $\angle ADB = \angle CBD$.

Conclusion: $AB = CD$

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $AB = CD$	1. Two triangles congruent with AB, CD as corresponding sides
2. Two triangles congruent with AB, CD as corresponding sides	2. $\triangle ABD \cong CDB$.
3. $\triangle ABD \cong \triangle CDB$	3. $AD = CB$, $BD = DB$, $\angle ADB = \angle CBD$.
4. $AD = CB$ by hypothesis.	
5. $BD = DB$ by reflexive property.	
6. $\angle ADB = \angle CBD$ by hypothesis.	

Proof:

STATEMENTS		AUTHORITIES
1. In $\triangle$'s ABD and CDB	$AD = CB$	1. Hypothesis
2.	$BD = DB$	2. Reflexive
3.	$\angle ADB = \angle CBD$	3. Hypothesis
4. $\triangle ABD \cong \triangle CDB$		4. sas
5. $AB = CD$		5. Definition

Steps in the Organization of a Deductive Proof.

1. Make a large labelled diagram.
2. Write the hypothesis and the conclusion.

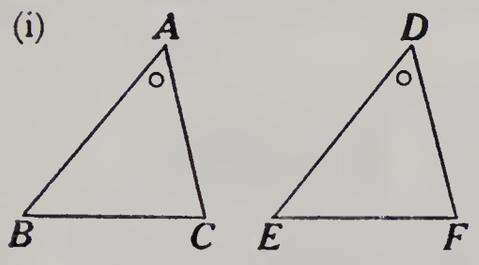
3. Mark all parts which are given equal in the hypothesis of the conditional statement to be proved. Note and mark all parts that can be proved equal as *obvious* consequents of the hypothesis and theorems.
4. Analyse the deduction in order to determine the definitions, postulates, and theorems which apply in the proof.
5. Write the proof in two columns.

Exercise 10-2

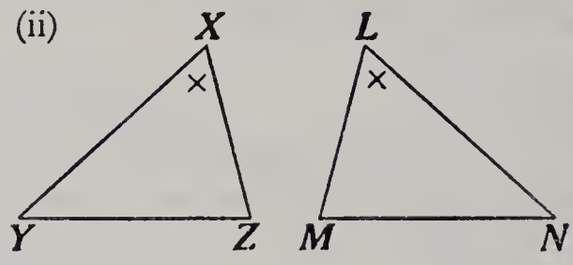
(A)

1. In the following pairs of triangles, name the pairs of equal sides sufficient for congruence by the sas postulate:

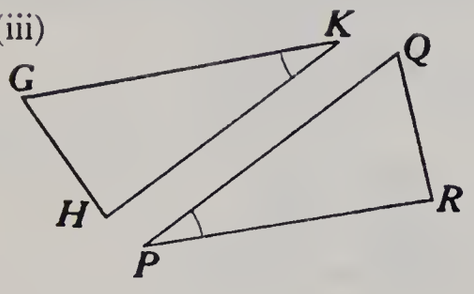
(i)



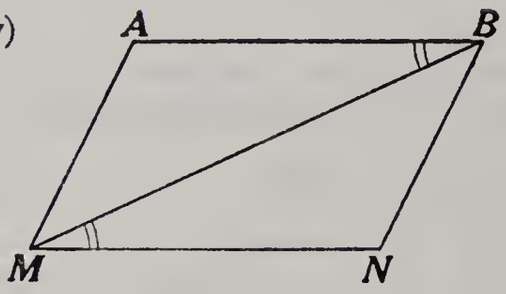
(ii)



(iii)

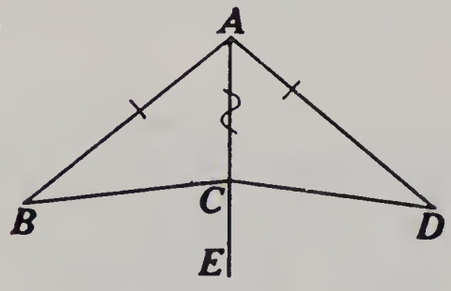


(iv)

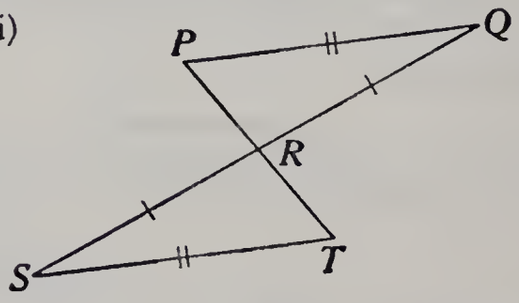


2. In the following diagrams, name the contained angles which must be equal for each pair of triangles to be congruent by the sas postulate:

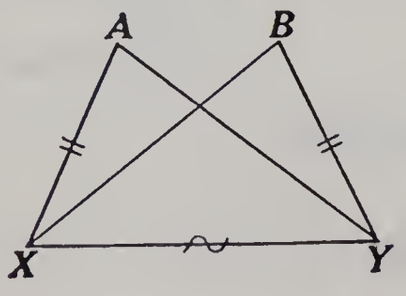
(i)



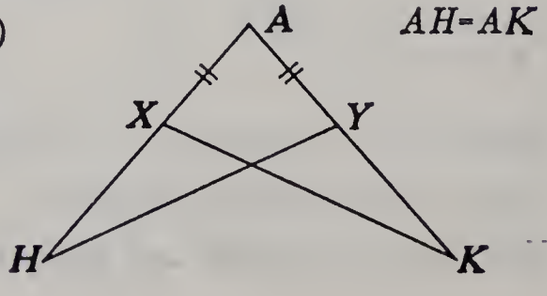
(ii)



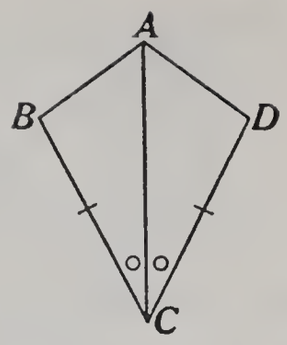
(iii)



(iv)



3. In the following proof, state the authorities required to complete the proof.



Hypothesis: Quadrilateral $ABCD$ in which $BC = DC$, $\angle BCA = \angle DCA$

Conclusion: $AB = AD$

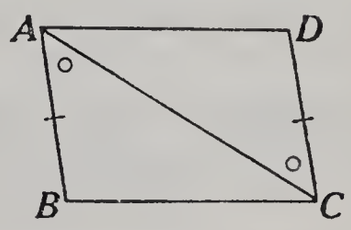
Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s ABC and ADC $\begin{cases} BC = DC \\ AC = AC \\ \angle BCA = \angle DCA \end{cases}$	1.
2.	2.
3.	3.
4. $\triangle ABC \cong \triangle ADC$	4.
5. $AB = AD$	5.

Give an oral analysis and proof for each of the following:

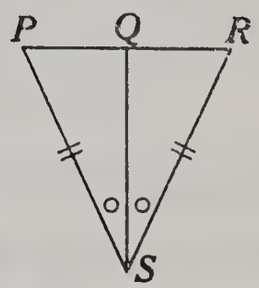
4. *Hypothesis:* $AB = CD$
$\angle BAC = \angle DCA$

Conclusion: $\angle B = \angle D$



5. *Hypothesis:* $PS = RS$
$\angle PSQ = \angle RSQ$
PQR is a line segment.

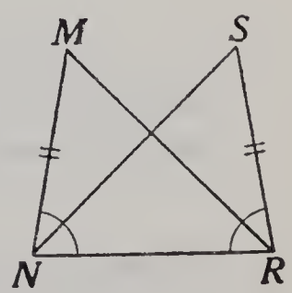
Conclusion: $\angle PQS = \angle RQS = 90°$



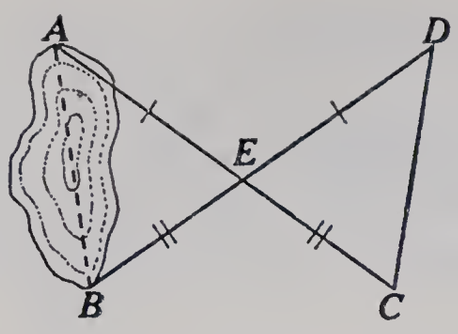
6. Give an oral analysis of the following deduction:

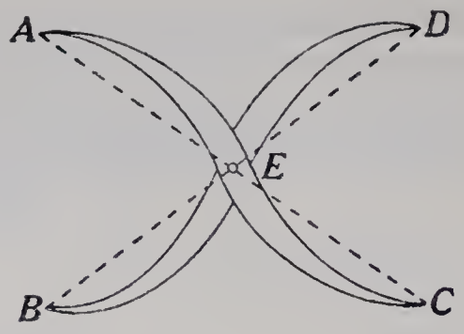
Hypothesis: $MN = SR$
$\angle MNR = \angle SRN$

Conclusion: $\angle M = \angle S$



7. A man wishes to find the distance AB across a small lake. He lays out stakes A, B, C, D, E as shown in the diagram at the top of page 308. What measurement will he make to find the length of AB? Justify your conclusion by geometric proof.





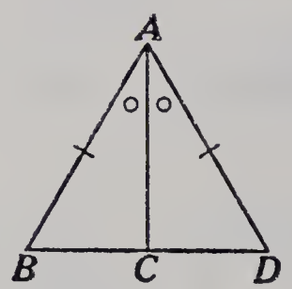
8. An instrument which may be used to measure the diameter of pipes or trees is shown in the diagram at the right above. It consists of two curved rods hinged at E so that AEC and BED are line segments and $AE = EC = BE = ED$. Prove that $AB = DC$ and explain how this instrument could be used to measure the diameter of a large pipe.

(B)

Give a written analysis and proof for each of the following:

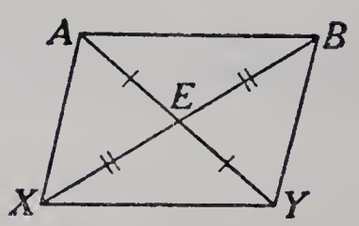
9. *Hypothesis:* $AB = AD$
$\angle BAC = \angle DAC$
BCD is a line segment.

Conclusion: (i) $BC = DC$
(ii) $\angle B = \angle D$



10. *Hypothesis:* $AE = YE$ and $XE = BE$

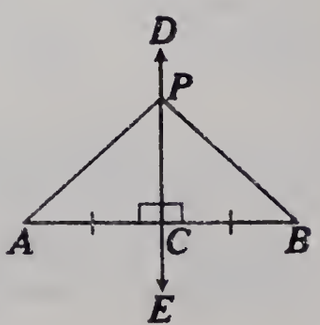
Conclusion: (i) $AX = YB$
(ii) $AB = YX$



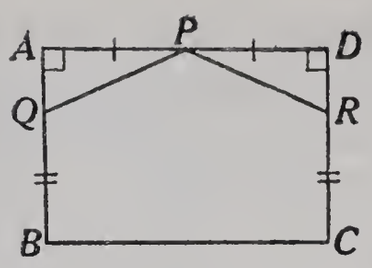
DEFINITION: *The right bisector of a given line segment is the line which bisects the given line segment at right angles.*

11. *Hypothesis:* DE is the right bisector of AB.
P is any point on DE.

Conclusion: $AP = BP$. Thus any point on the right bisector of a line segment is equidistant from the end points of the line segment.

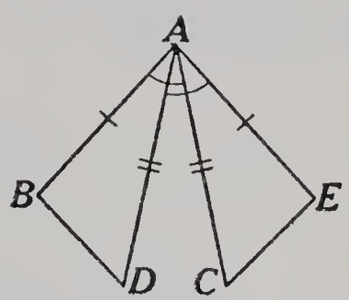


12. *Hypothesis:* $AP = DP$
$\angle A = \angle D = 90°$
$AB = DC$ and $BQ = CR$
Conclusion: $PQ = PR$



13. A, B, C, and D are points on the circumference of a circle with centre O such that $\angle AOB = \angle DOC$. Prove that $AB = DC$.

14. *Hypothesis:* $AB = AE$
$AD = AC$
$\angle BAC = \angle EAD$
Conclusion: $BD = EC$



***10·3* Analysis of multi-step deductions.** In the analysis of a multi-step deduction, a stage is often reached when two or more alternative methods of continuing the analysis are suggested. Sometimes each alternative leads to an independent proof of the deduction; sometimes only one proof is reached, since the other alternatives lead to a stage at which there is not sufficient information to continue the analysis. This is illustrated in the following examples.

Example 1. Examine the analysis of the following deduction; write a complete proof; compare your proof with that on page 487.

Hypothesis: P, $R \rightarrow \sim S$, $P \rightarrow \sim Q$, $T \rightarrow R$, $\sim U \rightarrow T$, $\sim Q \rightarrow R$

Conclusion: $\sim S$

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\sim S$	1. R.
2. R	2. T or $\sim Q$.
CASE I	
3. T	3. $\sim U$.
4. $\sim U$ (not sufficient information to continue the analysis).	
CASE II	
5. $\sim Q$	5. P.
6. P by hypothesis.	

Example 2. Examine the analysis of the following deduction; write a complete proof; compare your proof with that on page 487.

Hypothesis: In *Fig. 10-10*, D is on AB and E is on AC of $\triangle ABC$, so that $DB = EC$ and $\angle ADE = \angle AED$.

Conclusion: $BE = CD$

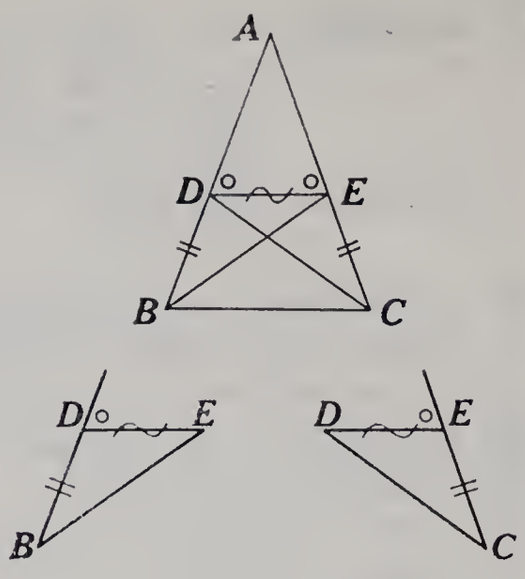


Fig. 10-10

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $BE = CD$	1. Two triangles congruent with BE and CD as corresponding sides.
2. Two triangles congruent with BE and CD as corresponding sides	2. (I) $\triangle BEA \cong \triangle CDA$. (II) $\triangle BEC \cong \triangle CDB$. (III) $\triangle BED \cong \triangle CDE$.
CASE I	
3. $\triangle BEA \cong \triangle CDA$	3. $BA = CA$, $EA = DA$, $\angle BAE = \angle CAD$.
4. $\angle BAE = \angle CAD$ by the reflexive property, but there is not sufficient information to prove $BA = CA$ and $EA = DA$.	
CASE II	
5. $\triangle BEC \cong \triangle CDB$	5. $BC = CB$, $EC = DB$, $\angle BCE = \angle CBD$.
6. $BC = CB$ by the reflexive property and $EC = DB$ by hypothesis, but there is not sufficient information to prove $\angle BCE = \angle CBD$.	

CASE III

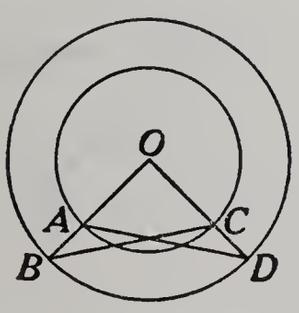
7. $\triangle BED \cong \triangle CDE$	7. $BD = CE$, $ED = DE$, $\angle BDE = \angle CED$.
8. $BD = CE$ by hypothesis.	
9. $ED = DE$ by the reflexive property.	
10. $\angle BDE = \angle CED$ by the Supplementary Angle Theorem.	

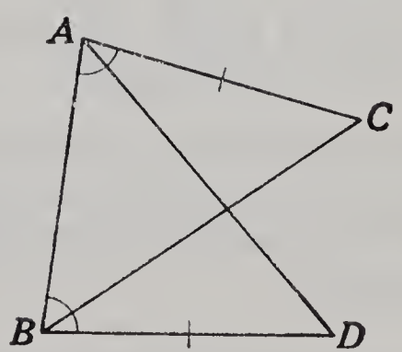
Exercise 10-3

(A)

Give an oral analysis of each of the following deductions:

1. *Hypothesis:* $T \rightarrow R,\ T,\ R \rightarrow \sim S$
   Conclusion: $\sim S$
2. *Hypothesis:* $\sim H,\ \sim J \rightarrow G,\ \sim H \rightarrow \sim J$
   Conclusion: G
3. Two circles are drawn with the same centre O. OAB and OCD are line segments as shown in the diagram at the left below. Prove $BC = DA$.



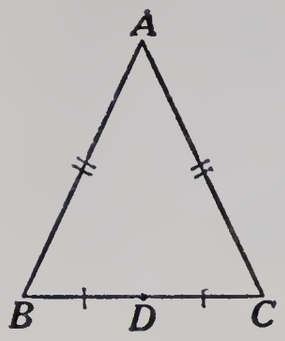


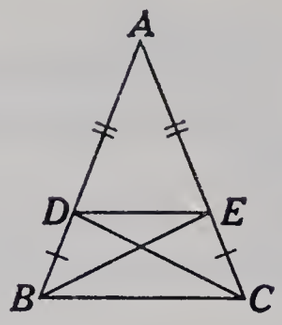
4. In $\triangle$'s ABC and BAD in the diagram at the right above, $AC = BD$ and $\angle BAC = \angle ABD$. Prove that $\angle CAD = \angle DBC$.

(B)

5. Write an analysis and proof of the following deduction:
   AEB and CED are line segments which intersect at E so that $AE = ED$ and $CE = BE$. Prove that $\triangle ACE \cong \triangle EBD$.
6. Write an analysis of the following deduction:
   ABC is a triangle in which $\angle B = \angle C$. AB is extended to D and AC is extended to E so that $BD = CE$. Prove $\angle ADC = \angle AEB$,

7. In the diagram at the left below, $AB = AC$, and D is the midpoint of BC, and $\angle B = \angle C$. Prove that AD bisects $\angle A$.





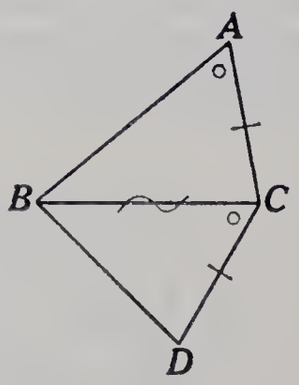
8. Write an analysis and complete proof of the following deduction:
   Hypothesis: $\triangle ABC$ with D on AB and E on AC so that $AD = AE$, $DB = EC$ (diagram at the right above)
   Conclusion: $BE = CD$

Write an analysis for deductions 9 *and* 10 *following:*

9. A, B, C, D are four points on a line such that $AB = BC = CD$. M is a point above or below the line such that $MB = MC$ and $\angle MBC = \angle MCB$. Prove $AM = DM$.

10. $ABCD$ is a quadrilateral with $AD = BC$ and $\angle A = \angle B = 90°$. AB is extended both ways to E and F so that $AE = BF$. Prove $\triangle ADE \cong \triangle BCF$ and $DF = CE$.

11. Justify the fact that the two triangles cannot be congruent by sas from the given facts marked on the diagram at the right.



Write an analysis and proof for each of the following deductions:

12. *Hypothesis:* $S \rightarrow \sim T, \sim T \rightarrow R, S$
    Conclusion: R

13. *Hypothesis:* (F or G) $\rightarrow \sim J, \sim J \rightarrow K$, ($F$ or G), $K \rightarrow$ (M or N)
    Conclusion: (M or N)

14. *Hypothesis:* (A and B) $\rightarrow C, C \rightarrow \sim D$, ($A$ and B), $\sim D \rightarrow \sim E$
    Conclusion: $\sim E$

15. *Hypothesis:* $\sim A \rightarrow C, A \rightarrow \sim B, D \rightarrow B, C \rightarrow B, \sim A$
    Conclusion: B

By analysis show that there is not enough information to prove the conclusion in each of the following:

16. *Hypothesis:* $A \rightarrow D,\ B \rightarrow \sim C,\ A, \sim D \rightarrow B$
Conclusion: $\sim C$

17. *Hypothesis:* $(N \text{ and } P) \rightarrow \sim Q,\ M \rightarrow (N \text{ and } P),\ \sim Q \rightarrow R$
Conclusion: R

10·4 **Triangle Congruence Postulate** (sss). Suppose two sets of three non-collinear points represented by $\{A, B, C\}$ and $\{D, E, F\}$ are selected such that

$$AB = DE$$
$$BC = EF$$
$$AC = DF$$

as shown in *Fig. 10-11.*

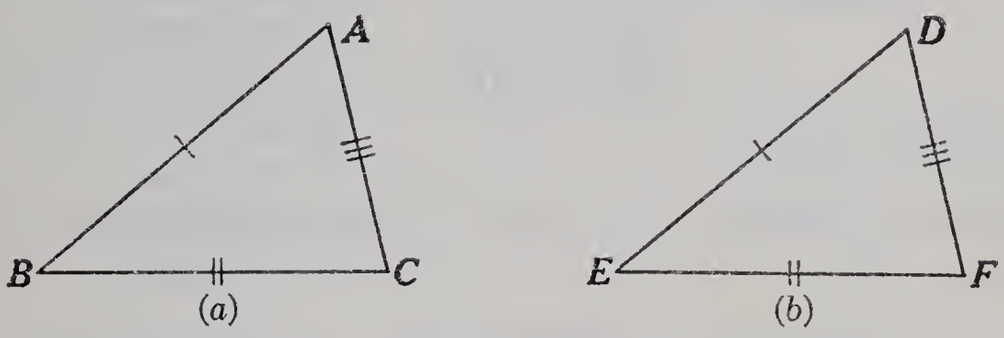


Fig. 10-11

Since $BC = EF$, there is a congruence matching of point set BC with point set EF such that $B \leftrightarrow E$ and $C \leftrightarrow F$. Since $BA = ED$, there is a congruence matching of BA with any radius of the circle with centre E and radius ED such that $B \leftrightarrow E$ and $A \leftrightarrow$ (a point on this circle). Since $CA = FD$, there is a congruence matching of CA with any radius of the circle with centre F and radius FD such that $C \leftrightarrow F$ and $A \leftrightarrow$ (a point on this circle). Since D is a point common to both of these circles, then point A is matched with point D. Thus, the given conditions provide a congruence matching of $\triangle ABC$ with $\triangle DEF$.

This illustration suggests the following postulate.

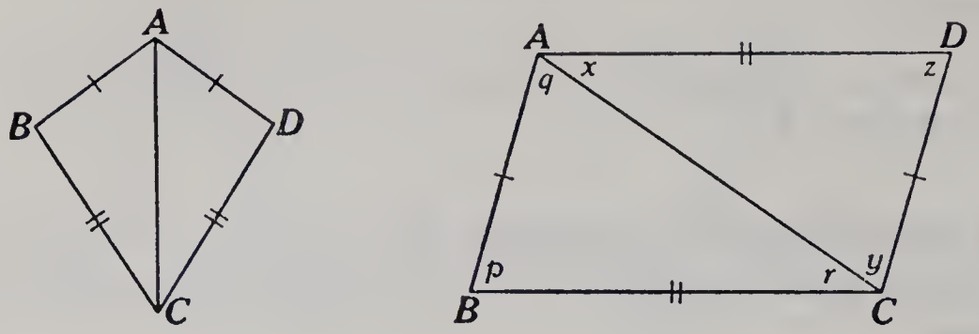
TRIANGLE CONGRUENCE POSTULATE (sss)

If two triangles have the three sides of one respectively equal to the three sides of the other, then the triangles are congruent.

This assumption may be referred to as the "sss (side, side, side) postulate" for congruence of triangles.

Example 1 emphasizes the sufficient conditions contained in the sss postulate.

Example 1. For each of the following diagrams, write down the sufficient conditions for the congruence of the two triangles by the sss postulate. Name the pairs of equal angles resulting from this congruence.



(*a*) (*b*)

Fig. 10-12

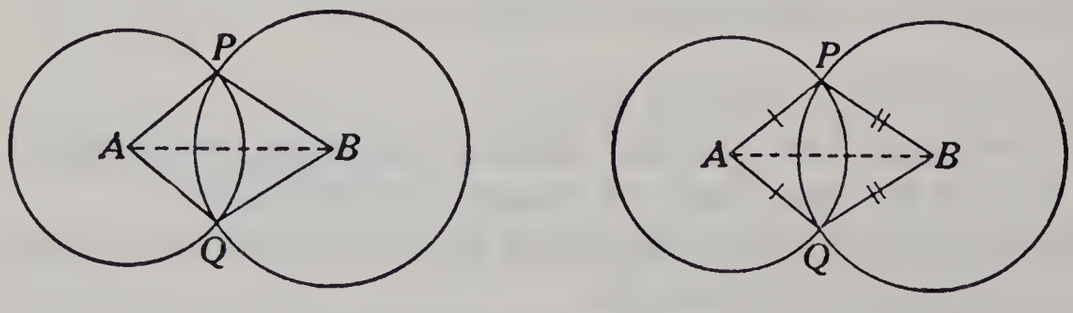
(a)

STATEMENTS	AUTHORITIES
1. In $\triangle$'s ABC and ADC $\begin{cases} AB = AD \\ BC = DC \\ AC = AC \end{cases}$	1. Hypothesis 2. Hypothesis 3. Reflexive
4. $\triangle ABC \cong \triangle ADC$	4. sss
5. $\angle ACB = \angle ACD$	5. Definition
6. $\angle BAC = \angle DAC$	6. Definition
7. $\angle B = \angle D$	7. Definition

(b)

STATEMENTS	AUTHORITIES
1. In $\triangle$'s ABC and CDA $\begin{cases} AB = CD \\ BC = DA \\ AC = CA \end{cases}$	1. Hypothesis 2. Hypothesis 3. Reflexive
4. $\triangle ABC \cong \triangle CDA$	4. sss
5. $\angle r = \angle x$	5. Definition
6. $\angle q = \angle y$	6. Definition
7. $\angle p = \angle z$	7. Definition

Example 2. In *Fig. 10-13* (*a*), if A and B are the centres of two intersecting circles, then line segment AB bisects $\angle PAQ$.



(*a*) (*b*)

Fig. 10-13

Hypothesis: Two circles with centres A and B intersecting on P and Q

Conclusion: $\angle PAB = \angle QAB$

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle PAB = \angle QAB$	1. Two triangles congruent with $\angle PAB$, $\angle QAB$ as corresponding angles.
2. Two triangles congruent with $\angle PAB$, $\angle QAB$ as corresponding angles	2. $\triangle PAB \cong \triangle QAB$.
3. $\triangle PAB \cong \triangle QAB$	3. $PA = QA$, $AB = AB$, $PB = QB$.
4. $PA = QA$ by definition.	
5. $AB = AB$ by the reflexive property.	
6. $PB = QB$ by definition.	

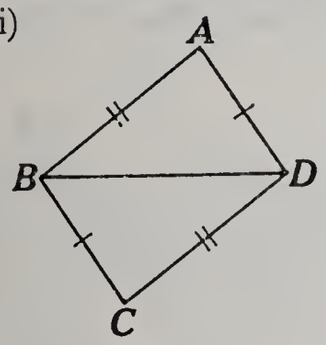
Proof: Write a complete proof; compare your proof with that on page 487.

Exercise 10-4

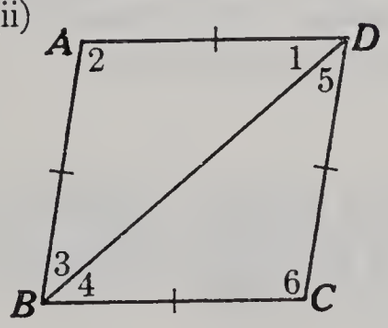
(A)

1. In the following diagrams, name the pairs of angles that are equal and justify your answers:

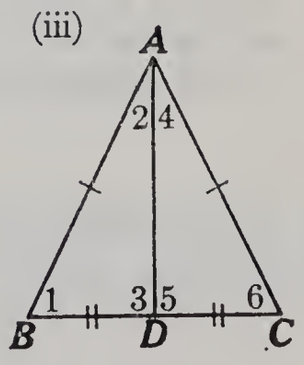
(i)



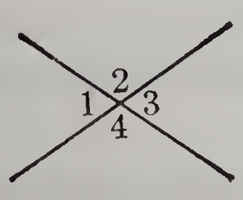
(ii)



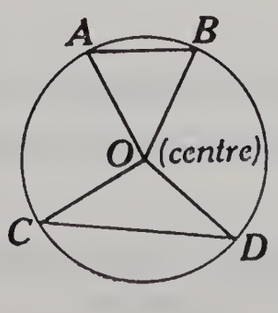
(iii)



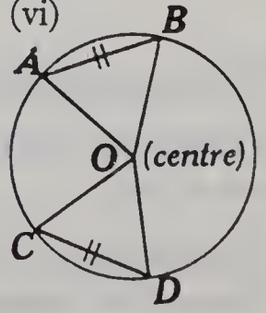
(iv)



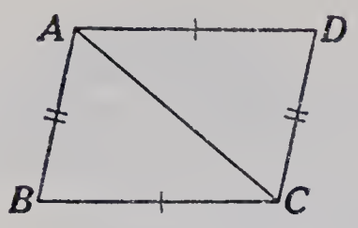
(v)

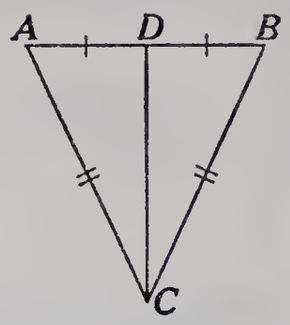


(vi)



2. In the diagram at the left, $AD = CB$, $AB = CD$. Prove that $\angle D = \angle B$.



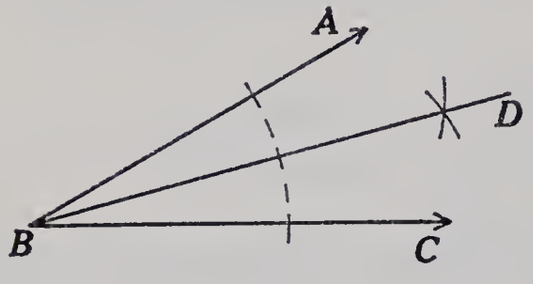


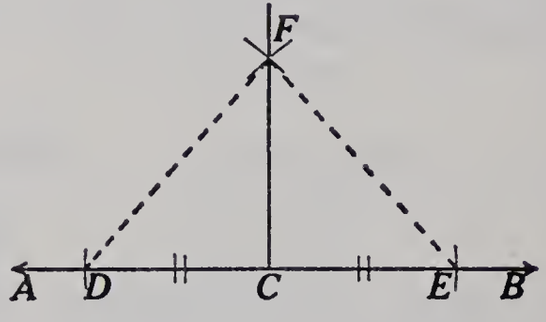
3. In the diagram at the right, $AC = BC$, $AD = BD$. Prove that $\angle ADC = \angle BDC = 90°$.

(B)

Write a complete proof of deductions 4 *and* 5:

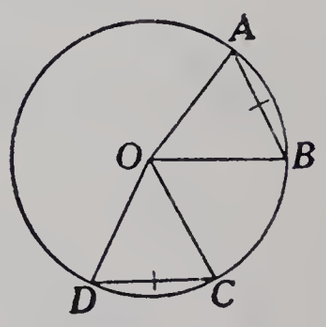
4. Write a proof to justify the ruler-compasses construction of the bisector of an angle (diagram at the left below). That is, prove BD bisects $\angle ABC$.





5. Write a proof to justify the ruler-compasses construction of a perpendicular to a line at a given point on the line. That is, prove $FC \perp AB$ (diagram at the right above).

6. In the accompanying diagram, AB and DC are equal chords in the circle with center O. Prove that $\angle AOB = \angle DOC$.



Write an analysis only of deductions 7 *and* 8:

7. Prove that the opposite angles of a quadrilateral with four equal sides are equal.

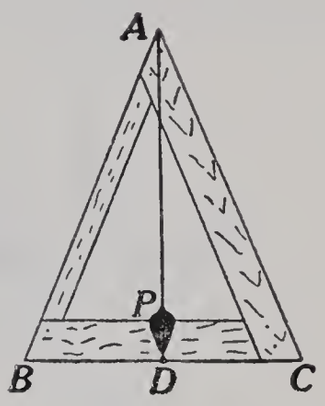
8. Prove that the diagonal of a quadrilateral with four equal sides bisects the angles through which it passes.

9. *Hypothesis:* ABC is an isosceles triangle in which $AB = AC$.

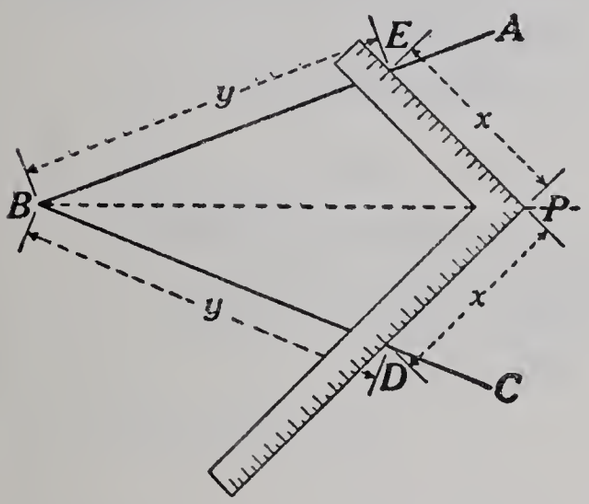
Conclusion: $\angle B = \angle C$

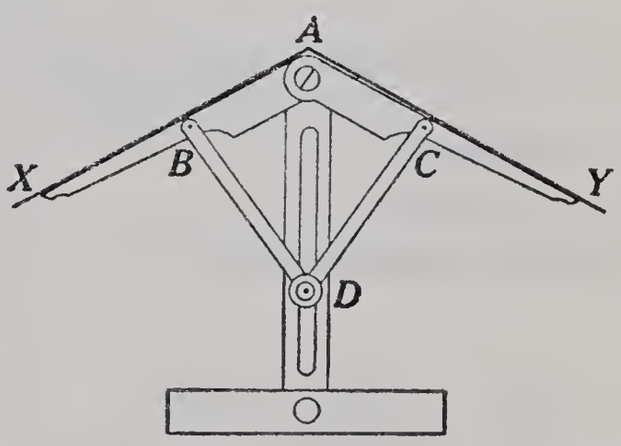
(HINT: Prove $\triangle ABC \cong \triangle ACB$.)

10. A carpenter wishes to test a joist to see if it is horizontal. Finding that he has forgotten his level, he nails three boards together to form $\triangle ABC$ having $AB = AC$. He marks D the midpoint of BC and then ties a plumb-line at A. If, when BC is placed on the joist, the bob P rests directly over D, prove that the joist is horizontal.



11. The diagram at the left below illustrates how a carpenter bisects $\angle ABC$ using a steel square. Study his method and prove it is correct.





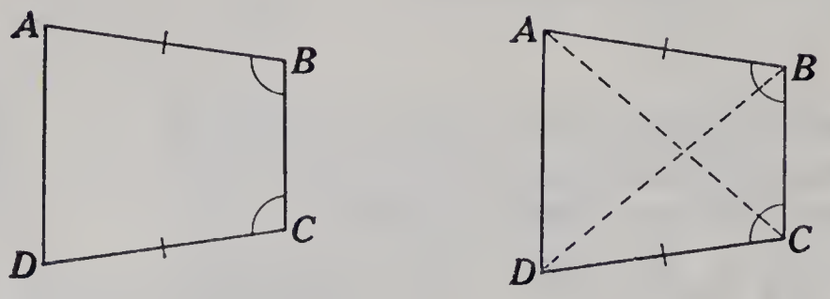
12. The diagram at the right above shows how a mechanical angle-divider is used to bisect $\angle XAY$. Study the construction of the instrument and explain why the centre of the slot AD is on the bisector of the angle.

***10·5* Multi-step deductions.** Examine the analysis of the following deduction carefully; write a complete proof and compare your proof with that on page 487.

Example. In *Fig. 10-14(a)* if $AB = DC$, $\angle ABC = \angle DCB$, then $\angle DAB = \angle ADC$.

Hypothesis: *Fig. 10-14*(*a*) in which $AB = DC$, $\angle ABC = \angle DCB$

Conclusion: $\angle DAB = \angle ADC$



(*a*) (*b*)

Fig. 10-14

Analysis:

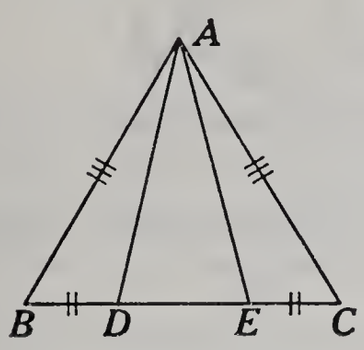
I CAN PROVE	IF I CAN PROVE
1. $\angle DAB = \angle ADC$	1. Two triangles congruent with $\angle DAB$, $\angle ADC$ as corresponding angles.
2. Two triangles congruent with $\angle DAB$, $\angle ADC$ as corresponding angles	2. $\triangle DAB \cong \triangle ADC$.
3. $\triangle DAB \cong \triangle ADC$	3. The conditions for sss.
4. The conditions for sss	4. $AB = DC$, $DA = AD$, $DB = AC$.
5. $AB = DC$ by hypothesis.	
6. $DA = AD$ by the reflexive property.	
7. $DB = AC$	7. Two triangles congruent with DB, AC as corresponding sides.
8. Two triangles congruent with DB, AC as corresponding sides	8. $\triangle DBC \cong \triangle ACB$.
9. $\triangle DBC \cong \triangle ACB$	9. $DC = AB$, $CB = BC$, $\angle DCB = \angle ABC$.
10. $DC = AB$ by hypothesis.	
11. $CB = BC$ by the reflexive property.	
12. $\angle DCB = \angle ABC$ by hypothesis.	

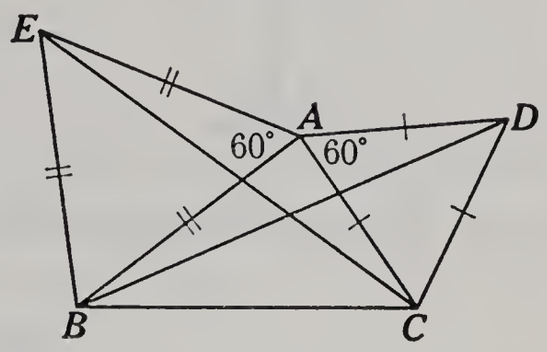
Exercise 10-5

(A)

1. Give an analysis of the following deduction:

If in the diagram at the left below, $AB = AC$, $AD = AE$, and $BD = CE$, prove $\angle ADE = \angle AED$.





(B)

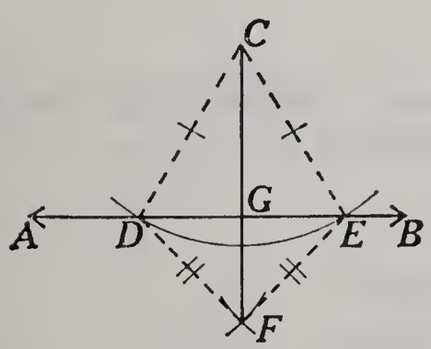
2. Write an analysis of the following deduction:

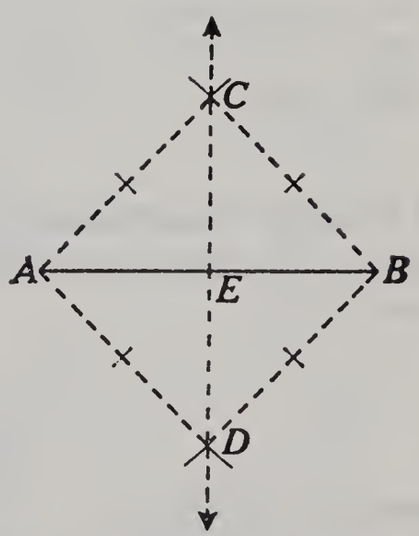
If, in the diagram at the right above, $EA = EB = BA$, $AC = CD = DA$, $\angle EAB = \angle DAC = 60°$, prove $EC = BD$.

3. Write an analysis and give a proof to justify the ruler-compasses methods of:

(i) constructing a perpendicular to a line from a point not in the line (diagram at the left below);

Prove $CG \perp AB$.



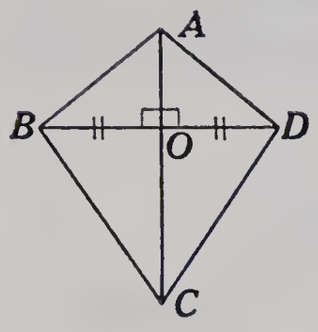


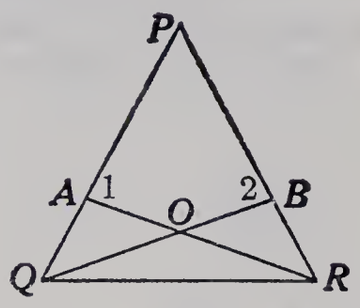
(ii) constructing the right bisector of a line segment (diagram at the right above).

Prove CD is the right bisector of AB.

Write an analysis of deductions 4 *and* 5 *below:*

4. In the diagram at the left below, if $BO = DO$ and the angles at O are each 90°, prove $\angle ABC = \angle ADC$.



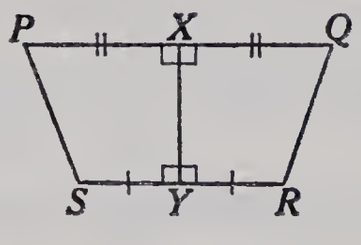


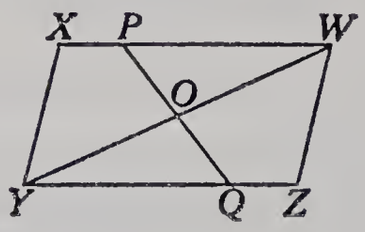
5. In the diagram at the right above, if $AQ = BR$ and $AR = BQ$, prove $\angle 1 = \angle 2$.

(C)

Write an analysis and proof of deductions 6 *and* 7 *below:*

6. In quadrilateral $PQRS$ (diagram left below), if $PX = QX$, $SY = RY$, $XY \perp PQ$, $XY \perp SR$, prove (i) $\angle P = \angle Q$ (ii) $PS = QR$.





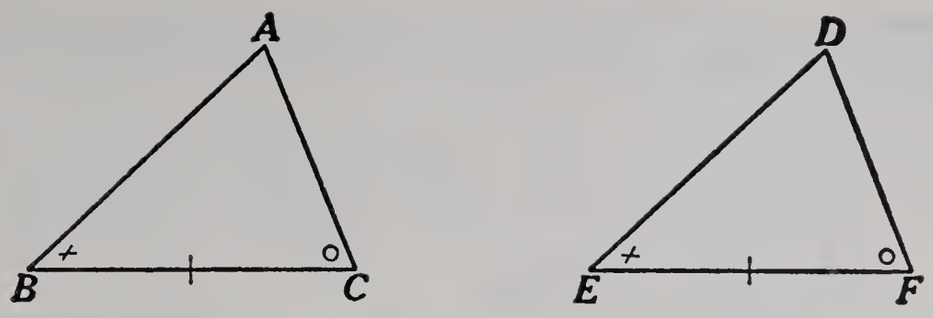
7. In quadrilateral $XYZW$ (diagram right above), if $XY = ZW$, $XW = ZY$, $OY = OW$, $XP = ZQ$, $\angle POQ$ is a straight angle, prove $OP = OQ$.

10·6 **Triangle Congruence Postulate (asa).** Suppose that two sets of non-collinear points in a plane represented by $\{A, B, C\}$ and $\{D, E, F\}$ are such that

$$BC = EF$$
$$\angle ABC = \angle DEF$$
$$\angle ACB = \angle DFE$$

as shown in *Fig. 10-15.*

Since $BC = EF$, there is a congruence matching of point set BC with point set EF such that $B \leftrightarrow E$ and $C \leftrightarrow F$. Since $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$, there is a congruence matching of these pairs of equal angles such that $B \leftrightarrow E$, $C \leftrightarrow F$, and A corresponds to a point on the ray



(*a*) (*b*)

Fig. 10-15

on ED and the ray on FD. Since D is the only point common to both of these rays, then $A \leftrightarrow D$. Thus, the given conditions are sufficient to provide a congruence matching of $\triangle ABC$ with $\triangle DEF$.

This illustration suggests the following postulate.

TRIANGLE CONGRUENCE POSTULATE (asa)

If two triangles have two angles and the contained side of one respectively equal to two angles and the corresponding side of the other, then the two triangles are congruent.

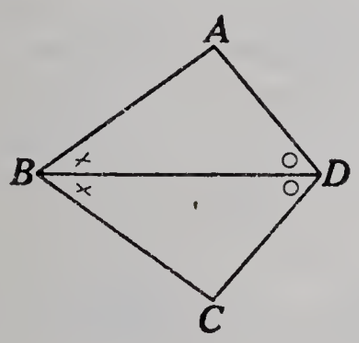
This postulate may be referred to as the "asa (angle, side, angle) postulate".

Exercise 10-6

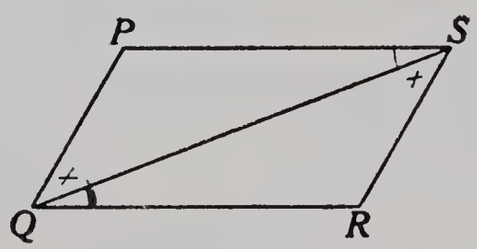
(A)

In each of the following diagrams, name a pair of congruent triangles; justify your answer:

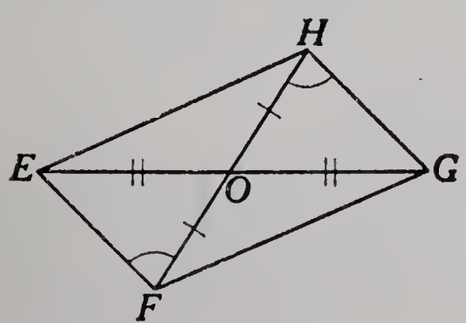
1.



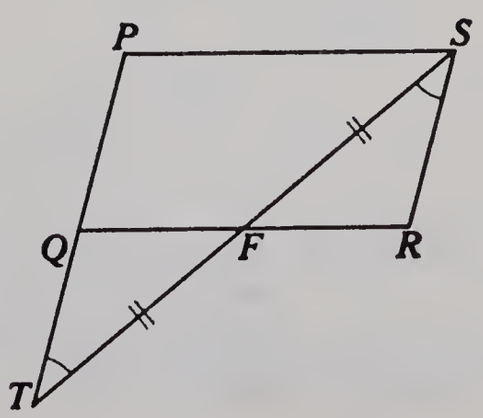
2.



3.

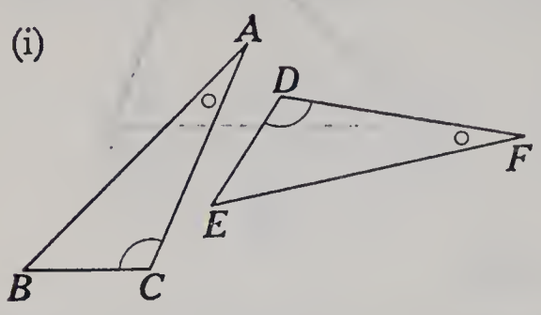


4.

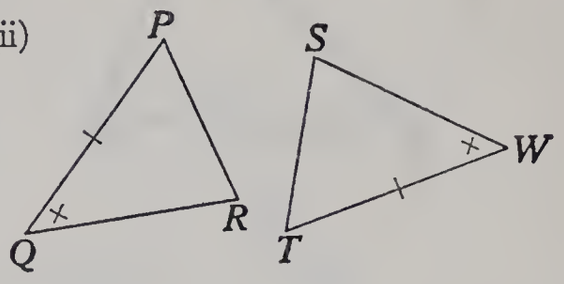


5. In each of the following pairs of triangles, state any third condition required for congruence.

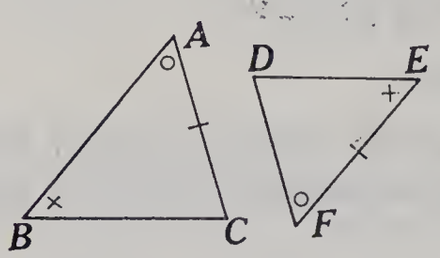
(i)

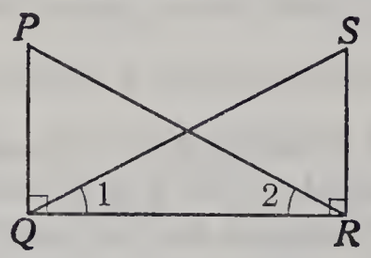


(ii)



6. Explain why it is not possible to say that the two triangles illustrated at the left below are congruent.





(B)

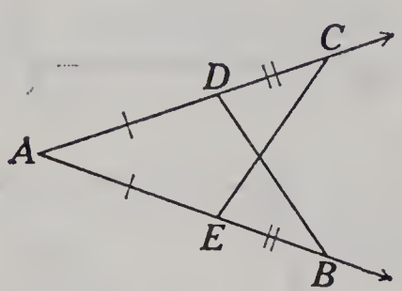
Write an analysis for deductions 7, 8, *and* 9:

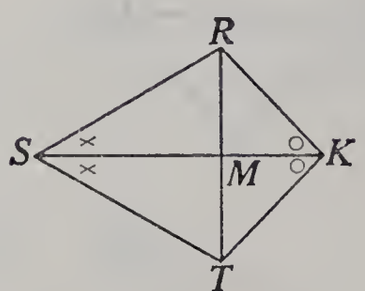
7. *Hypothesis:* Figure at right above in which $\angle 1 = \angle 2$, $PQ \perp QR$, $SR \perp QR$

Conclusion: $PQ = SR$

8. *Hypothesis:* Rays AC and AB with D on AC, E on AB, so that $AD = AE$, $DC = EB$, $\angle C = \angle B$ (diagram left below)

Conclusion: $CE = BD$

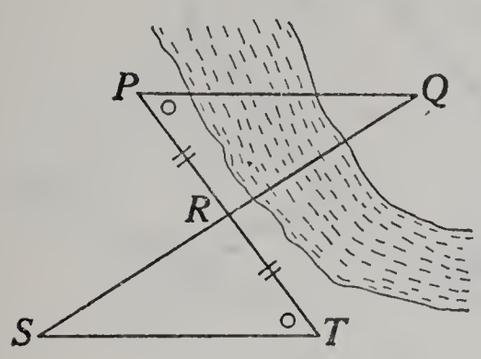


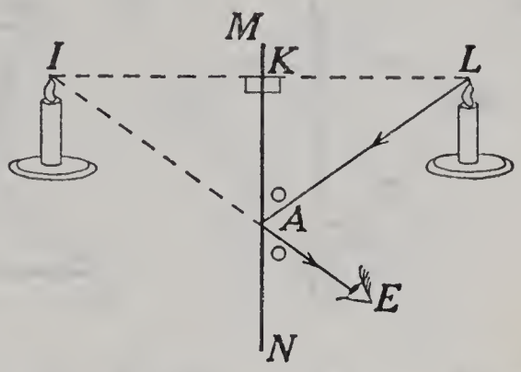


9. *Hypothesis:* Quadrilateral $RSTK$ with diagonals SK and RT intersecting on M, and $\angle RSM = \angle TSM$, $\angle RKM = \angle TKM$ (diagram right above)

Conclusion: $RM = TM$

10. The distance PQ across a river (diagram left below) was determined as follows. Stakes were set up at P and T. A third stake was placed at R, half-way between P and T. Then $\angle T$ was constructed equal to $\angle P$ and stake S located so that it was in line with R and Q. Prove that distance PQ across the river is given by the length of TS.





11. A ray of light from a candle, L, strikes a mirror, MN, at A and is reflected to the eye, E, making $\angle MAL = \angle NAE$. The image of the flame as seen by the eye appears to be at a point I such that $IKL \perp NM$ (diagram right above). Prove that $IK = LK$ and hence that the image is as far behind the mirror as the object is in front.

***10·7* Summary of basic authorities.** The basic authorities which are used to deduce angles equal, lines equal, triangles congruent are summarized in the following table:

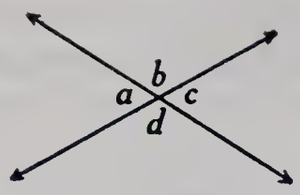
AUTHORITIES TO PROVE

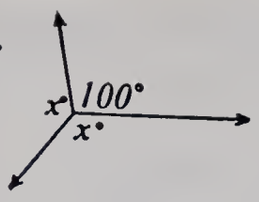
ANGLE = ANGLE	LINE SEGMENT = LINE SEGMENT	$\triangle \cong \triangle$
Complementary Angle Theorem	sas postulate	sas postulate
Supplementary Angle Theorem	asa postulate	sss postulate
Vertical Angle Theorem		asa postulate
sas postulate		
sss postulate		
asa postulate		

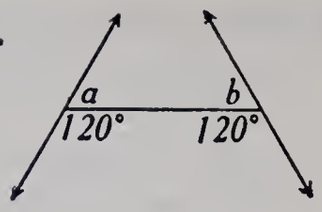
Practice Exercise 10-7

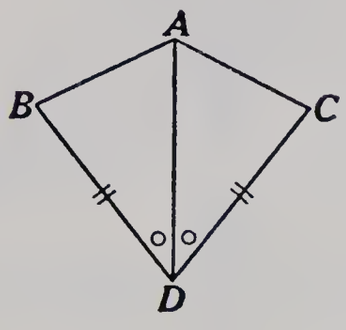
(A)

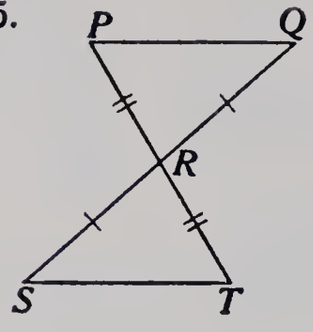
State all the facts you can about each diagram; justify your answers. Join points where necessary.

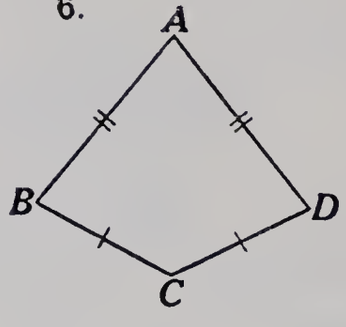
1. 

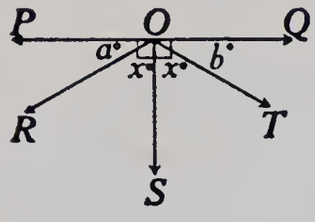
2. 

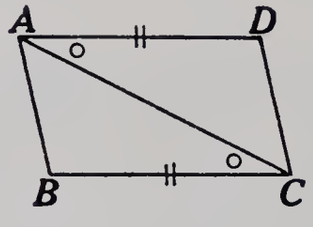
3. 

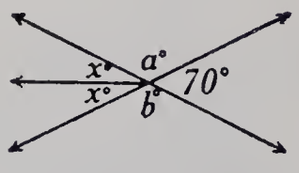
4. 

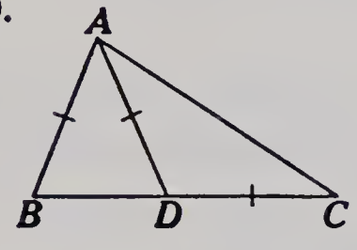
5. 

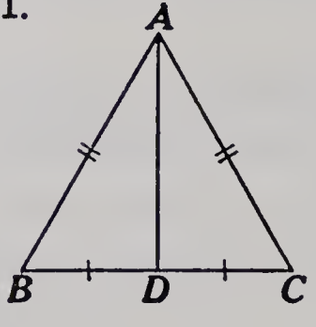
6. 

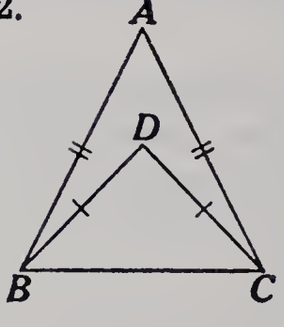
7. 

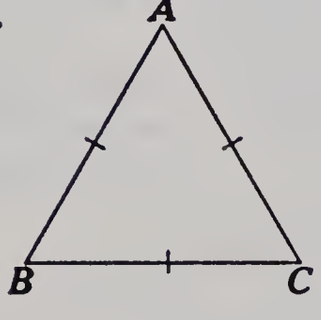
8. 

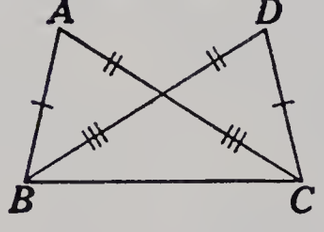
9. 

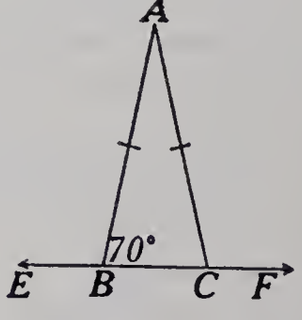
10. 

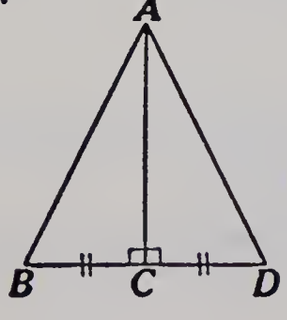
11. 

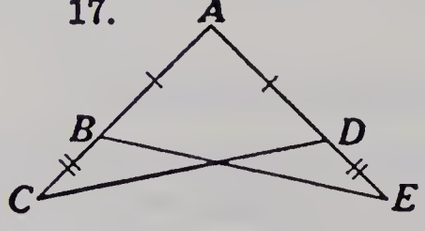
12. 

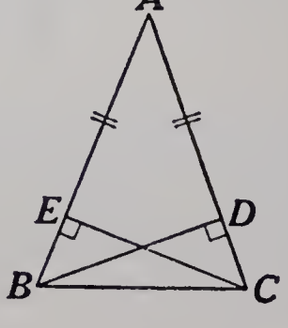
13. 

14. 

15. 

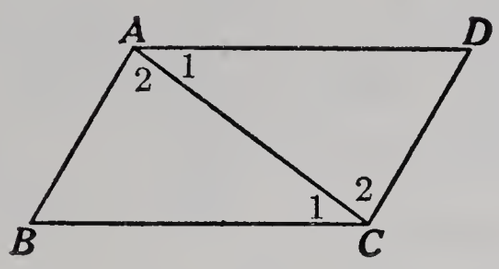
16. 

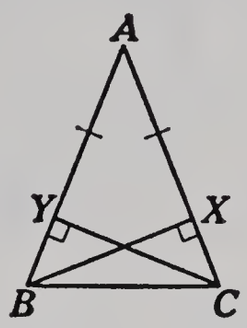
17. 

18. 

19. *Hypothesis:* $ABCD$ is a quadrilateral in which $\angle DAC = \angle BCA$, $\angle BAC = \angle DCA$ (diagram left below).

Conclusion: (i) $\angle B = \angle D$
(ii) $BA = DC$
(iii) $BC = DA$



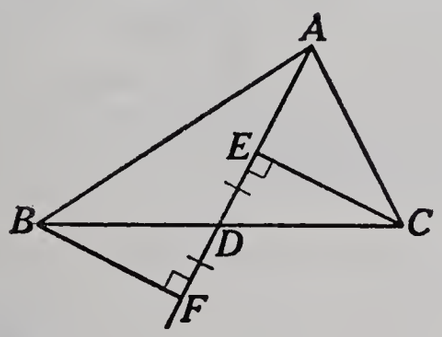


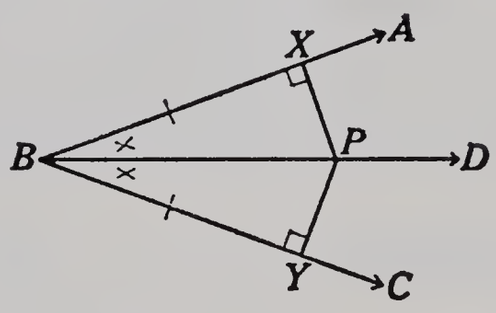
20. *Hypothesis:* ABC is a triangle in which $BX \perp AC$, $CY \perp AB$, $AX = AY$ (diagram right above).

Conclusion: $BX = CY$

21. *Hypothesis:* ABC is a triangle. $CE \perp AD$, $BF \perp AD$ extended, $ED = FD$ (diagram left below).

Conclusion: $BF = CE$



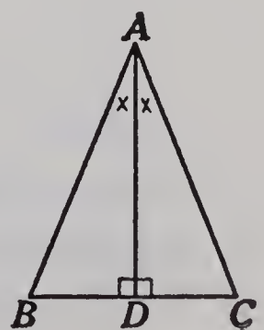


22. *Hypothesis:* BD is the bisector of $\angle ABC$. P is a point on BD and $PX \perp AB$, $PY \perp BC$, $BX = BY$ (diagram right above).

Conclusion: $PX = PY$

23. *Hypothesis:* ABC is a triangle in which AD bisects $\angle BAC$ and $AD \perp BC$.

Conclusion: (i) $AB = AC$
(ii) $BD = CD$



***10·8* Practical implications of congruent triangles.** The triangle congruence postulates indicate that, if certain parts of a triangle are given, the triangle is fixed as to size and shape.

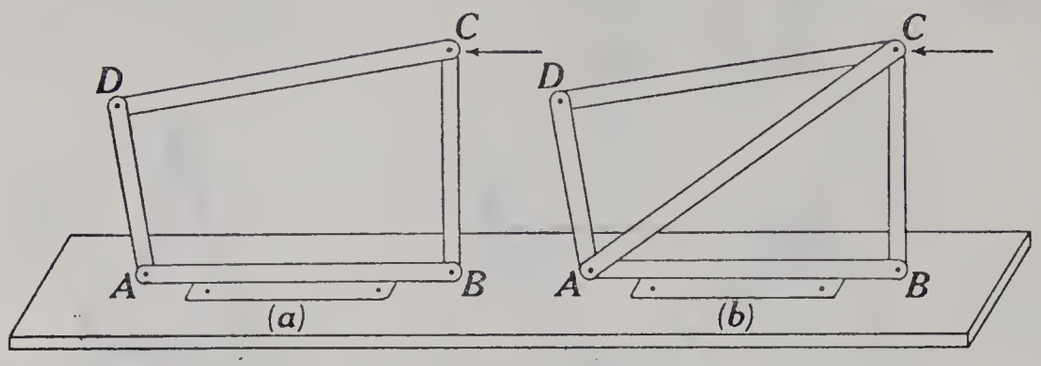
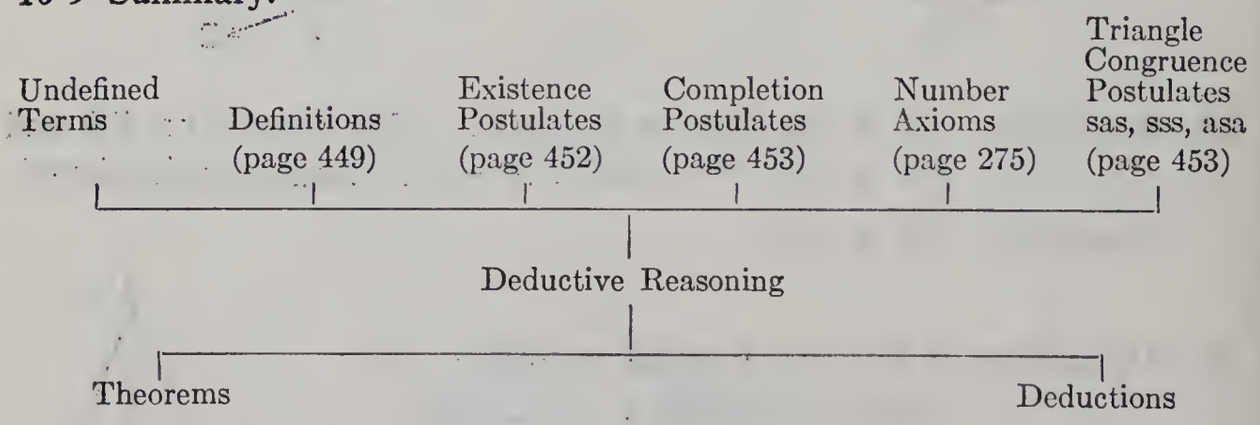


Fig. 10-16

In *Fig. 10-16* (*a*), four parts AB, BC, CD, and DA of a framework are pinned as shown; at A, B, C, and D. A slight push to the left at C causes distortion in the framework. Even if the joints are welded or nailed at A, B, C, and D, stability depends on the strength of the weld or joint. If an additional part AC is introduced, as in *Fig. 10-16* (*b*), thus creating $\triangle ABC$ and $\triangle DAC$, then rigidity is obtained. In this case the shape of the framework depends on the lengths of AB, BC, and CA in $\triangle ABC$. Thus, if the lengths of the three sides of a triangle are fixed, the triangle is fixed as to size and shape.

It should also be noted that in numerous forms of building construction and in the manufacture of many goods and machinery, rigidity in framework is obtained by using the triangle as a unit of design. This is commonly seen in steel bridges and arches, in hydro transmission towers, windmill stands, scaffolding, and barn construction.

***10·9* Summary.**



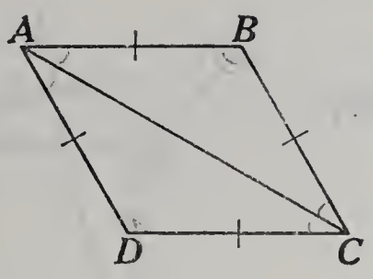
1. Complementary Angle Theorem
2. Supplementary Angle Theorem
3. Vertical Angle Theorem

Review Exercise 10-8

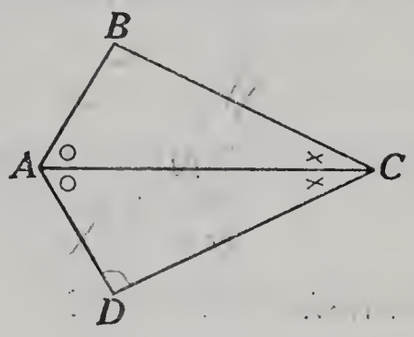
(A)

1. $\triangle ABC \cong \triangle DEF$. Without making a diagram, name the equal angles and the equal sides.
2. $\triangle RST \cong \triangle RSQ$. Without making a diagram name: (i) the equal angles (ii) the equal sides (iii) the side common to both triangles.
3. (a) State the authority for congruence of triangles in each of the following diagrams:

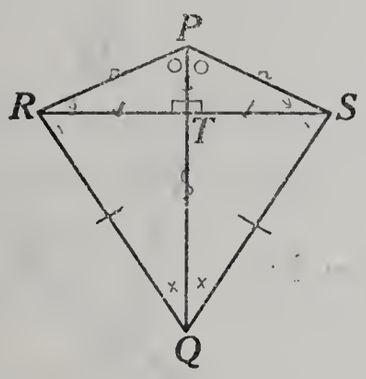
(i)



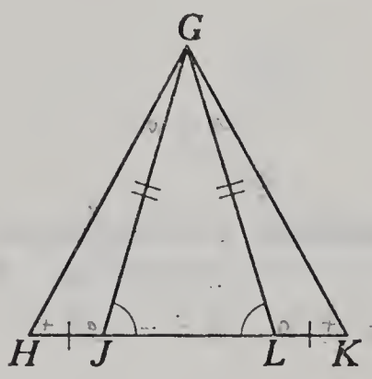
(ii)



(iii)



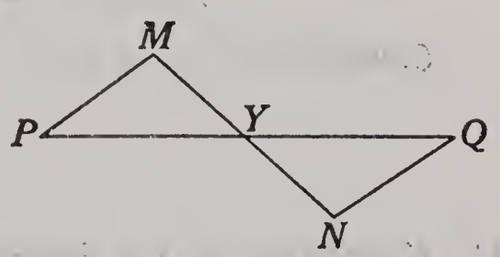
(iv)



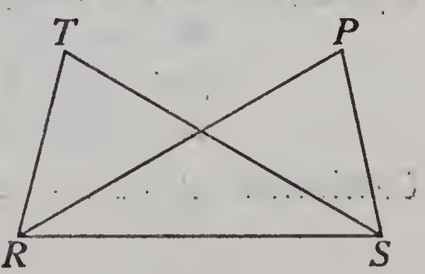
(b) For each diagram in (a) name pairs of equal lines and pairs of equal angles not marked in the diagrams.

Give a proof for each of the following:

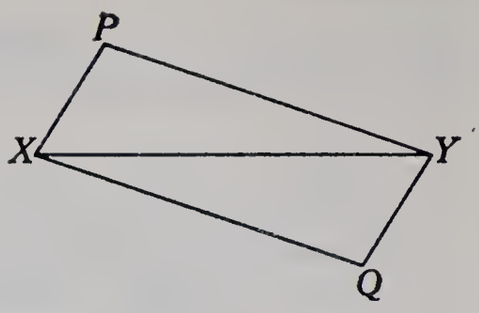
4. *Hypothesis:* $P \rightarrow Q, \sim R \rightarrow P, \sim R$
   Conclusion: Q
5. *Hypothesis:* PQ and MN bisect each other at Y.
   Conclusion: $\triangle PYM \cong \triangle QYN$



6. *Hypothesis:* $RT = SP, TS = PR$
   Conclusion: $\triangle RTS \cong \triangle SPR$



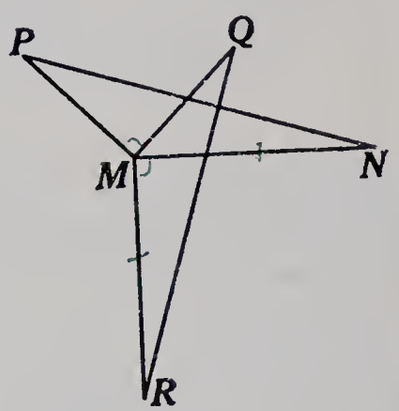
7. *Hypothesis:* $\angle PXY = \angle QYX$
$\angle PYX = \angle QXY$
Conclusion: $\triangle PXY \cong \triangle QYX$



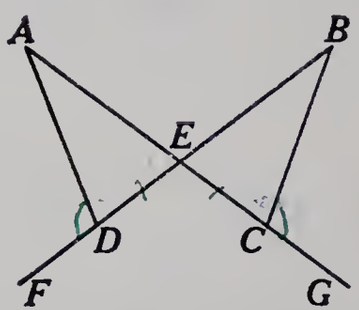
(B)

Write an analysis and proof for each of the following deductions:

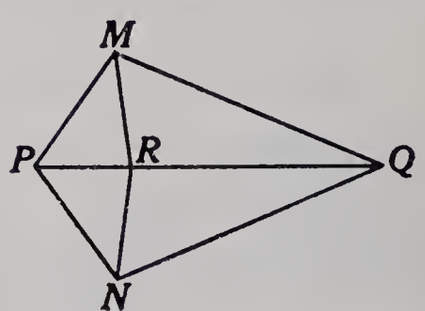
8. *Hypothesis:* In the diagram, $PM \perp QM$, $MN \perp MR$, $PM = MQ$, $MN = MR$.
Conclusion: $PN = QR$



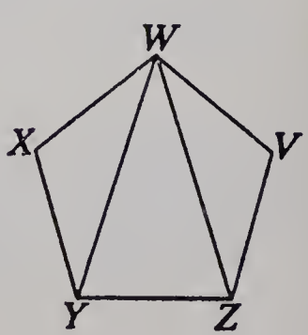
9. *Hypothesis:* In the diagram, $\angle ADF = \angle BCG$ and $ED = EC$.
Conclusion: $AE = BE$



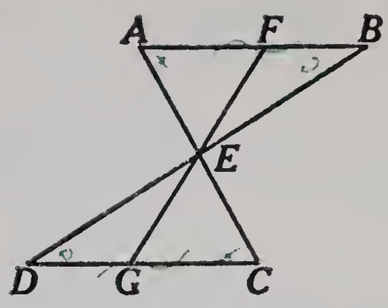
10. *Hypothesis:* In the diagram, $MQ = NQ$, $\angle MQR = \angle NQR$.
Conclusion: $PM = PN$



11. *Hypothesis:* In the diagram, $WX = XY = YZ = ZV = VW$ and $\angle WXY = \angle XYZ = \angle YZV = \angle ZVW = \angle VWX$.
Conclusion: $\angle WYZ = \angle WZY$

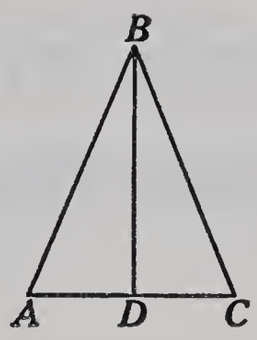


12. *Hypothesis:* $AB = CD$, $\angle BAC = \angle DCA$, $\angle BDC = \angle DBA$
Conclusion: $FE = GE$

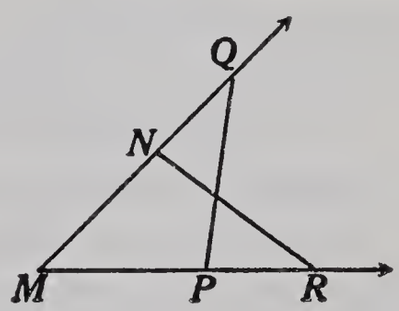


Show by analysis that in each of the following the hypothesis is insufficient to prove the conclusion:

13. *Hypothesis:* BD bisects $\angle ABC$.
Conclusion: $AD = CD$



14. *Hypothesis:* $MQ = MR$, $PQ = NR$
Conclusion: $MN = MP$



Chapter XI

CONVERSE STATEMENTS

CONVERSE THEOREMS

11·1 **Converse statements.** Consider the implication:

(1) *If John is born in Ontario, then John is born in Canada.*
If the hypothesis and the conclusion of this statement are interchanged, the following statement is formed.

(2) *If John is born in Canada, then John is born in Ontario.*

This second statement (2) is called the *converse* of statement (1); also, statement (1) is the converse of statement (2). It should be noted that statement (2) is not true even though the given implication (1) is true.

> DEFINITION: *When two conditional statements are so related that the hypothesis and conclusion consist of single statements which are interchanged, then each conditional statement is said to be the converse of the other.*

Consider the implication:

(1) *If John is the father of Tom, then Tom is the son of John.*
and its converse:

(2) *If Tom is the son of John, then John is the father of Tom.*
It should be noted that if statement (1) is true, then the converse statement (2) is also true.

Consider the implication:

(1) *If an animal is a dog, then the animal has four legs.*
and its converse:

(2) *If an animal has four legs, then the animal is a dog.*
Converse statement (2) is false, even though statement (1) is true.

An implication and its converse may be represented symbolically as follows:

IMPLICATION	CONVERSE
If P then Q	If Q then P
$P \rightarrow Q$	$Q \rightarrow P$

The statements $P \rightarrow Q$ and $Q \rightarrow P$ do not necessarily state the same thing. Thus, even though $P \rightarrow Q$ has been proven to be true, it must *not* be inferred that $Q \rightarrow P$ is also true. It is still necessary to prove the implication $Q \rightarrow P$ before it may be accepted as a theorem.

It should be recalled that all definitions are conditional statements. One of the important characteristics of a correct definition is that the statement formed by reversing the subject and predicate is also true. This characteristic can now be stated more precisely by saying that the converse of a correct definition is also true. This is illustrated in the statement of the following definition and its converse.

DEFINITION	CONVERSE
If a triangle has three equal sides, then it is an equilateral triangle.	If a triangle is equilateral, then it has three equal sides.

Example. State the converse of each of the following conditional statements; assess the truth of the converse and justify your assessment:

(i) If the measurement of an angle is less than 90°, then it is an acute angle.

(ii) If a geometric figure is a square, then it is a rectangle.

Solution.

(i) If an angle is an acute angle, then it has a measurement which is less than 90°. By our understanding of a correct definition this converse is true.

(ii) If a geometric figure is a rectangle, then it is a square.

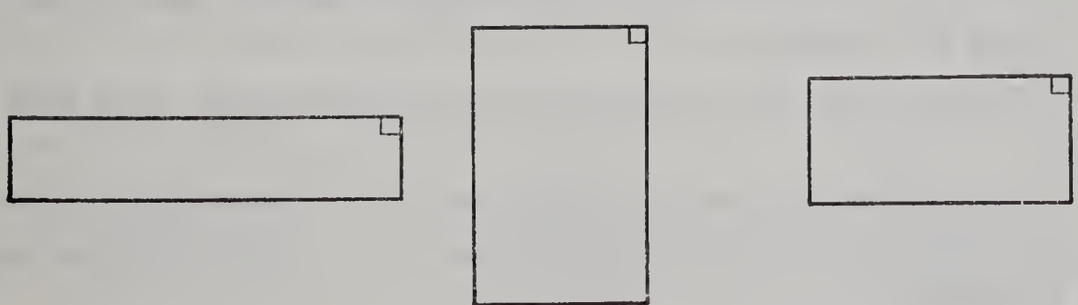


Fig. 11-1

Our experience suggests that there are many rectangles which are not squares, such as those shown above. Thus, this converse is false.

Exercise 11-1

(A)

Copy and complete the following table by writing in the second column the converse of the implication in the first column; assess the truth of the converse:

IMPLICATION	CONVERSE	TRUTH OF CONVERSE
1. If angles are right angles, then they are equal angles.	**1.** If angles are equal angles, then they are right angles.	False
2. If $x = 4$, then $3x - \frac{1}{2} = 14\frac{1}{2}$.	**2.**	
3. If an animal is a dog, then it has four legs.	**3.**	
4. If a geometric figure is a square, then it is a rhombus.	**4.**	
5. If a polygon is a square, then it is a parallelogram.	**5.**	
6. If it is raining, then it is cloudy.	**6.**	
7. If a bicycle has a weak chain, then it is unsafe.	**7.**	
8. If $x = 3$, then $2x + 5 = 11$.	**8.**	
9. If two lines are perpendicular, then they intersect at right angles.	**9.**	
10. If A, then B.	**10.**	

(B)

Write the converse of each of the following implications, if there is one; assess the truth of the converse statement in each case; justify your assessment:

11. If two sides and the contained angle of one triangle are respectively equal to two sides and the contained angle of another triangle, the two triangles are congruent.

12. If two angles are the supplements of the same angle, then they are equal.

13. If a triangle has two unequal sides, then it has two unequal angles.

14. If an angle is part of a given angle, then it is less than the given angle.

15. It is snowing.

16. If two angles are complementary, then their angle sum is 90°.

17. *Hypothesis:* $\triangle ABC$ in which $\angle A = \angle B = \angle C$
Conclusion: $BC = AC = AB$

18. *Hypothesis:* Lines AB and CD intersect on E.
Conclusion: $\angle AED = \angle CEB$

11·2 "If and only if" connective; biconditional sentences.

The conditional sentence:

(1) *If a number is an even number, then the number is a multiple of 2.*
and its converse,

(2) *If a number is a multiple of 2, then the number is an even number.*
involve the following two simple sentences:

(a) *The number is an even number.*
(b) *The number is a multiple of 2.*

The *if and only if* connective may be used to combine the simple sentences (a) and (b) into a compound sentence which contains the conditions expressed in both (1) and (2). Thus, the statement, *A number is an even number if and only if the number is a multiple of 2,* includes the implication in (1) and its converse (2). Since this sentence contains two conditions it is called a *biconditional sentence.*

Example 1. Write a biconditional sentence which includes the following conditional sentence and its converse:
If a triangle has two equal sides, then it has two equal angles.

Solution. The biconditional sentence is:
A triangle has two equal sides if and only if it has two equal angles.

Example 2. Write the two converse implications expressed by the biconditional sentence:
A polygon is a triangle if and only if it has three sides.

Solution. The two converse implications are:
(1) If a polygon is a triangle, then it has three sides.
(2) If a polygon has three sides, then it is a triangle.

In general, if P and Q represent any two statements, then implication, converse, and biconditional statements may be represented as follows:

IMPLICATION	CONVERSE	BICONDITIONAL
$P \rightarrow Q$	$Q \rightarrow P$	$P \leftrightarrow Q$
If P then Q	If Q then P	P if and only if Q
P implies Q	Q implies P	P is equivalent to Q

It should be noted that before an *if and only if* statement is accepted as true, it must be shown that *both* the statements included in the biconditional sentence are true.

Exercise 11-2

(A)

State the two conditional statements contained in each of the following biconditional statements:

1. John will become tanned if and only if he gets more sunshine.
2. Iron will melt if and only if it is heated to 1535°C.
3. A geometric figure is a triangle if and only if it has three vertices.
4. $3 \times 4 = 22$ if and only if the numerals are base 5.
5. $AB + BC = AC$ if and only if B is between A and C.
6. $3x + 7 = 13$ if and only if $x = 2$.
7. A if and only if B.
8. Two lines are parallel if and only if they have no point in common.
9. A figure is a square if and only if it is a rectangle.
10. Two triangles are congruent if and only if two angles and the contained side of one are respectively equal to two angles and the corresponding side of the other.

State the converse of each of the following statements and then express each pair of statements by a biconditional sentence:

11. If the set of all factors of a number has exactly two members, then the number is a prime number.
12. If a set has precisely no members, then it is a null set.
13. If zero is defined to be a multiple of 2, then zero is an even number.
14. If two sides of a triangle are equal, then the angles opposite these sides are equal.
15. If a point is on the right bisector of a given line segment, then it is equidistant from the end points of the line segment.
16. If a point is on the bisector of an angle, then it is equidistant from the sides of the angle.
17. If points are on a line parallel to a given line, then they are the same distance from the given line.
18. If points are on a circle, then they are the same distance from a fixed point.

11·3 **Converse theorems; "if and only if" theorems.** Since many of the theorems of geometry often involve several statements in the conclusion and several statements in the hypothesis, a converse theorem is defined as follows:

DEFINITION: *A converse theorem is a theorem that is formed from a given theorem by interchanging any number of statements in the hypothesis of the given theorem with an equal number of statements in the conclusion.*

It should always be remembered that the truth of a converse does not follow from the original theorem. The truth of a converse of a given theorem must be established before it is accepted as a theorem.

In geometry, it is convenient to combine a theorem and its converse theorem into a single statement by using the *if and only if* form. The combined theorem may be referred to as an *if and only if* theorem. To establish an *if and only if* theorem it is necessary to establish:

(i) the implication,

(ii) the converse of the implication.

The theorems in this chapter are *if and only if* theorems.

***11·4* The Isosceles Triangle Theorem.** We have discovered by inductive reasoning the following relations in an isosceles triangle.

IMPLICATION	CONVERSE
(i) If two sides of a triangle are equal, then the angles opposite these sides are equal.	(ii) If two angles of a triangle are equal, then the sides opposite these angles are equal.

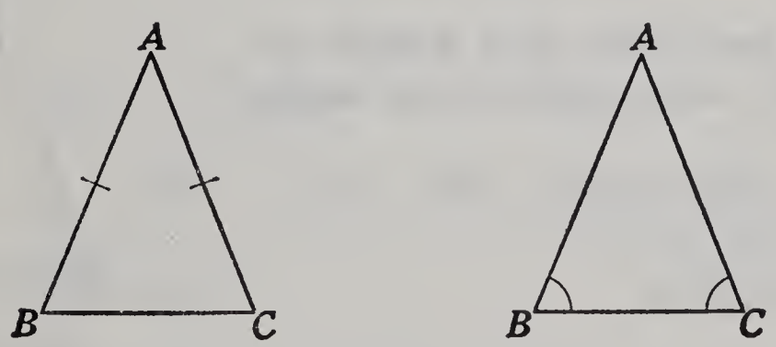


Fig. 11-2

Hypothesis: $AB = AC$ → *Hypothesis:* $\angle B = \angle C$

Conclusion: $\angle B = \angle C$ → *Conclusion:* $AB = AC$

These two relations expressed by a biconditional sentence form the *Isosceles Triangle Theorem.* A complete demonstration of the proof of each part of this theorem follows.

THEOREM 4

Isosceles Triangle Theorem

Two angles of a triangle are equal if and only if the sides opposite these angles are equal.

PART I

If two sides of a triangle are equal, then the angles opposite these sides are equal.

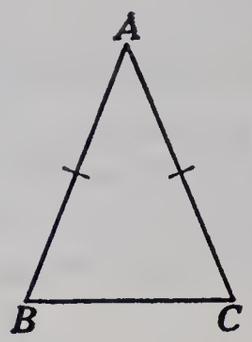


Fig. 11-3

Hypothesis: $\triangle ABC$ in which $AB = AC$ (*Fig. 11-3*)

Conclusion: $\angle ABC = \angle ACB$

Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s ABC and ACB $\begin{cases} AB = AC \\ AC = AB \\ \angle BAC = \angle CAB \end{cases}$	1. Hypothesis 2. Hypothesis 3. Reflexive
(Note the order of correspondence.)	
4. $\triangle ABC \cong \triangle ACB$	4. sas
5. $\angle ABC = \angle ACB$	5. Definition

PART II

Converse: If two angles of a triangle are equal, then the sides opposite these angles are equal.

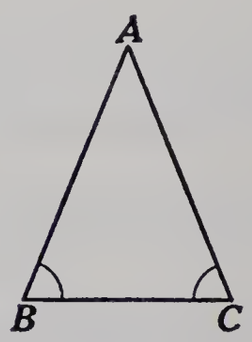


Fig. 11-4

Hypothesis: $\triangle ABC$ in which $\angle B = \angle C$ (*Fig. 11-4*)

Conclusion: $AB = AC$

Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s ABC and ACB $\begin{cases} \angle ABC = \angle ACB \\ \angle ACB = \angle ABC \\ BC = CB \end{cases}$	1. Hypothesis 2. Hypothesis 3. Reflexive
(Note the order of correspondence.)	
4. $\triangle ABC \cong \triangle ACB$	4. asa
5. $AB = AC$	5. Definition

Corollary: **A triangle is equilateral if and only if it is equiangular.**

Example. In $\triangle ABC$, *Fig. 11-5*, $AD = BD = DC$. Prove $\angle BAC = \angle 1 + \angle 4$

Hypothesis: $\triangle ABC$ in which $AD = BD = DC$

Conclusion: $\angle BAC = \angle 1 + \angle 4$

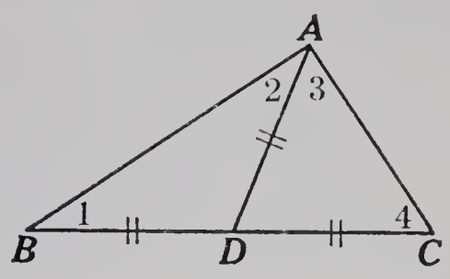


Fig. 11-5

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle BAC = \angle 1 + \angle 4$	1. $\angle 1 + \angle 4 = \angle 2 + \angle 3$.
2. $\angle 1 + \angle 4 = \angle 2 + \angle 3$	2. $\angle 1 = \angle 2$, $\angle 4 = \angle 3$.
3. $\angle 1 = \angle 2$, $\angle 4 = \angle 3$ by the Isosceles Triangle Theorem.	

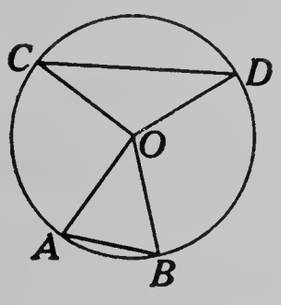
Proof: Write a complete proof; compare your proof with that on page 488

Exercise 11-3

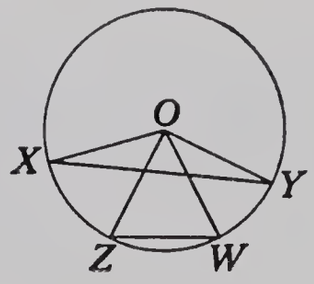
(A)

1. In each of the following diagrams, if O is the centre of the circle, name the pairs of equal angles, giving reasons:

(i)

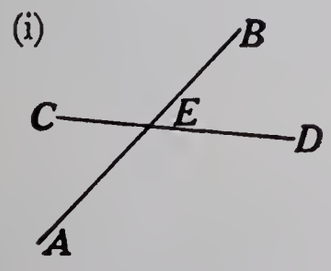


(ii)

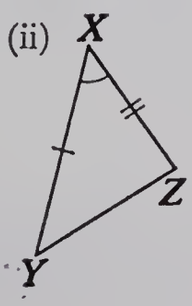


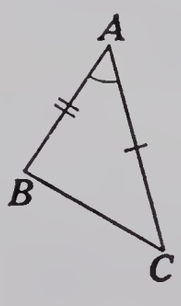
2. In the following diagrams name the pairs of equal angles, giving reasons:

(i)

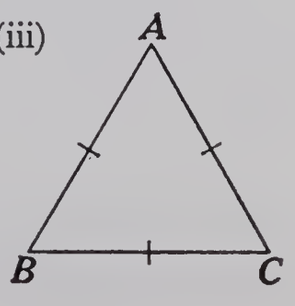


(ii)



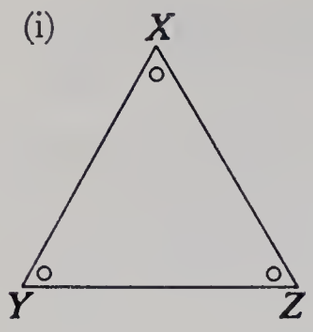


(iii)

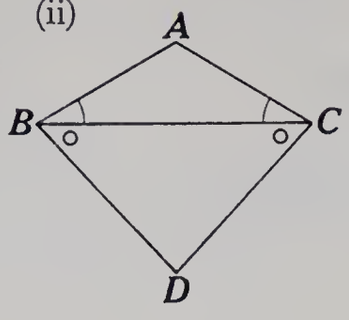


3. In each of the following diagrams, name the equal line segments, giving authorities:

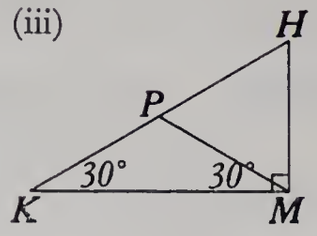
(i)



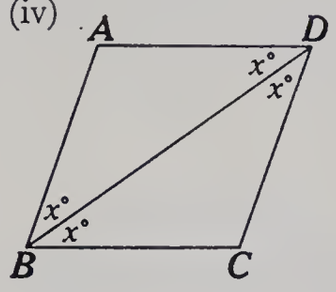
(ii)



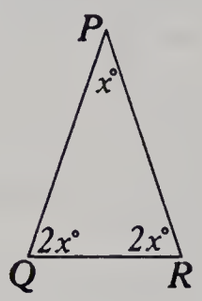
(iii)



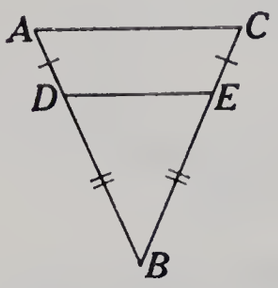
(iv)

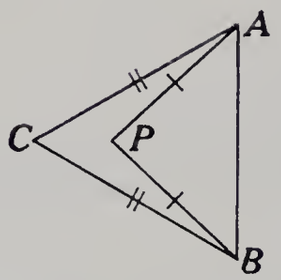


(v)



4. In the diagram at the left, prove $AB = BC$ and $\angle A = \angle C$.

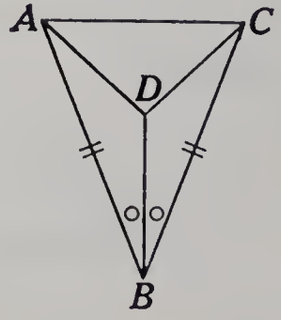




5. In the diagram at the right, prove $\angle CAP = \angle CBP$.

6. State all the authorities now established that prove angles equal.

(B)



7. Write an analysis of the following deduction:
In the accompanying diagram, $AB = BC$ and $\angle ABD = \angle CBD$.
Prove that $\angle DAC = \angle DCA$.

8. Write an analysis of the following deduction:
In quadrilateral $PQRS$, if $PQ = PS$ and $\angle Q = \angle S$, then $RQ = RS$.

Write a complete proof of each of the following deductions:

9. ABC is a triangle with $AB = AC$. BC is extended to D, and CB is extended to E so that $CD = BE$. Prove that $AE = AD$.

10. Prove that the midpoint of the base of an isosceles triangle is equidistant from the midpoint of the other two sides.

11. If a triangle is equilateral, then the line segments joining the midpoints of the sides are equal.

12. In $\triangle PQR$, $PQ = PR$. QR is extended its own length to S, and PQ is extended its own length to T. Prove that $TR = PS$.

***11·5* Historical note on Theorem 4.** It is believed that Part II of Theorem 4 was first proved by Thales (*ca.* 600 B.C.). His method of proof was later adopted by Euclid (*ca.* 300 B.C.) in his text on geometry called *The Elements*. His proof, which follows, differed greatly from the one on page 336.

In $\triangle ABC$ (*Fig. 11-6*), $AB = AC$. It is required to prove $\angle ABC = \angle ACB$.

Let F and G represent points on AB and AC extended, respectively, so that $CG = BF$. Join CF and BG.

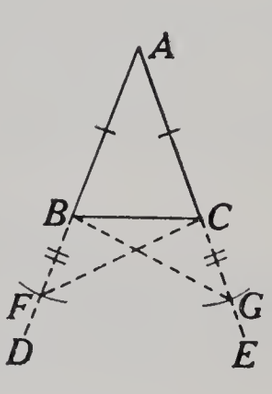


Fig. 11-6

Now try to prove: (i) $\triangle AFC \cong \triangle AGB$

(ii) $\triangle BFC \cong \triangle CGB$

(iii) $\angle ABC = \angle ACB$

For various reasons this deduction, with the proof suggested above, was known as *pons asinorum* or bridge of fools. It probably represented the limit in mathematical attainment for many students in the Middle Ages and hence formed the bridge across which fools could not hope to pass. It was also suggested that *Fig. 11-7*, as used by Euclid, resembled the simplest form of truss-bridge design, one that even the poorest student could make.

***11·6* The Right Bisector Theorem.** The conditional statement:

If a point is on the right bisector of a line segment, then it is equidistant from the end points of the line segment.

and its converse,

If a point is equidistant from the end points of a line segment, then it is on the right bisector of the line segment.

are combined in the biconditional statement of the *Right Bisector Theorem*,

THEOREM 5

Right Bisector Theorem

A point is equidistant from the end points of a line segment if and only if it is on the right bisector of the line segment.

PART I

If a point is on the right bisector of a line segment, then it is equidistant from the end points of the line segment.

Hypothesis: CED is the right bisector of AB and P is a point on CD (*Fig. 11-7*).

Conclusion: P is equidistant from A and B; that is, $PA = PB$.

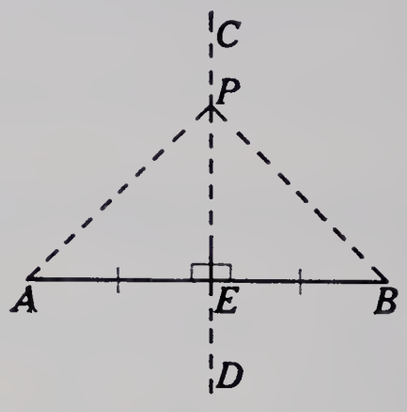


Fig. 11-7

Proof: Copy and complete the authorities column of this proof; compare your completed proof with the one on page 488.

STATEMENTS	AUTHORITIES
1. Join PA and PB.	1.
2. In $\triangle$'s PAE and PBE $\left\{ \begin{array}{r} AE = BE \\ PE = PE \\ \angle PEA = \angle PEB \end{array} \right.$	2.
3.	3.
4.	4.
5. $\triangle PAE \cong \triangle PBE$	5.
6. $PA = PB$	6.

PART II

Converse: **If a point is equidistant from the end points of a line segment, then it is on the right bisector of the line segment.**

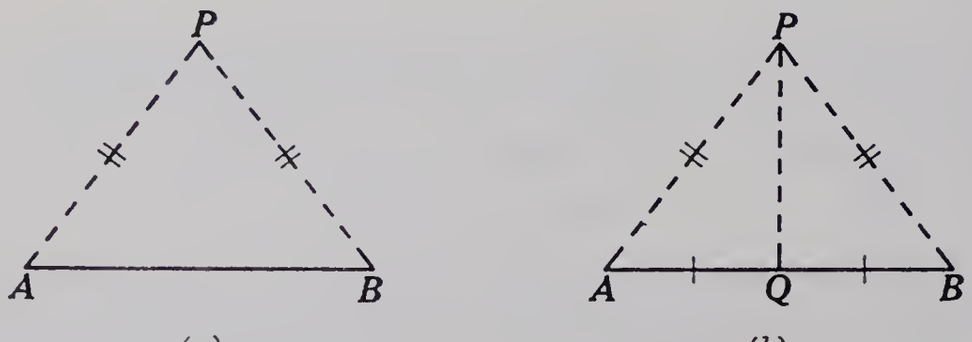


Fig. 11-8

Hypothesis: A point P and line segment AB such that $PA = PB$, *Fig. 11-8* (*a*)

Conclusion: P is on the right bisector of AB.

Analysis:

I CAN PROVE	IF I CAN PROVE
1. P is on the right bisector of AB	1. P and a second position of P, that is, Q, *Fig. 11-8* (*b*), determine the right bisector of AB (since two points determine a line).
2. PQ is the right bisector of AB	2. $\angle PQA = \angle PQB = 90°$.
3. $\angle PQA = \angle PQB = 90°$	3. $\triangle PAQ \cong \triangle PBQ$.
4. $\triangle PAQ \cong \triangle PBQ$ by the sss postulate.	

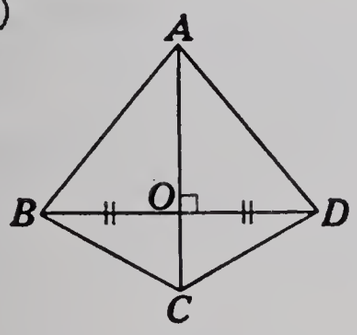
Proof: Write a complete proof; compare your proof with that on page 488.

Exercise 11-4

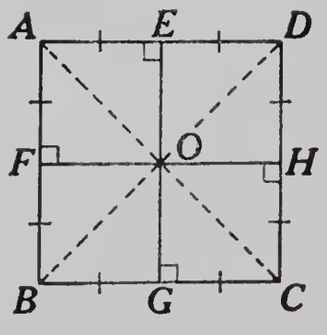
(A)

1. Study each of the following marked diagrams and state, giving reasons, any additional facts which are true:

(i)

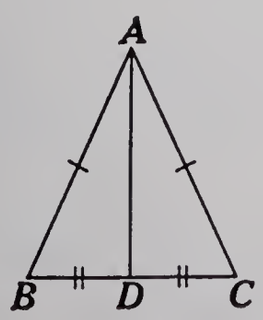


(ii)

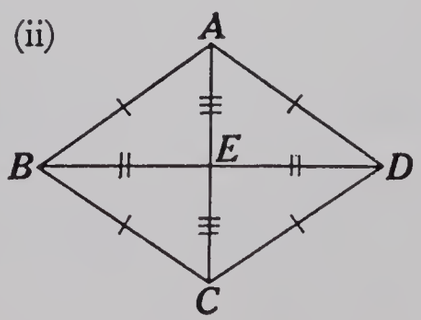


2. State all the facts you can about each of the following diagrams, giving reasons:

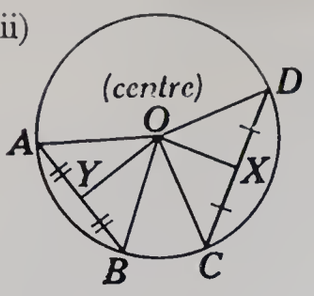
(i)



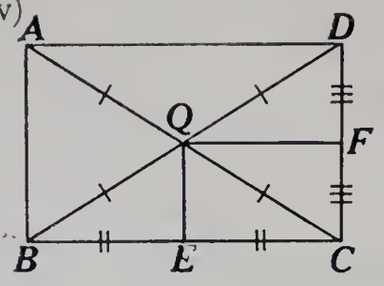
(ii)



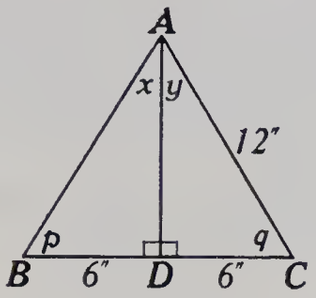
(iii)



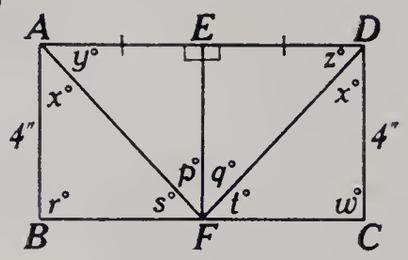
(iv)



(v)



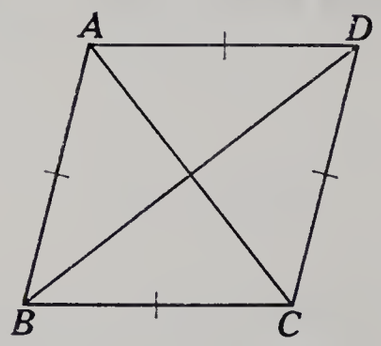
(vi)



(B)

3. Examine carefully the analysis of the following deduction; then write a complete proof of the deduction:

The diagonals of a quadrilateral with four equal sides bisect each other at right angles.



Hypothesis: Quadrilateral $ABCD$ in which $AB = BC = CD = DA$

Conclusion: AC bisects BD at right angles; BD bisects AC at right angles.

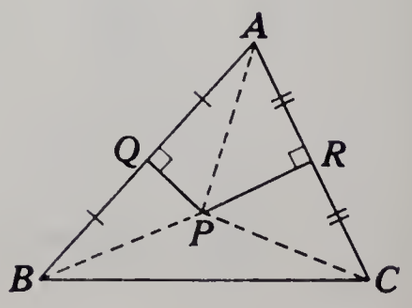
Analysis:

I CAN PROVE	IF I CAN PROVE
1. AC bisects BD at right angles	1. A and C are on the right bisector of BD.
2. A and C are on the right bisector of BD	2. A and C are equidistant from B and D.
3. A and C are equidistant from B and D, since $AB = AD$ and $BC = DC$ by definition.	

Give a complete proof of each of the following deductions:

4. AB and XY bisect each other at right angles on E. Prove that the quadrilateral formed by joining the end points of the line segments has all sides equal.

5. In the diagram at the right, ABC is any triangle. The right bisectors of AB and AC meet at P.
Prove $AP = BP = CP$.



6. Prove that the right bisectors of the sides of a triangle are concurrent.
7. AB and BC are two chords in a circle. The right bisectors of AB and BC meet on O. Prove $AO = BO = CO$.

***11·7* Multiple converses (supplementary).** *A converse theorem* has been defined as follows:

> *A converse theorem is a theorem that is formed from a given theorem by interchanging any number of statements in the hypothesis of the given theorem with an equal number of statements in the conclusion.*

Using this definition, it is possible to write five converses of the following deduction.

Hypothesis: $\triangle ABC$ in which $AB = AC$, $AD \perp BC$, *Fig. 11-9*
Conclusion: $BD = CD$, $\angle BAD = \angle CAD$

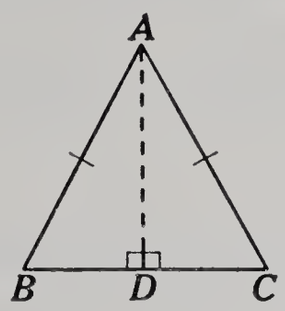


Fig. 11-9

Converse 1:
Hypothesis: $\triangle ABC$ in which $BD = CD$, $AD \perp BC$
Conclusion: $AB = AC$, $\angle BAD = \angle CAD$

Converse 2:
Hypothesis: $\triangle ABC$ in which $AB = AC$, $\angle BAD = \angle CAD$
Conclusion: $BD = CD$, $AD \perp BC$

Converse 3:
Hypothesis: $\triangle ABC$ in which $\angle BAD = \angle CAD$, $AD \perp BC$
Conclusion: $BD = CD$, $AB = AC$

Converse 4:
Hypothesis: $\triangle ABC$ in which $AB = AC$, $BD = CD$
Conclusion: $AD \perp BC$, $\angle BAD = \angle CAD$

Converse 5:
Hypothesis: $\triangle ABC$ in which $BD = CD$, $\angle BAD = \angle CAD$
Conclusion: $AB = AC$, $AD \perp BC$

Exercise 11-5

(B)

1. Write a proof for the given theorem in section 11·7 page 343, and write a proof for each of converses 1 to 4 of this theorem. Draw a diagram to show that $\triangle ABC$ in converse 5 is not necessarily isosceles.
2. State all possible converses of the following theorem:

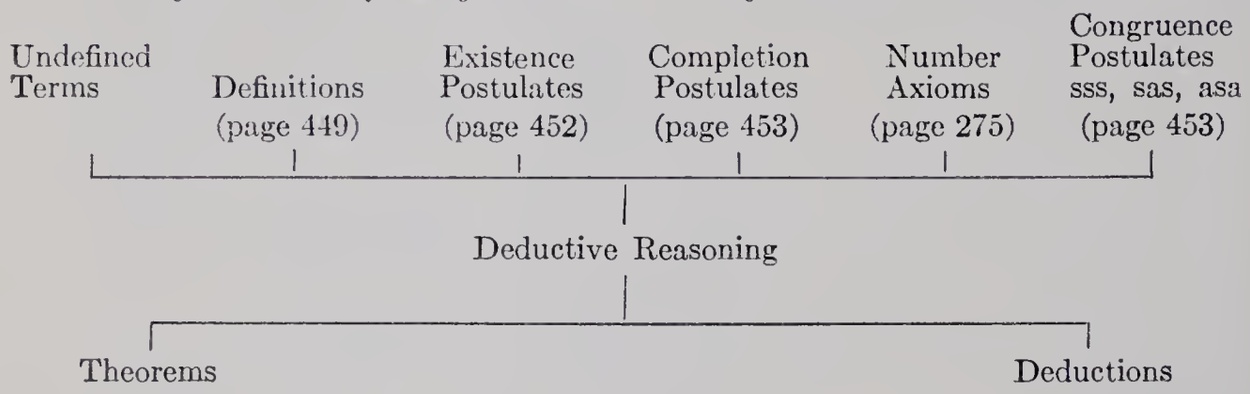
   Hypothesis: ABC and DEF are two triangles such that $AB = DE$, $\angle BAC = \angle EDF$, $AC = DF$.

   Conclusion: $BC = EF$

   Draw a diagram to illustrate each and assess the conclusion of each.

11·8 Summary.

I. *Organization of our geometric knowledge*



Angle
1. Complementary Angle Theorem
2. Supplementary Angle Theorem
3. Vertical Angle Theorem

Basic Biconditional
4. Isosceles Triangle Theorem
5. Right Bisector Theorem

II. *Summary of authorities to prove*

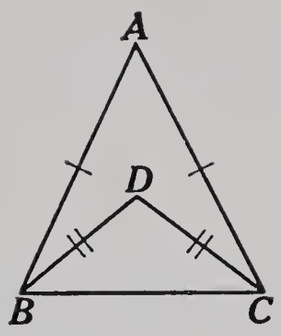
ANGLE = ANGLE	LINE SEGMENT = LINE SEGMENT	$\triangle \cong \triangle$
Complementary Angle Th.	Isosceles Triangle Th.	sas postulate
Supplementary Angle Th.	Right Bisector Th.	sss postulate
Vertical Angle Th.	sas postulate	asa postulate
Isosceles Triangle Th.	asa postulate	
sas postulate		
sss postulate		
asa postulate		

Review Exercise 11-6

(B)

1. *Hypothesis:* In the accompanying diagram, $AB = AC$, $DB = DC$.

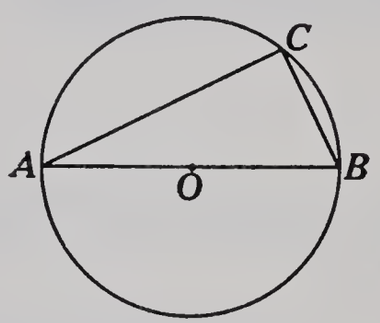
   Conclusion: $\angle ABD = \angle ACD$



2. Write a complete proof for the corollary to the Isosceles Triangle Theorem: "A triangle is equilateral if and only if it is equiangular".
3. If $\triangle PQR$ is equiangular, prove $\triangle PQR \cong \triangle RPQ$.

4. *Hypothesis:* In the accompanying diagram, O is the centre of the circle.

   Conclusion: $\angle ACB = \angle CAB + \angle CBA$



5. If two circles intersect, the line segment which joins their centres is the right bisector of the line segment joining their points of intersection.
6. $\triangle MNP$ is an isosceles triangle with $NM = MP$. The bisectors of $\angle MNP$ and $\angle MPN$ meet MP and MN at R and S, respectively, and intersect on O. Prove $\angle MRO = \angle MSO$.
7. PQR is an isosceles triangle with $PQ = PR$. M and N are points on PQ and PR, respectively, such that $PM = PN$. QN and RM intersect on S. Prove that $\triangle SQR$ and $\triangle SMN$ are isosceles triangles.
8. $ABCD$ is a quadrilateral with $AB = AD$ and $CB = CD$.

   Prove: (i) $\angle ABC = \angle ADC$;

   (ii) AC is the right bisector of DB;

   (iii) AC bisects $\angle DAB$ and $\angle DCB$.

Chapter XII

INDIRECT PROOF

INEQUALITIES

12·1 **Contradictory statements, negation, indirect proof.** The two sentences:

1. *It is cold.*
2. *It is not cold.*

are *contradictory* sentences. Each of these statements is called the *contradiction* or *negation* of the other. If either of these is true, the other is false, and, conversely, if either of these is false, the other is true. This assumption is a fundamental rule of inference in logical reasoning, and it is referred to as the *Law of Contradiction*. This law may be stated:

(i) If a statement is true, then its negation is false; and
(ii) if a statement is false, then its negation is true.

The Law of Contradiction may be expressed using symbols as follows:

(i) If P is true, then $\sim P$ is false; and
(ii) if P is false, then $\sim P$ is true;

where P represents any statement.

The following table contains a list of statements. Copy the table and write the contradictory statement (negation) in the column indicated; compare your solutions with those on page 488.

STATEMENT	CONTRADICTORY STATEMENT
1. $\triangle ABC$ is scalene.	1.
2. John is studying.	2.
3. $AB \perp CD$	3.
4. $\angle ABC = 60°$	4.

5. It is not raining.	5.
6. $\angle ABC \neq \angle DEF$	6.
7. $\angle ABC > \angle DEF$	7.
8. AB is not parallel to CD.	8.
9. 5 is greater than 10.	9.
10. Jean is the sister of Betty.	10.

The Law of Contradiction leads to a new method of proof, *proof by contradiction*, which is also called the *indirect method* of proof. The plan in this method of proof is to show that the contradictory (negation) of a conclusion to be proved is false. Then we infer by the Law of Contradiction that the conclusion is true.

The following examples illustrate the indirect method of proof.

Example 1.

Hypothesis: $A \rightarrow B$, $B \rightarrow C$, $\sim C$

Conclusion: $\sim A$

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\sim A$	1. A is false.
2. A is false	2. The assumption that A is true leads to a contradiction of a known fact.
3. The assumption A is true leads to the logical consequent B; B leads to the logical consequent C, which is a contradiction of the given premise $\sim C$.	

Proof:

STATEMENTS	AUTHORITIES
1. A is true.	1. Assumption of the negation of the conclusion $\sim A$
2. $A \rightarrow B$	2. Hypothesis
3. B	3. Law of Detachment 1, 2
4. $B \rightarrow C$	4. Hypothesis
5. C	5. Law of Detachment 3, 4
6. A is false.	6. 5 contradicts the hypothesis.
7. $\sim A$	7. Law of Contradiction

Example 2.

Hypothesis: $\triangle ABC$, *Fig. 12-1*, in which $\angle B \neq \angle C$

Conclusion: $AB \neq AC$

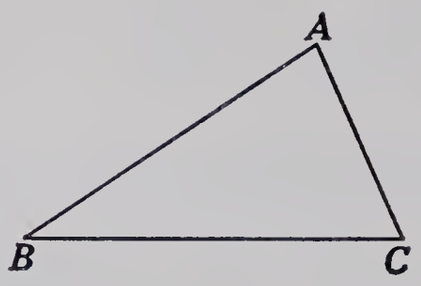


Fig. 12-1

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $AB \neq AC$	1. $AB = AC$ is false.
2. $AB = AC$ is false	2. The assumption $AB = AC$ leads to a conclusion which contradicts a known fact.
3. The assumption $AB = AC$ leads to the logical consequent $\angle B = \angle C$ by the Isosceles Triangle Theorem, which is a contradiction of the premise $\angle B \neq \angle C$.	

Proof:

STATEMENTS	AUTHORITIES
1. $AB = AC$	1. Assumption of the negation of the conclusion $AB \neq AC$
2. $\angle B = \angle C$	2. Isosceles Triangle Theorem
3. $AB = AC$ is false.	3. 2 contradicts the hypothesis.
4. $AB \neq AC$	4. Law of Contradiction

Steps in an indirect proof.

1. Introduce the contradictory statement (negation) of the desired conclusion as a new premise.
2. From this new premise, together with the given premises, deduce a contradiction of a known fact.
3. State that the desired conclusion is true by the Law of Contradiction.

Example 3. If in $\triangle ABC$, *Fig. 12-2*, $BD = DC$ and AD is not perpendicular to BC, then $AB \neq AC$.

Hypothesis: $\triangle ABC$ in which $BD = DC$, AD is not perpendicular to BC.

Conclusion: $AB \neq AC$

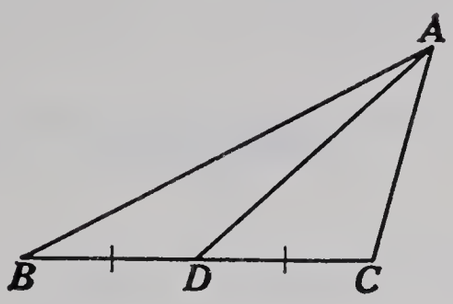


Fig. 12-2

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $AB \neq AC$	1. $AB = AC$ is false.
2. $AB = AC$ is false	2. The assumption that $AB = AC$ leads to a conclusion which contradicts a known fact.
3. The assumption $AB = AC$ leads to the conclusion that $\triangle ADB \cong \triangle ADC$ by the sss postulate; this leads to the conclusion $\angle ADB = \angle ADC = 90°$ and $AD \perp BC$, which contradicts the hypothesis, AD is not perpendicular to BC.	

Proof: Write a complete proof; compare your proof with that on page 489.

Example 4. If in $\triangle ABC$, *Fig. 12-3*, $AC \neq AB$, then $\angle B \neq \angle C$.

Hypothesis: $\triangle ABC$ in which $AC \neq AB$

Conclusion: $\angle B \neq \angle C$

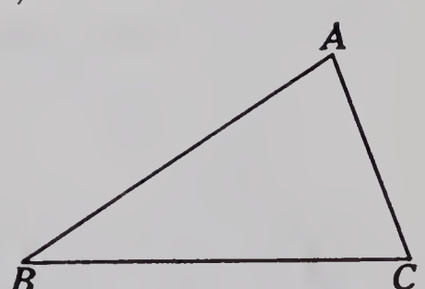


Fig. 12-3

Write a complete analysis and proof using the indirect method; compare them with those given on page 489.

There is no general rule which can tell you exactly when to use an indirect proof rather than a direct proof. Usually an indirect proof is used when the given premises do not seem to provide a starting point. In the indirect proof, the additional premise which is the negation of the required conclusion provides a starting point. When we are required to prove inequalities or converse statements, the indirect proof is used frequently.

Exercise 12-1

(A)

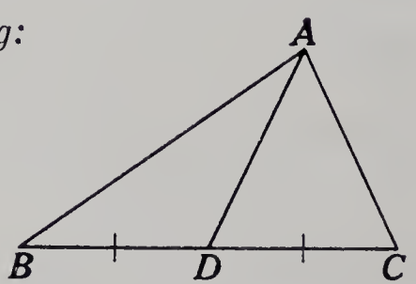
State the negation of the statements in 1 *to* 10*:*

1. It is snowing.
2. $\angle ABC = \angle DEF$.
3. $l_1 \parallel l_2$.
4. AB is greater than CD.
5. $AB \neq CD$.
6. AB is not less than CD.
7. $AB \nparallel CD$.
8. $PQ \perp XY$.
9. $\angle ABC = 90°$.
10. CD is the right bisector of AB.
11. State the Law of Contradiction.
12. State the three steps in an indirect proof.

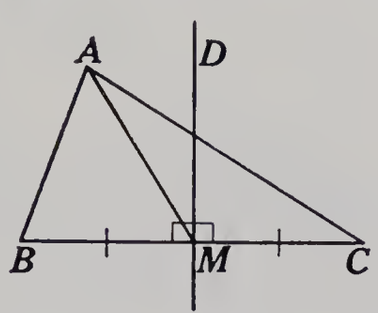
(B)

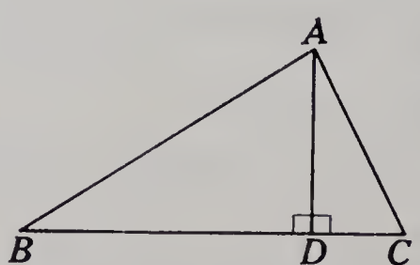
Write indirect proofs for each of the following:

13. In $\triangle ABC$ in the accompanying diagram, if $AB \neq AC$, $BD = DC$, prove AD is not perpendicular to BC.



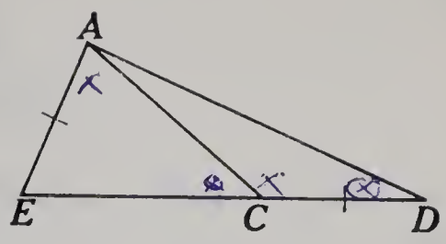
14. In $\triangle ABC$, if $DM \perp BC$, $BM = MC$, A is not on DM, prove $AB \neq AC$ (diagram at left below).





15. In $\triangle ABC$, if $AB \neq AC$, $AD \perp BC$, then $BD \neq DC$ (diagram at right above).

16. In $\triangle AEC$, if EC is extended to D so that $CD = AE$, then $\angle ACD \neq \angle EAC$.



***12·2* The fundamental triangle inequality theorem.** In *Fig. 12-4*, the angle ACD is called an *exterior angle* of $\triangle ABC$.

DEFINITION: *If C is between B and D, then $\angle ACD$ is an exterior angle of $\triangle ABC$.*

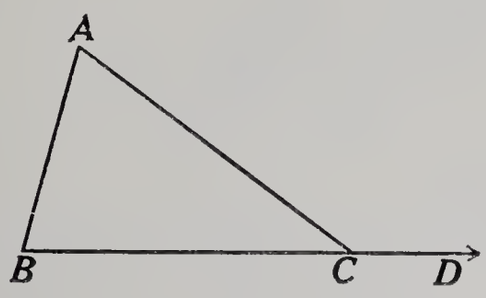


Fig.12-4

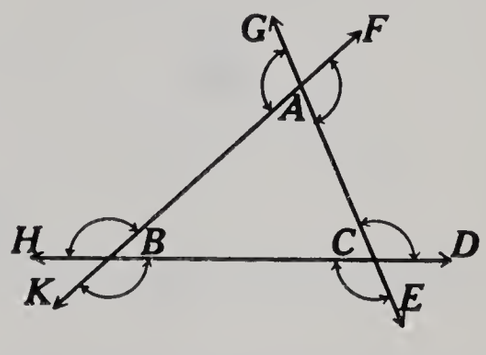


Fig. 12-5

Every triangle has six exterior angles, as shown in *Fig. 12-5*, two at each vertex of the triangle. At vertex C, the exterior angles are $\angle ACD$ and $\angle BCE$; at vertex A, $\angle FAC$ and $\angle GAB$; at vertex B, $\angle ABH$ and $\angle CBK$. The six exterior angles form three pairs of vertical angles.

DEFINITION: *$\angle BAC$ and $\angle ABC$ are called the remote interior angles of exterior angles ACD and BCE of $\triangle ABC$.*

Similarly, the remote interior angles of $\angle ABH$ and $\angle CBK$ are $\angle BAC$ and $\angle ACB$. The remote interior angles of $\angle BAG$ and $\angle CAF$ are $\angle ABC$ and $\angle ACB$.

Our experimental study of angle relations in a triangle has led, by means of inductive reasoning, to the following conjecture about the relative size of an exterior angle and its interior remote angles. In *Fig. 12-5*, $\angle ACD > \angle BAC$ and $\angle ACD > \angle ABC$.

This conjecture leads to the fundamental triangle inequality theorem: the *Exterior Angle Theorem.*

THEOREM 6

Exterior Angle Theorem

An exterior angle of a triangle is greater than either remote interior angle.

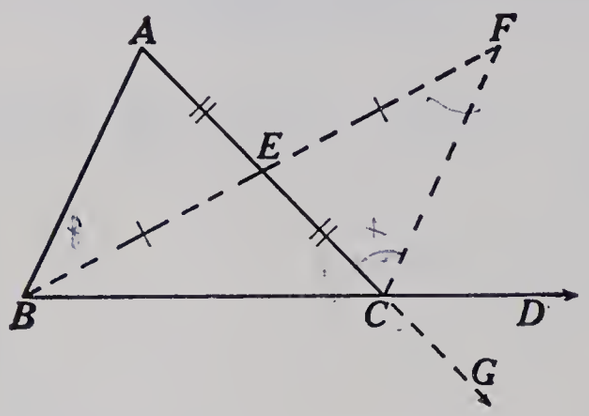


Fig. 12-6

Hypothesis: $\triangle ABC$ in which C is between B and D, *Fig. 12-6*

Conclusion: $\angle ACD > \angle A$ and $\angle ACD > \angle ABC$

Proof:

STATEMENTS	AUTHORITIES
1. E is the midpoint of AC, and F a point on ray BE so that $EB = EF$.	1. Existence postulates
2. In $\triangle$'s AEB, CEF $\left\{\right.$ $AE = CE$	2. Definition
3. $EB = EF$	3. Definition
4. $\left.\angle AEB = \angle CEF\right.$	4. Vertical Angle Theorem
5. $\triangle AEB \cong \triangle CEF$	5. sas
6. $\angle A = \angle ECF$	6. Definition
7. $\angle ACD > \angle ECF$	7. Completion
8. $\angle ACD > \angle A$	8. Replacement
9. $\angle BCG = \angle ACD$	9. Vertical Angle Theorem
10. $\angle BCG > \angle A$	10. Replacement
11. Similarly, $\angle BCG > \angle ABC$ and $\angle ACD > \angle ABC$	11. Authorities 1 to 10 above

Exercise 12-2

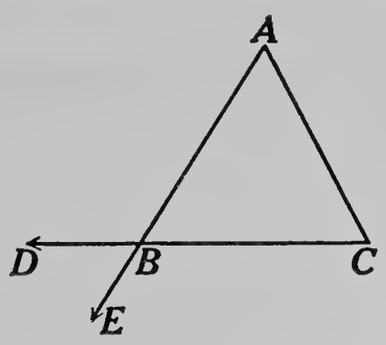
(A)

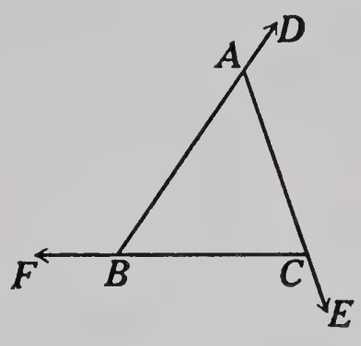
1. In the figure at the left below:

(i) name two exterior angles;

(ii) compare the measurement of each of these exterior angles with the measurement of $\angle A$ and $\angle C$;

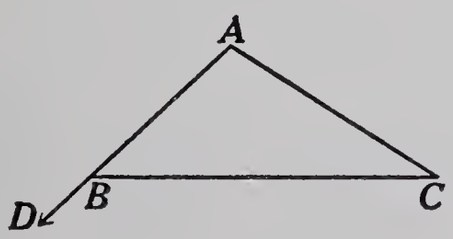
(iii) state the relationship of each exterior angle to $\angle ABC$.

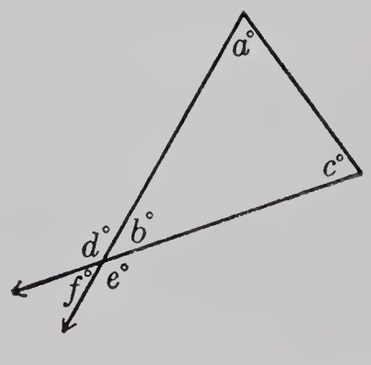




2. Name the remote interior angles of each of the exterior angles in the figure at the right above.

3. In the diagram at the left below, compare the measurement of $\angle A$ with that of $\angle DBC$.





4. If a, b, c, d, e, f are the measurements, in degrees, of the angles indicated in the diagram at the right above, state and justify all known relations between these.

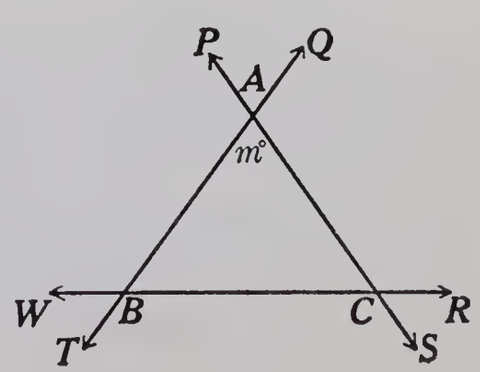
(B)

5. The measurement of $\angle BAC$ in the accompanying figure is m degrees.

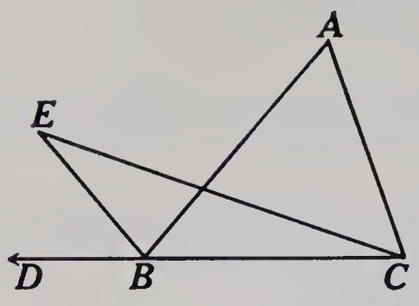
(i) Name six exterior angles of $\triangle ABC$.

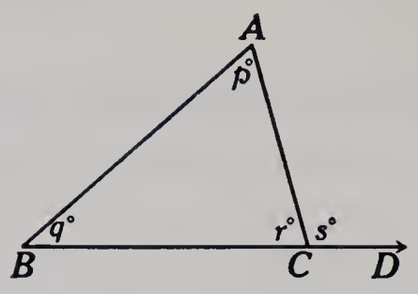
(ii) Name the remote interior angles of each exterior angle.

(iii) Name four angles whose degree measurements are greater than m.



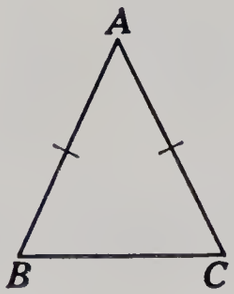
6. In the figure, left below, prove that $\angle ABD > \angle E$.

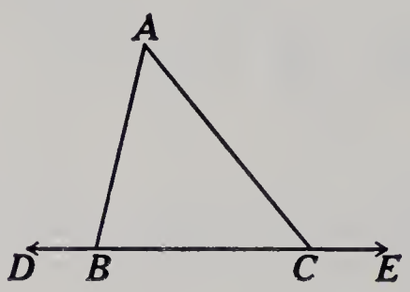




7. In the figure at the right above, p, q, r, and s represent the measures, in degrees, of the angles indicated. Prove that $p + r < 180$.

8. In the diagram at the left below, ABC is an isosceles triangle having $AB = AC$. Prove that $\angle B$ is acute.





9. Prove that the sum of the exterior angles in the figure at the right above is greater than a straight angle.

12·3 Summary.

Undefined Terms	Definitions (page 449)	Existence Postulates (page 452)	Completion Postulates (page 453)	Number Axioms (page 275)	Congruence Postulates sss, sas, asa (page 453)

Deductive Reasoning

Theorems — Deductions

Angle
1. Complementary Angle Theorem
2. Supplementary Angle Theorem
3. Vertical Angle Theorem

Basic Biconditional
4. Isosceles Triangle Theorem
5. Right Bisector Theorem

Inequality
6. Exterior Angle Theorem

Chapter XIII

PARALLEL LINES

13·1 Parallel lines, transversal, existence postulate.

a. *Parallel lines. If two distinct lines lie in the same plane and have no point in common, then the two lines are parallel.*

The symbol || is used to represent the words, "is parallel to", and the symbol ∦ represents the words, "is not parallel to".

b. *Parallel line segments. If two distinct line segments are segments of parallel lines, then the two line segments are parallel.*

Thus, two distinct line segments are parallel if they are on the same plane and do not meet however far they are produced in either direction.

c. *Parallel postulate. If l is any line and A is any point not on l, then there is one and only one line on A which is parallel to line l (Fig. 13-1).*

This is Euclid's statement of the parallel postulate and is often referred to as "Euclid's Fifth Postulate".

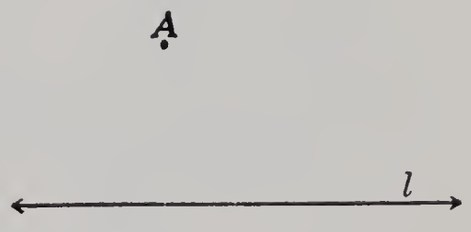


Fig. 13-1

The statement of the parallel postulate in the form:

Two intersecting straight lines cannot both be parallel to the same straight line.

was expressed in a textbook written by the English mathematician, John Playfair, which was published in 1858. This statement of the postulate is often referred to as "Playfair's Axiom".

d. *Transversal. If a line intersects two or more distinct lines, then it is a transversal.*

13·2 **Classification of angles formed by a transversal.**

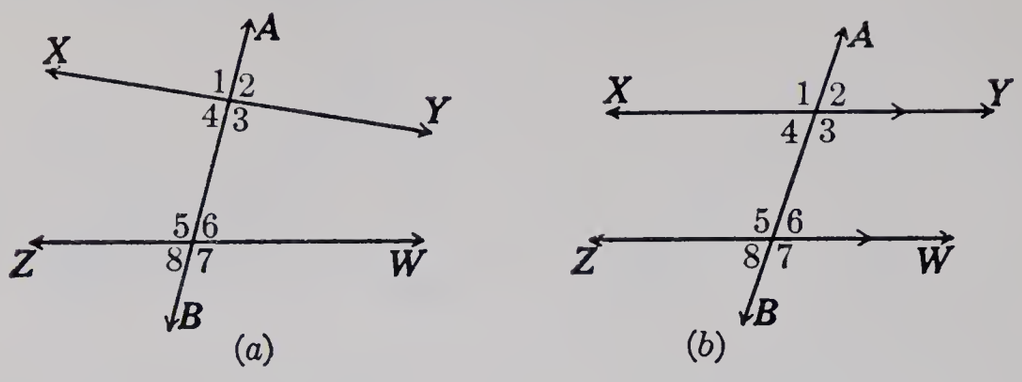


Fig. 13-2

In *Fig. 13-2*, AB intersects lines XY and ZW on distinct points; thus AB is a *transversal.* The arrows on the ends of lines XY and ZW indicate their endless nature. The arrows between X and Y, Z and W, in *Fig. 13-2* (*b*), indicate line $XY \parallel$ line ZW.

Classification of the angles formed by transversal AB (Fig. 13-2)

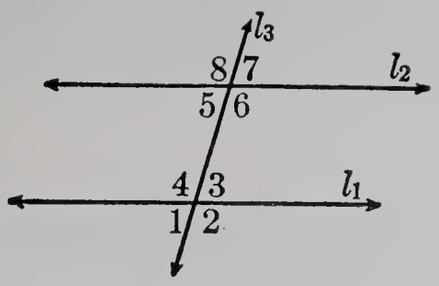
1. Exterior angles: $\angle 1$, $\angle 2$, $\angle 7$, $\angle 8$
2. Interior angles: $\angle 3$, $\angle 4$, $\angle 5$, $\angle 6$
3. Pairs of exterior angles on the same side of the transversal:
   (i) $\angle 2$, $\angle 7$ (ii) $\angle 1$, $\angle 8$
4. Pairs of interior angles on the same side of the transversal:
   (i) $\angle 3$, $\angle 6$ (ii) $\angle 4$, $\angle 5$
5. Pairs of alternate angles:
   (i) $\angle 3$, $\angle 5$ (ii) $\angle 4$, $\angle 6$
   (NOTE: Alternate angles are on opposite sides of the transversal.)
6. Pairs of corresponding angles:
   (i) $\angle 1$, $\angle 5$ (ii) $\angle 2$, $\angle 6$
   (iii) $\angle 8$, $\angle 4$ (iv) $\angle 7$, $\angle 3$
   (NOTE: Pairs of corresponding angles are on the same side of the transversal.)

Exercise 13-1

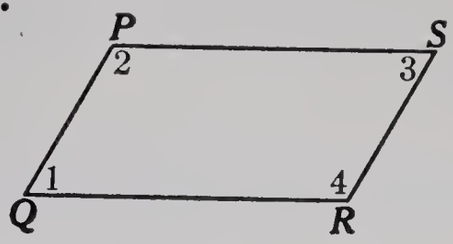
(B)

In each of the following, name all transversals and for each transversal make a list of all pairs of alternate angles, corresponding angles, and interior angles on the same side of the transversal which are labelled:

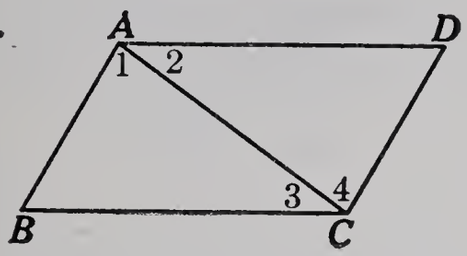
1.



2.



3.



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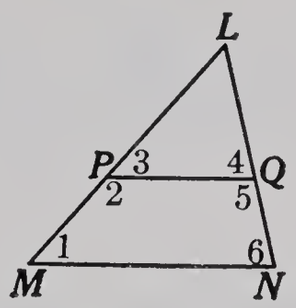
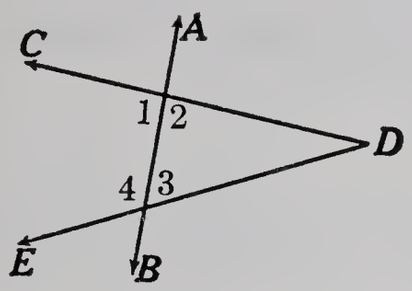
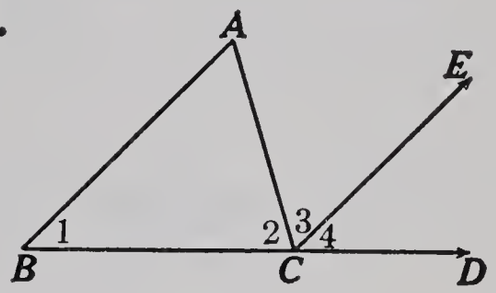


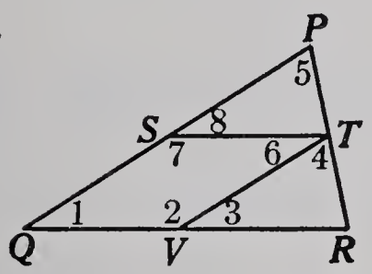
5.



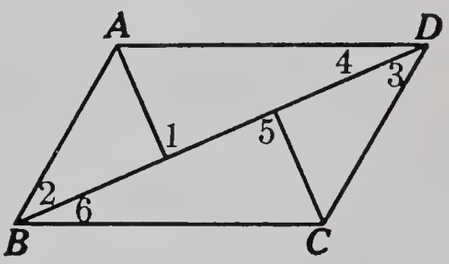
6.



7.

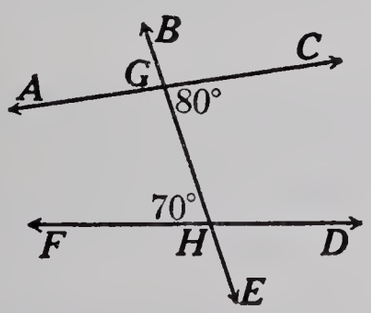


8.

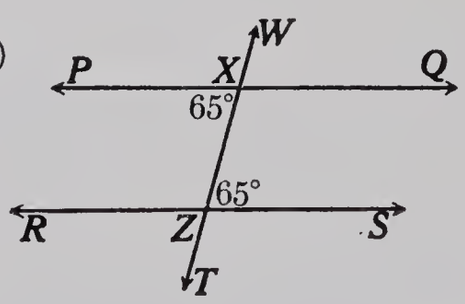


9. For the following diagrams, list the measurements of each of the angles formed by the lines and the transversal; justify your answers:

(i)

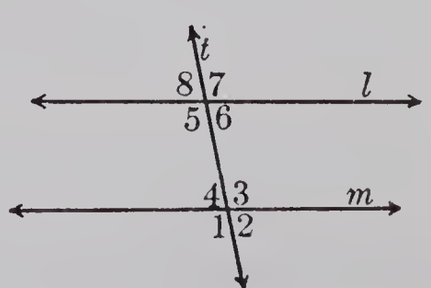


(ii)



10. Two lines, l and m, are cut by a transversal, t, as in the diagram at the right. If $\angle 6 = \angle 4$, prove:

(i) $\angle 5 = \angle 3$
(ii) $\angle 4 = \angle 8$
(iii) $\angle 1 = \angle 7$
(iv) $\angle 3$ and $\angle 6$ are supplementary.



***13·3* The Parallel Line Theorem.** We have discovered by inductive reasoning the following conjectures about angle pairs formed by a transversal.

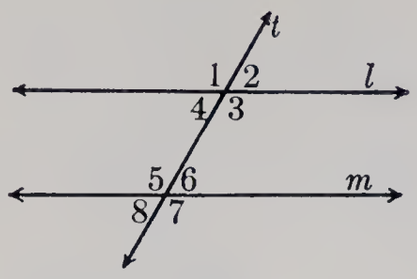


Fig. 13-3

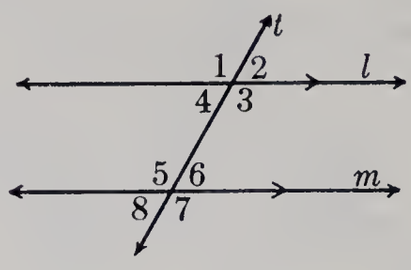


Fig. 13-4

CONJECTURE I

If alternate angles are equal, then the two lines are parallel. That is, if $\angle 4 = \angle 6$ (or $\angle 3 = \angle 5$), then $l \parallel m$ (*Fig. 13-3*).

Conversely:

If two lines are parallel, then the pairs of alternate angles are equal. That is, if $l \parallel m$, then $\angle 4 = \angle 6$ and $\angle 3 = \angle 5$ (*Fig. 13-4*).

CONJECTURE II

If corresponding angles are equal, then the two lines are parallel. That is, if $\angle 2 = \angle 6$ (or $\angle 1 = \angle 5$, or $\angle 4 = \angle 8$, or $\angle 7 = \angle 3$), then $l \parallel m$ (*Fig. 13-3*).

Conversely:

If two lines are parallel, then pairs of corresponding angles are equal. That is, if $l \parallel m$, then $\angle 1 = \angle 5$, $\angle 2 = \angle 6$, $\angle 4 = \angle 8$, $\angle 7 = \angle 3$ (*Fig. 13-4*).

CONJECTURE III

If the sum of the measurements of two interior angles on the same side of the transversal is 180°, then the lines are parallel. That is, if $\angle 3 + \angle 6 = 180°$ (or $\angle 4 + \angle 5 = 180°$), then $l \parallel m$ (*Fig. 13-3*).

Conversely:

If two lines are parallel, then the sum of the measurements of the two interior angles on the same side of the transversal is 180°. That is, if $l \parallel m$, then $\angle 3 + \angle 6 = 180°$ and $\angle 4 + \angle 5 = 180°$ (*Fig. 13-4*).

Although these conjectures have been discovered, they have not been proved by deductive reasoning. All of these relations are contained in the "if and only if" statement of the Parallel Line Theorem, which is considered in two parts in the pages following.

13·4 The Parallel Line Theorem, Part I.

THEOREM 7

Parallel Line Theorem

If a transversal meets two lines, the two lines are parallel if and only if:

(i) alternate angles are equal;

(ii) corresponding angles are equal;

(iii) interior angles on the same side of the transversal are supplementary.

PART I

(i) If a transversal meets two lines so that alternate angles are equal, then the two lines are parallel.

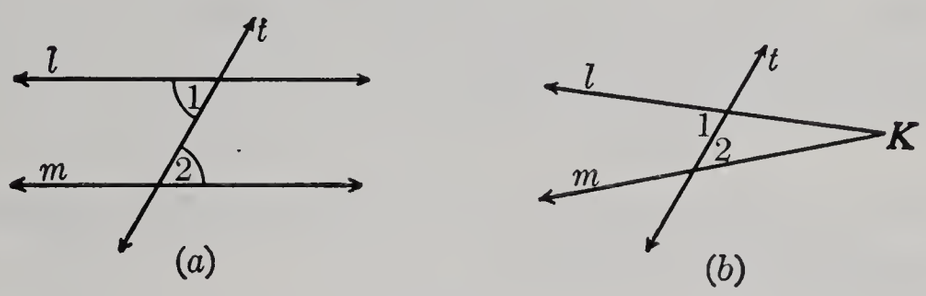


Fig. 13-5

Hypothesis: Transversal t meeting lines l and m such that $\angle 1 = \angle 2$

Conclusion: $l \parallel m$

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $l \parallel m$	1. $l \nparallel m$ is false.
2. $l \nparallel m$ is false	2. The assumption that $l \nparallel m$ leads to a contradiction of a known fact.
3. $l \nparallel m$ leads to a contradiction	3. The assumption that l and m have a common point K, *Fig. 13-5* (*b*), leads to a contradiction.
4. In *Fig. 13-5* (*b*) that $\angle 1 > \angle 2$ by the Exterior Angle Theorem, which contradicts the hypothesis.	

Proof: Write a complete proof; compare your proof with that on page 489.

(ii) **If a transversal meets two lines so that corresponding angles are equal, then the two lines are parallel.**

Hypothesis: Transversal t which meets l, m, so that $\angle 3 = \angle 2$ (*Fig. 13-6*)

Conclusion: $l \parallel m$

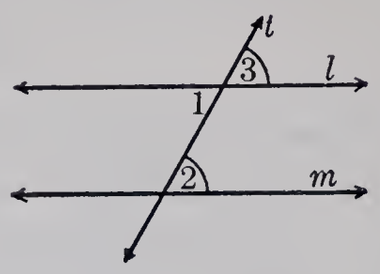


Fig. 13-6

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $l \parallel m$	1. Alternate angles 1 and 2 equal.
2. $\angle 1 = \angle 2$	2. $\angle 1 = \angle 3$.
3. $\angle 1 = \angle 3$ by the Vertical Angle Theorem.	

Proof: Write a complete proof from the above analysis; compare your proof with that on page 490.

(iii) **If a transversal meets two lines so that interior angles on the same side of the transversal are supplementary, then the two lines are parallel.**

Hypothesis: Transversal t which meets l, m, so that $\angle 4 + \angle 2 = 180°$ (*Fig. 13-7*)

Conclusion: $l \parallel m$

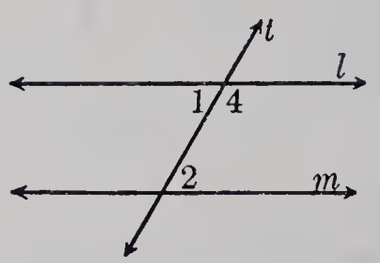


Fig. 13-7

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $l \parallel m$	1. Alternate angles 1 and 2 are equal.
2. $\angle 1 = \angle 2$	2. $\angle 1 + \angle 4 = \angle 2 + \angle 4$.
3. $\angle 1 + \angle 4 = \angle 2 + \angle 4$	3. $\angle 1 + \angle 4 = 180°$.
4. $\angle 1 + \angle 4 = 180°$ by definition.	

Proof: Write a complete proof from the above analysis; compare your proof with that on page 490.

Corollary: If lines are perpendicular to the same line, then they are parallel to each other.

Write an analysis and proof of this corollary; compare it with that on page 490.

Authorities used in proving lines parallel:

(i) equal alternate angles;
(ii) equal corresponding angles;
(iii) supplementary angles.

In some problems involving parallel line segments, the line segments meet the transversal without crossing it. In such cases it is sometimes difficult to locate corresponding angles, alternate angles, or two supplementary interior angles on the same side of the transversal. By extending the line segments, as shown in *Fig. 13-8*, the angle relationships necessary for parallel lines become more apparent. The horizontal line segments in each case are parallel by the Parallel Line Theorem.

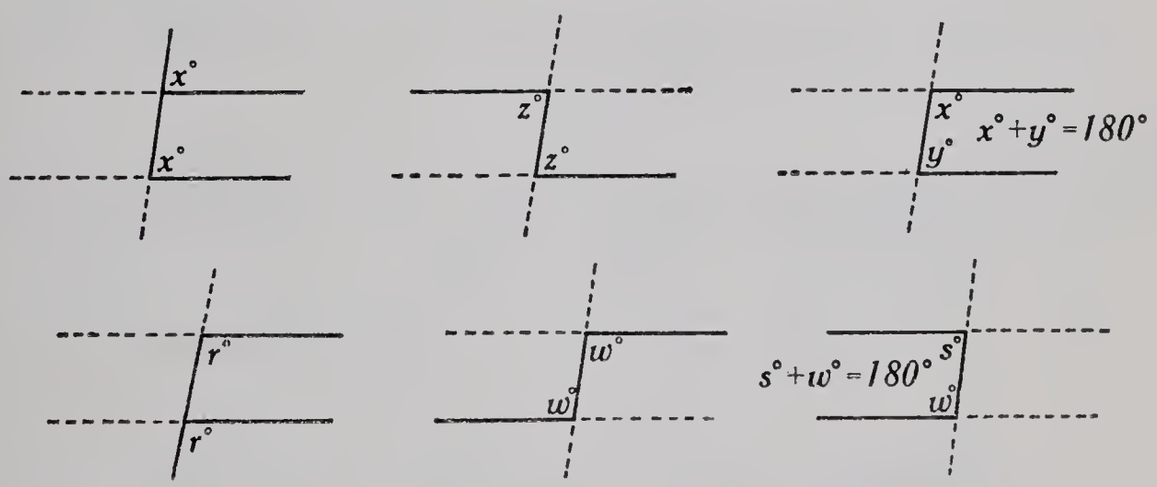


Fig. 13-8

Example 1. In each diagram of *Fig. 13-9* state, giving reasons, which line segments are parallel:

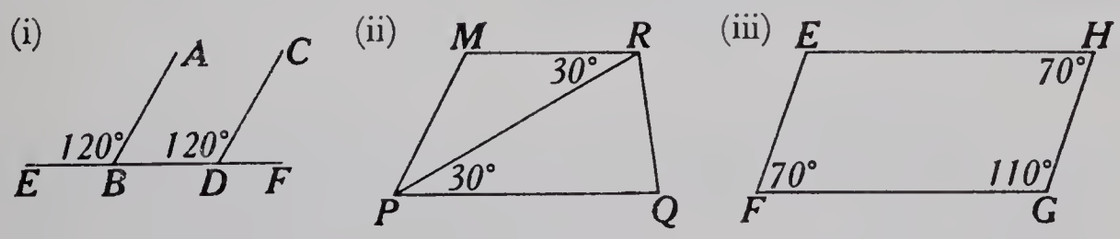
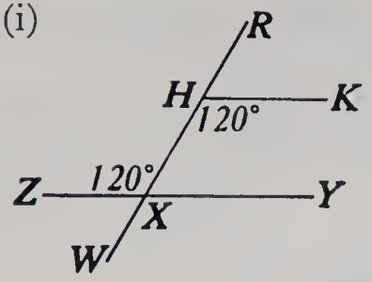


Fig. 13-9

STATEMENTS	AUTHORITIES
(i) $AB \parallel CD$	(i) Equal corresponding angles
(ii) $MR \parallel PQ$	(ii) Equal alternate angles
(iii) $EH \parallel FG$, $EF \parallel HG$	(iii) Supplementary angles

Example 2. In each diagram of *Fig. 13-10* state, giving reasons, which line segments are parallel:

(i)



(ii)

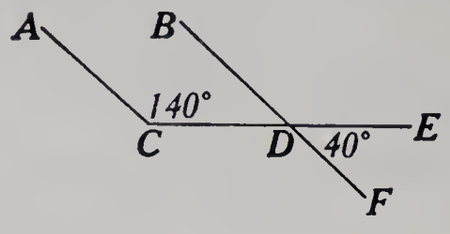


Fig. 13-10

STATEMENTS	AUTHORITIES
(i) $HK \parallel ZXY$	(i) Equal alternate angles
(ii) 1. $\angle EDF = \angle BDC = 40°$	(ii) 1. Vertical angles
2. $\angle ACD + \angle CDB = 140° + 40° = 180°$	2. Addition
3. $AC \parallel BD$	3. Supplementary angles

DEFINITION: *If a quadrilateral has both pairs of opposite sides parallel, then it is a parallelogram.*

Example 3.

Hypothesis: Quadrilateral $ABCD$, *Fig. 13-11*, in which $AB = DC$, $AD = BC$

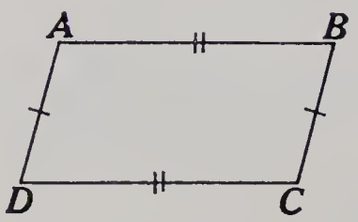


Fig. 13-11

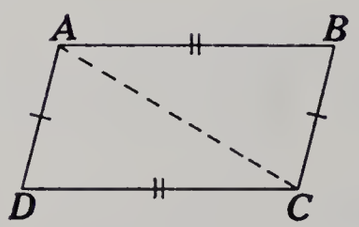


Fig. 13-12

Conclusion: $ABCD$ is a parallelogram.

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $ABCD$ is a parallelogram	1. $AB \parallel DC$ and $AD \parallel BC$.
2. $AB \parallel DC$	2. $\angle BAC = \angle DCA$.
3. $\angle BAC = \angle DCA$	3. $\triangle BAC \cong \triangle DCA$.
4. $\triangle BAC \cong \triangle DCA$ by sss.	
5. $AD \parallel BC$ in a similar manner.	

Proof: Write a complete proof; compare your proof with that on page 490.

Example 4. Write an analysis and proof of the following deduction; compare your analysis and proof with that on page 491.

Hypothesis: $\triangle ABC$ in which E is the midpoint of AC, D is on AB, DE is extended so that $EF = DE$ (*Fig. 13-13*)

Conclusion: $FC \parallel AB$

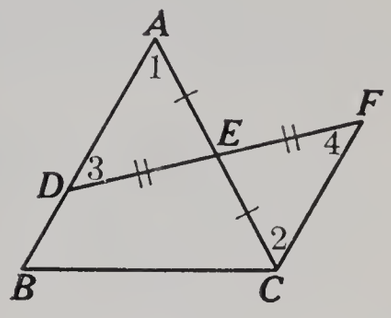


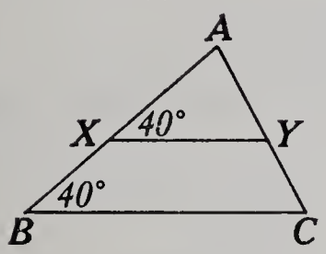
Fig. 13-13

Exercise 13-2

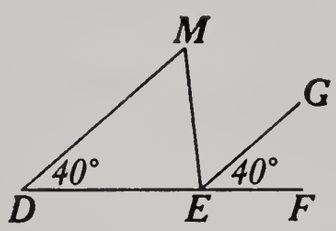
(A)

1. In each of the following diagrams, state, giving reasons, the line segments which are parallel:

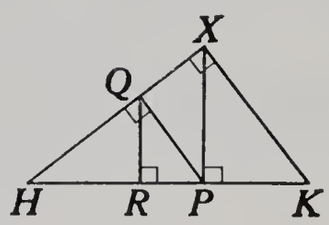
(i)



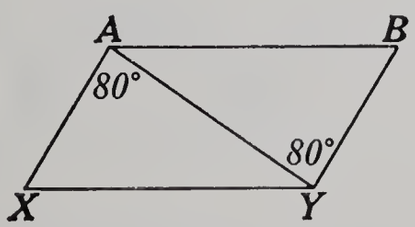
(ii)



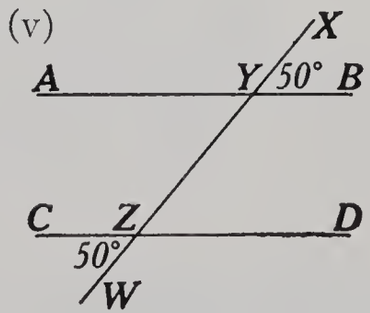
(iii)



(iv)



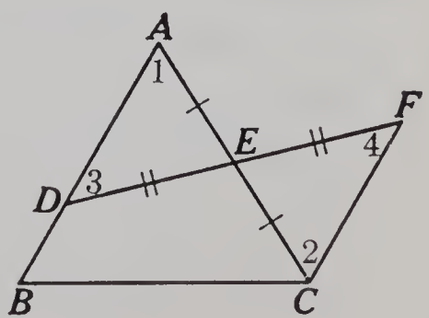
(v)



2. In the following demonstration of proof, state the authority which justifies the statement in each line of the proof:

Hypothesis: $\triangle ABC$ in which E is the midpoint of AC, D is on AB, DE is extended so that $EF = DE$

Conclusion: $FC \parallel AB$

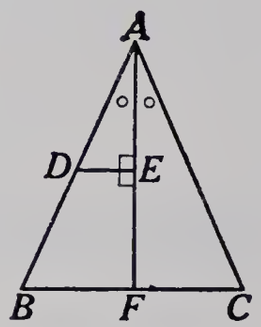


Proof:

STATEMENTS		AUTHORITIES
1. In $\triangle$'s AED and CEF	$AE = CE$	1.
2.	$DE = FE$	2.
3.	$\angle AED = \angle CEF$	3.
4. $\triangle AED \cong \triangle CEF$		4.
5. $\angle 1 = \angle 2$		5.
6. $FC \parallel AB$		6.

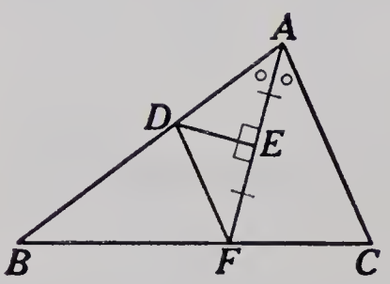
(B)

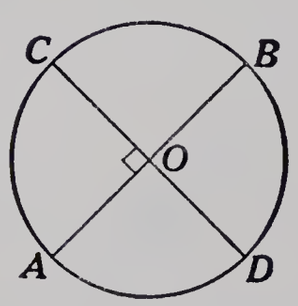
3. Write an analysis of the following deduction:
In the diagram, $AB = AC$, AF bisects $\angle BAC$, $DE \perp AF$.
Prove $DE \parallel BC$.



Write a complete proof for each of the following deductions:

4. In the diagram, left below, AF bisects $\angle BAC$, DE is the right bisector of AF. Prove $DF \parallel AC$.





5. In a given circle (right above), AB and CD are perpendicular diameters. Prove that $AC \parallel BD$.

6. Prove that a quadrilateral $ABCD$ with all sides equal is a parallelogram: that is, $AB \parallel CD$ and $AD \parallel BC$.

7. Prove that a square is a parallelogram.

8. E and D are the midpoints of AB and AC, respectively, in $\triangle ABC$. CE is extended to G so that $GE = EC$. Similarly, BD is extended to F so that $FD = DB$. Prove (i) $AG \parallel BC$ (ii) $AF \parallel BC$ and hence $\angle GAF$ is a straight angle.

9. In $\triangle ABC$, D and E are the midpoints of AB and AC, respectively. DE is extended to F so that $FE = ED$. Prove $FC \parallel AB$.

10. In quadrilateral $PQRS$, X and Y are the midpoints of PQ and SR, respectively, $\angle PSR = \angle QRS$, $PS = QR$. Prove that (i) $XY \perp PQ$ (ii) $XY \perp SR$ (iii) $PQ \parallel SR$.

13·5 The Parallel Line Theorem, Part II.

THEOREM 7

Parallel Line Theorem

If a transversal meets two lines, the two lines are parallel if and only if:

(i) **alternate angles are equal;**
(ii) **corresponding angles are equal;**
(iii) **interior angles on the same side of the transversal are supplementary.**

PART II

Converses: (i) If a transversal meets two parallel lines, then alternate angles are equal.

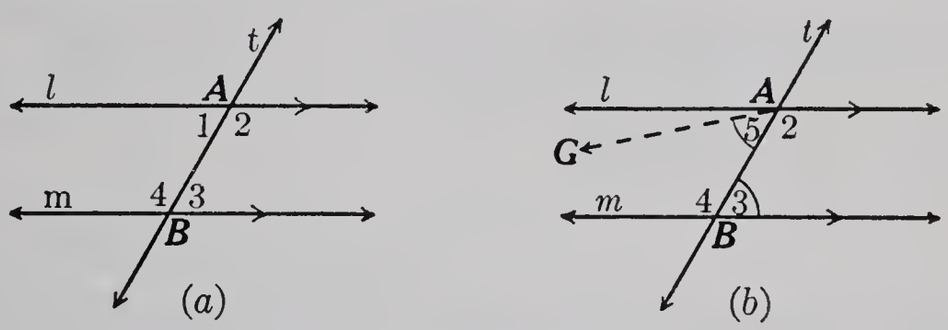


Fig. 13-14

Hypothesis: Transversal t meets two parallel lines, l and m, at A and B, respectively, *Fig. 13-14* (*a*).

Conclusion: $\angle 1 = \angle 3$, $\angle 2 = \angle 4$

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle 1 = \angle 3$	1. $\angle 1 \neq \angle 3$ is false.
2. $\angle 1 \neq \angle 3$ is false	2. $\angle 1 \neq \angle 3$ leads to a contradiction of a known fact.
3. $\angle 1 \neq \angle 3$ leads to a contradiction	3. Ray AG such that $\angle 5 = \angle 3$ leads to a contradiction.
4. Ray AG is parallel to line m by the Parallel Line Theorem, which contradicts the parallel postulate.	

Proof: Write a complete proof; compare your proof with that on page 491.

(ii) **If a transversal meets two parallel lines, then corresponding angles are equal.**

Hypothesis: Transversal t meets parallel lines l, m (*Fig. 13-15*).

Conclusion: $\angle 1 = \angle 2$

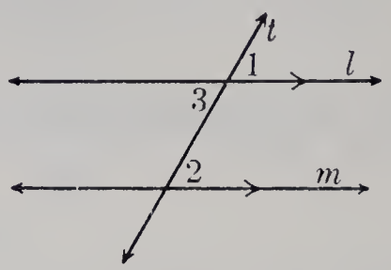


Fig. 13-15

Proof: Write a complete proof; compare your proof with that on page 491.

(iii) **If a transversal meets two parallel lines, then interior angles on the same side of the transversal are supplementary.**

Hypothesis: Transversal t meets parallel lines l and m (*Fig. 13-16*).

Conclusion: $\angle 1 + \angle 2 = 180°$

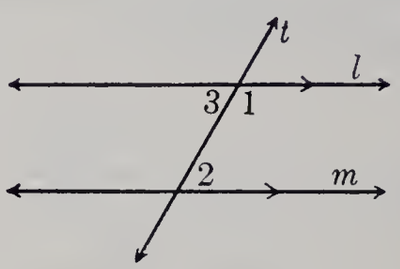


Fig. 13-16

Proof: Write a complete proof; compare your proof with that on page 491.

Corollary: **If lines are parallel to the same line, then the lines are parallel to each other.**

Example 1. Parallel line segments are indicated in the diagrams by arrowheads. In each diagram (*Fig. 13-17*), state the pairs of equal angles.

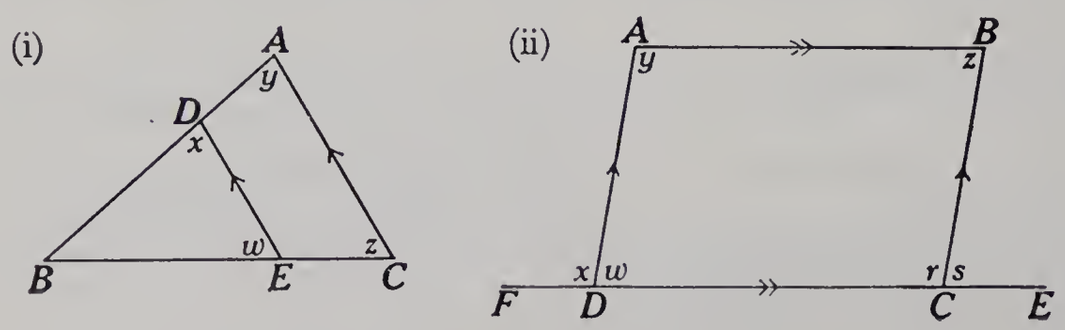


Fig. 13-17

Solution.

(i) $\left.\begin{array}{l}\angle x = \angle y \\ \angle w = \angle z\end{array}\right\}$ corresponding angles

(ii) $\left.\begin{array}{l}\angle x = \angle r \\ \angle w = \angle s\end{array}\right\}$ corresponding angles

$\left.\begin{array}{l}\angle x = \angle y \\ \angle z = \angle s\end{array}\right\}$ alternate angles

Example 2. In *Fig. 13-18*, prove that AD bisects the exterior $\angle EAC$ of $\triangle ABC$.

Hypothesis: $\triangle ABC$ in which $AB = AC$ and $AD \parallel BC$

Conclusion: $\angle 1 = \angle 2$

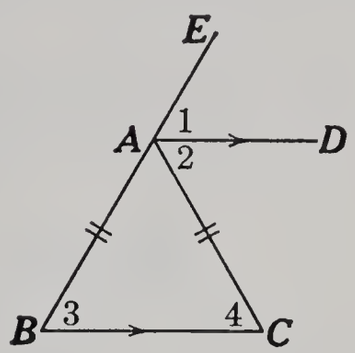


Fig. 13-18

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle 1 = \angle 2$	1. $\angle 1 = \angle 3$, $\angle 2 = \angle 4$, and $\angle 3 = \angle 4$.
2. $\angle 1 = \angle 3$, $\angle 2 = \angle 4$ by the Parallel Line Theorem.	
3. $\angle 3 = \angle 4$ by the Isosceles Triangle Theorem.	

Proof: Write a complete proof; compare your proof with that on page 492.

Example 3. In any parallelogram prove that the opposite angles are equal.

Hypothesis: $\parallel^{gm} DEFG$ (*Fig. 13-19*) in which $DE \parallel GF$ and $DG \parallel EF$

Conclusion: (i) $\angle GDE = \angle EFG$
(ii) $\angle E = \angle G$

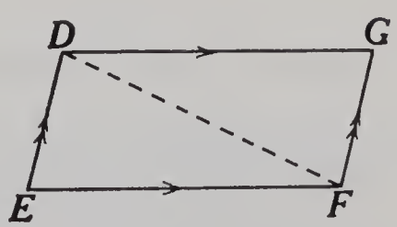


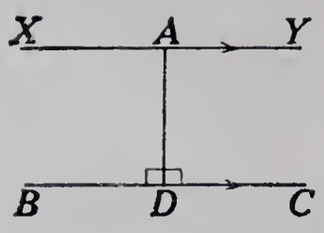
Fig. 13-19

Write an analysis and proof of this deduction; compare your solution with that on page 492.

Exercise 13-3

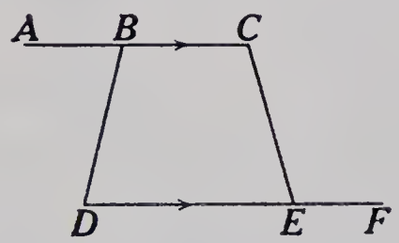
(A)

1. In the diagram, a perpendicular is drawn from A to BC. If $XAY \parallel BC$, prove that AD is perpendicular to XY.

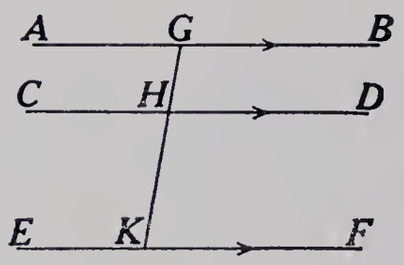


2. In each of the following diagrams, state, giving reasons, pairs of angles that are equal:

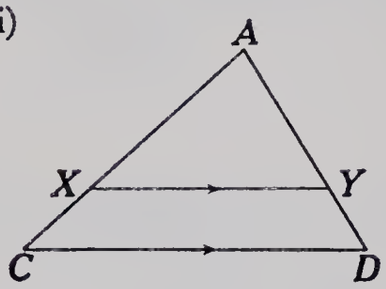
(i)



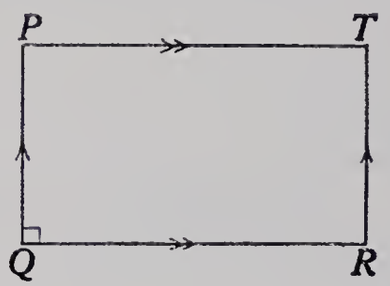
(ii)



(iii)

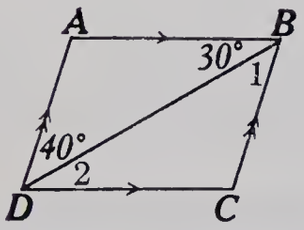


(iv)

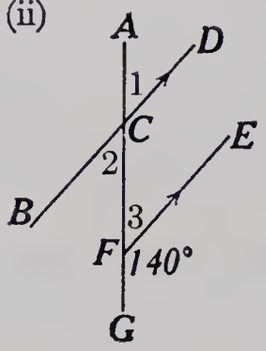


3. In the following, state the measurements of $\angle 1$, $\angle 2$, $\angle 3$:

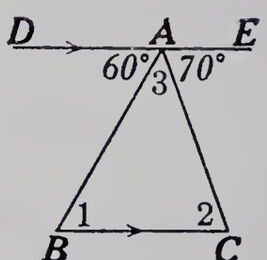
(i)



(ii)

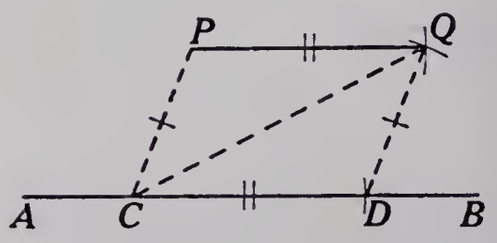


(iii)



4. State the parallel postulate.

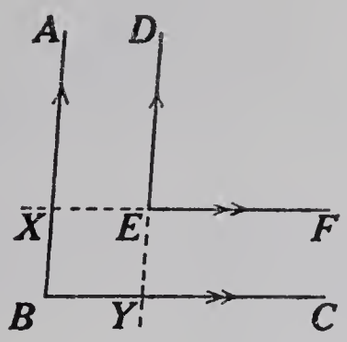
5. In the accompanying diagram, the ruler-compasses construction used in a physical model to draw a line segment parallel to a given line segment AB on a given point P is demonstrated. Prove the construction method by proving $PQ \parallel CD$.

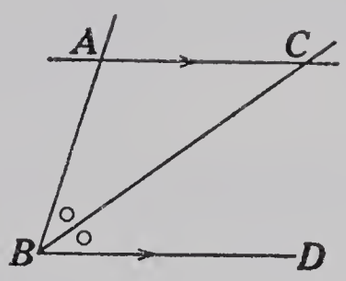


(B)

Write a complete proof of each of the following deductions:

6. In the diagram at the left below, $DE \parallel AB$ and $EF \parallel BC$. Prove $\angle ABC = \angle DEF$.





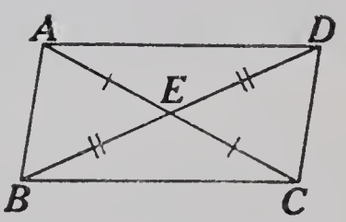
7. In the diagram at the right above, prove $AB = AC$.

8. AB and CD are perpendicular diameters of a circle. Prove $ABDC$ is a parallelogram with all sides equal.

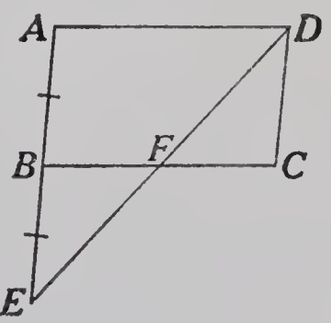
9. If in a quadrilateral one pair of opposite sides is equal and parallel, then the other pair of sides is equal and parallel.

10. If one angle of a parallelogram is 90°, then each of the other three angles is also 90°.

11. If $ABCD$ is a quadrilateral whose diagonals bisect each other at E, as illustrated, then $ABCD$ is a parallelogram.



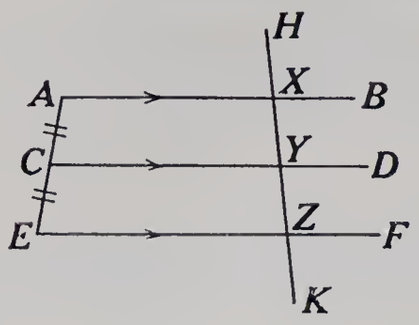
12. In the quadrilateral $ABCD$, $AB \parallel CD$ and $AB = CD$. AB is extended to E so that $AB = BE$ as illustrated. Prove that $BD = EC$ and $BD \parallel EC$.

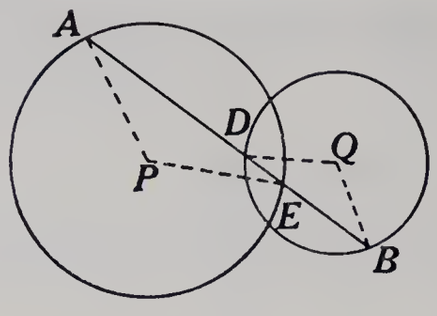


13. If the bisector of an exterior angle of a triangle is parallel to the opposite side, then the triangle has two equal angles.

14. In $\triangle XYZ$, D is on XY, E on XZ, F on ZY such that $DE \parallel YZ$ and $EF \parallel XY$. Prove that the three angles of $\triangle XDE$ are respectively equal to the three angles of $\triangle EFZ$. Thus $\triangle XDE$ is equiangular to $\triangle EFZ$.

15. In the diagram at the left below, if $\angle A = \angle HXB$, prove $\angle ACY = \angle FZK$.





16. P and Q are the centres of two intersecting circles. Two parallel radii, AP and BQ, are drawn. AB intersects the circles on D and E, as illustrated at the right above. Prove $PE \parallel DQ$.

17. $ABCD$ is a parallelogram. The bisectors of angles B and D meet AD, BC on E and F, respectively. Prove $BE \parallel DF$.

18. $ABCD$ is a quadrilateral with $AB = CD$ and $BC = AD$. The midpoints of AD, AB, BC, and CD are E, F, G, and H, respectively. Prove that $EF = GH$.

***13·6* Historical note on the calculation of the circumference of the earth.** While Euclid was Professor of Mathematics at the University of Alexandria (*ca.* 300 B.C.), he wrote his textbook on geometry, *The Elements*. In it the Parallel Line Theorem was known as Theorem 29.

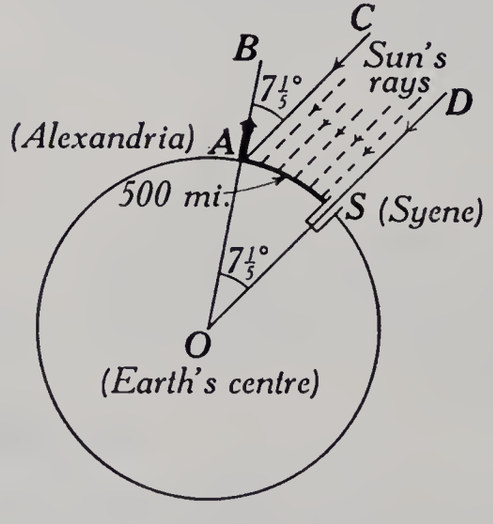


Fig. 13-20

Later, at the same University, Eratosthenes (*ca.* 240 B.C.), whose official position was librarian, geographer, and poet, undertook to calculate the circumference of the earth. To do this he used Euclid's Theorem 29. On a certain day at noon, the sun shone vertically into a deep well at Syene (now Aswan) on the Nile. At the same moment, at Alexandria 500

miles north of Syene, the shadow of a high pillar showed that the sun was 7° 12′ south of the vertical.

In *Fig. 13-20*, $\angle BAC = 7\frac{1}{5}°$. CA and DSO are parallel rays of the sun, and BO is a transversal making $\angle BAC = \angle BOD = 7\frac{1}{5}°$.

At O, the centre of the earth, there are $\frac{360}{7\frac{1}{5}}$ or 50 such angles. Thus, the circumference of the earth must be 50 × arc AS, or 50 × 500 miles, or 25,000 miles. This is a remarkably fine result to have been found over 2,000 years ago using very ordinary instruments.

***13·7* Parallelogram Theorems.** The two theorems following establish the basic properties of a parallelogram.

THEOREM 8

Quadrilateral-Parallelogram Theorem

If two opposite sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.

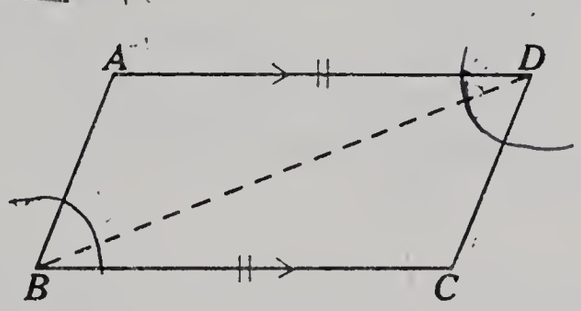


Fig. 13-21

Hypothesis: Quadrilateral $ABCD$ in which $AD = BC$ and $AD \parallel BC$ (*Fig. 13-21*)

Conclusion: $ABCD$ is a parallelogram.

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $ABCD$ is a parallelogram	1. $AB \parallel CD$.
2. $AB \parallel CD$	2. Alternate angles ABD and CDB are equal.
3. $\angle ABD = \angle CDB$	3. $\triangle ABD \cong \triangle CDB$.
4. $\triangle ABD \cong \triangle CDB$ by sas	4. $\angle ADB = \angle CBD$.
5. $\angle ADB = \angle CBD$ by the Parallel Line Theorem (equal alternate angles).	

Proof: Write a complete proof; compare your proof with that on page 493.

THEOREM 9

Parallelogram Theorem

In any parallelogram:

(i) **the opposite angles are equal;**

(ii) **the opposite sides are equal;**

(iii) **the diagonals bisect each other.**

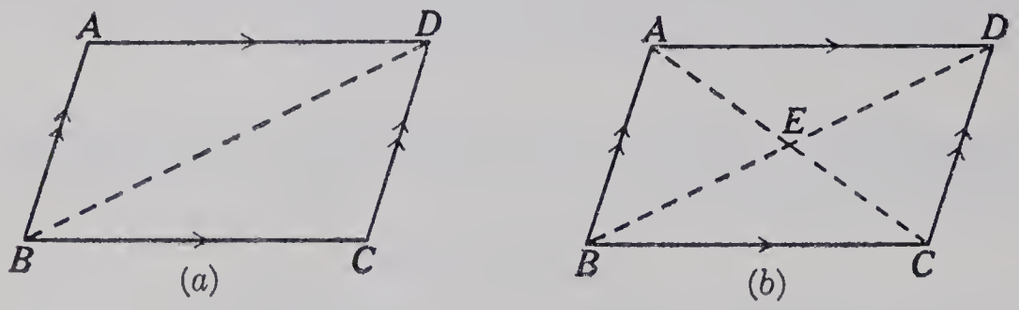


Fig. 13-22

Hypothesis: A parallelogram $ABCD$ in which AC and BD intersect at E, *Fig. 13-22* (*b*)

Conclusion: (i) $\angle A = \angle C$, $\angle ABC = \angle ADC$

(ii) $AD = BC$, $AB = DC$

(iii) BD and AC bisect each other.

Analysis:

PART (i) and (ii)

I CAN PROVE	IF I CAN PROVE
1. $\angle A = \angle C$, $\angle ABC = \angle ADC$, $AD = BC$, $AB = DC$	1. $\triangle ABD \cong \triangle CDB$.
2. $\triangle ABD \cong \triangle CDB$ by asa	2. $\angle ADB = \angle CBD$ and $\angle ABD = \angle CDB$.
3. $\angle ADB = \angle CBD$ and $\angle ABD = \angle CDB$ by the Parallel Line Theorem (equal alternate angles).	

PART (iii)

I CAN PROVE	IF I CAN PROVE
1. BD and AC bisect each other	1. $AE = EC$ and $BE = ED$.
2. $AE = EC$ and $BE = ED$	2. $\triangle AED \cong \triangle CEB$.
3. $\triangle AED \cong \triangle CEB$ by asa ($AD = CB$, proved in part (ii))	3. $\angle DAE = \angle ECB$ and $\angle ADE = \angle CBE$.
4. $\angle DAE = \angle ECB$ and $\angle ADE = \angle CBE$ by the Parallel Line Theorem (equal alternate angles).	

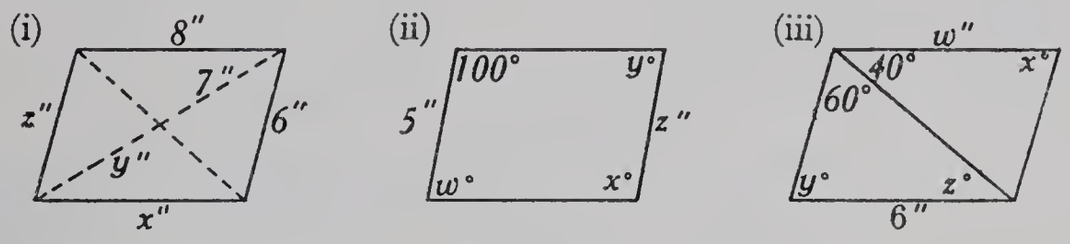
Proof: Write a complete proof; compare your proof with that on page 493.

Corollary: **The perpendicular distance between any two parallel lines is constant.**

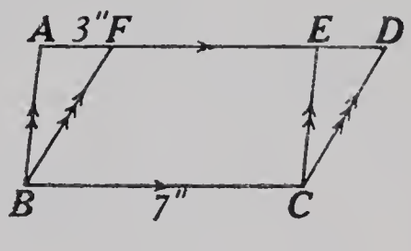
Exercise 13-4

(A)

1. In each of the parallelograms below, state the numbers represented by x, y, z, and w:



2. In the accompanying diagram, $BC = 7$ in., $AF = 3$ in.; find the lengths of AE, FE, FD, and ED.

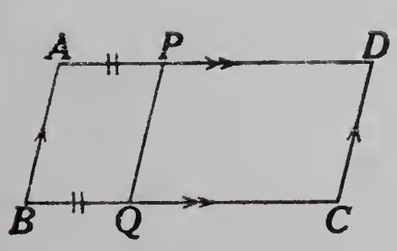


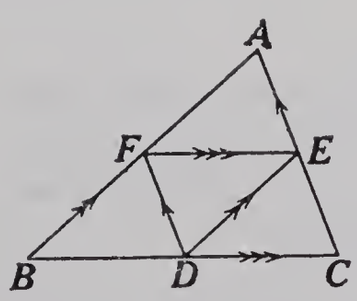
3. Quote the authorities now available for proving two straight lines parallel.

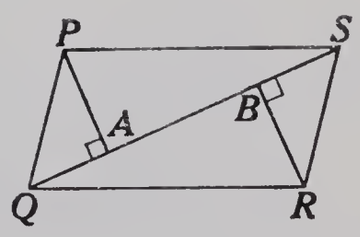
(B)

Write a complete proof for each of the following:

4. In parallelogram $ABCD$, left below, $AP = BQ$. Prove $PQ = DC$.







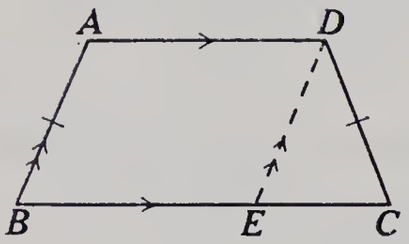
5. In the centre diagram above, prove four triangles congruent.

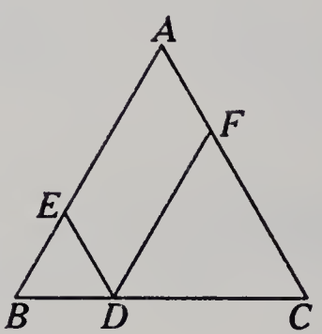
6. In parallelogram $PQRS$ at the right above, prove that the perpendiculars from P and R to the diagonal QS are equal. Prove also that $PARB$ is a parallelogram.

DEFINITION: *A rhombus is a parallelogram with a pair of adjacent sides equal.*

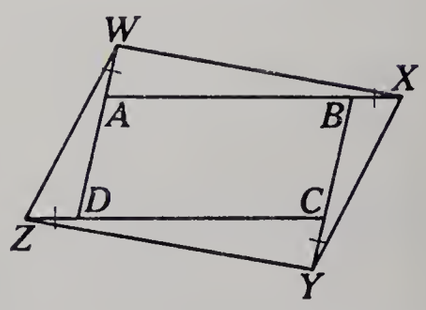
Write a complete proof for each of the following:

7. A rhombus has all sides equal.
8. $PQRS$ is a parallelogram. X and Y are the midpoints of PS and QR, respectively. Prove $PYRX$ is a parallelogram.
9. The quadrilateral formed by joining the midpoints of the sides of any parallelogram is also a parallelogram.
10. $ABCD$ is a parallelogram and E is the point of intersection of the diagonals. Prove that a line segment on E and terminated by the sides is bisected by E.
11. $ABCD$ is a parallelogram. E is the midpoint of AD, and F is the midpoint of BC. Prove that CE and AF trisect the diagonal BD.
12. $XYPQ$ is a rhombus. YX is extended its own length to F, and QP is extended its own length to E. Prove that $FYEQ$ is a parallelogram.
13. In an isosceles trapezoid, $ABCD$ has $AD \parallel BC$, $AB = DC$, $DE \parallel AB$ (diagram left below). Prove (i) $\angle ABC = \angle DCB$ (ii) $\angle BAD = \angle ADC$.





14. In the diagram at the right above, $AB = AC$, $ED \parallel AC$, $DF \parallel AB$. Prove $AE + ED + AF + FD = AB + AC$.
15. The diagonals of a rhombus are perpendicular to each other.
16. In the diagram at the right, $ABCD$ is a parallelogram with sides extended so that $AW = BX = CY = DZ$. Prove that $WXYZ$ is a parallelogram.



17. D and E are the midpoints of AB and AC, respectively, of any triangle ABC. Prove $DE \parallel BC$ and equal to one-half of BC.

***13·8* The angle sum of a triangle.** The angle sum of a triangle is the sum of the measurements of the angles of a triangle. By inductive reasoning

the conjecture was made that the angle sum of a triangle is 180°. This important relation of Euclidean geometry is established in the following theorem.

THEOREM 10

Triangle Angle Sum Theorem

The angle sum of a triangle is 180°.

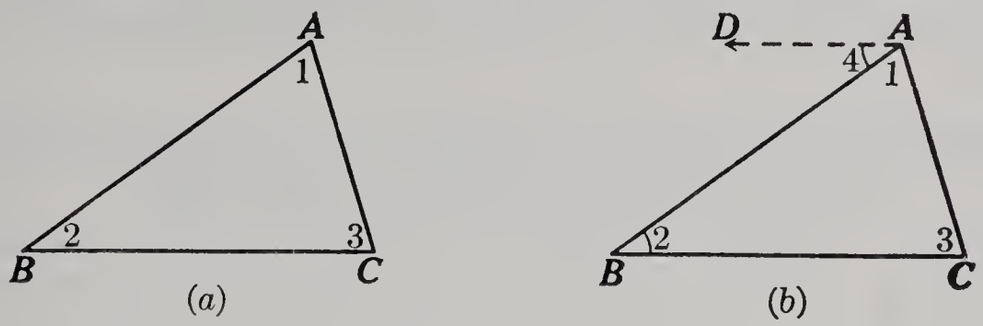


Fig. 13-23

Hypothesis: $\triangle ABC$, *Fig. 13-23 (a)*

Conclusion: $\angle 1 + \angle 2 + \angle 3 = 180°$

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle 1 + \angle 2 + \angle 3 = 180°$	1. $\angle 1 + \angle 4 + \angle 3 = 180°$ in *Fig. 13-23 (b)*.
2. $\angle 1 + \angle 4 + \angle 3 = 180°$	2. Ray $AD \parallel BC$.
3. Ray $AD \parallel BC$ by the Parallel Line Theorem.	

Proof: Write a complete proof; compare your proof with that on page 493.

Corollary 1: **An exterior angle of a triangle is equal to the sum of its remote interior angles.**

Hypothesis: $\triangle ABC$ and exterior angle 4, *Fig. 13-24*

Conclusion: $\angle 4 = \angle 1 + \angle 2$

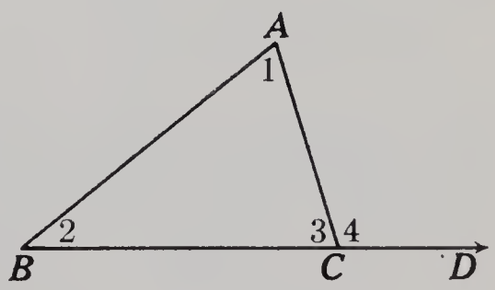


Fig. 13-24

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle 4 = \angle 1 + \angle 2$	1. $\angle 4 + \angle 3 = \angle 1 + \angle 2 + \angle 3$.
2. $\angle 4 + \angle 3 = 180°$ by definition. $\angle 1 + \angle 2 + \angle 3 = 180°$ by the Triangle Angle Sum Theorem.	

Proof: Write a complete proof; compare your proof with that on page 494.

Corollary 2: **If two angles of one triangle are respectively equal to two angles of another triangle, the third angles are equal.**

Analyse this deduction and write a complete proof; compare your proof with that on page 494.

Corollary 3: **The acute angles of a right triangle are complementary.**

Write a complete proof of this deduction and compare it with that on page 494.

Corollary 4: **If one angle of a triangle is equal to the sum of the other two angles, then that angle is a right angle.**

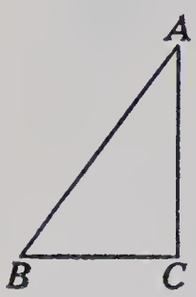


Fig. 13-25

Hypothesis: $\triangle ABC$ in which $\angle C = \angle A + \angle B$, *Fig. 13-25*

Conclusion: $\angle C$ is a right angle.

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle C$ is a right angle	1. $\angle C = 90°$.
2. $\angle C = 90°$	2. $2\ \angle C = 180°$.
3. $2\ \angle C = 180°$	3. $2\ \angle C = \angle A + \angle B + \angle C$.
4. $2\ \angle C = \angle A + \angle B + \angle C$	4. $\angle C = \angle A + \angle B$.
5. $\angle C = \angle A + \angle B$ by hypothesis.	

Proof: Write a complete proof; compare your proof with that on page 494.

Corollary 5: **Each angle of an equilateral triangle has a measurement of 60°.**

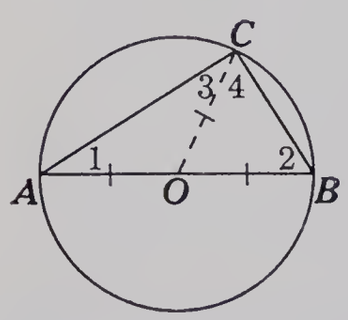


Fig. 13-26

Example 1. In *Fig. 13-26*, C is any point on a circle with AOB as diameter. Prove $\angle ACB = 90°$, that is, the angle in a semicircle is a right angle.

Hypothesis: C is any point on a circle with AOB as diameter.

Conclusion: $\angle ACB = 90°$

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle ACB = 90°$	1. $\angle 3 + \angle 4 = 90°$.
2. $\angle 3 + \angle 4 = 90°$	2. $\angle 3 + \angle 4 = \angle 1 + \angle 2$.
3. $\angle 3 + \angle 4 = \angle 1 + \angle 2$	3. $\angle 3 = \angle 1$, $\angle 4 = \angle 2$.
4. $\angle 3 = \angle 1$, $\angle 4 = \angle 2$ by the Isosceles Triangle Theorem.	

Proof: Write a complete proof; compare your proof with that on page 495.

Example 2. In *Fig. 13-27*, ABC is a triangle in which the angles B and C are each double the vertical angle A. The bisector of $\angle ACB$ meets AB at D. Prove that $\angle ADC = 3\ \angle A$.

Hypothesis: $\triangle ABC$ in which $\angle w = \angle y + \angle z = 2\angle x$, $\angle y = \angle z$

Conclusion: $\angle ADC = 3\ \angle x$

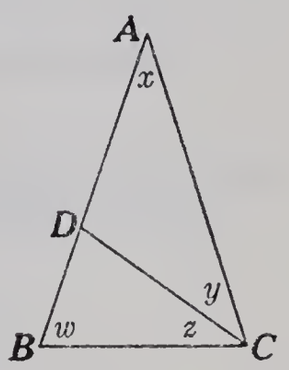


Fig. 13-27

Write a complete analysis and proof; compare your solution with the one given on page 495.

Example 3. Using ruler and compasses, construct angles of 30° and 75°.

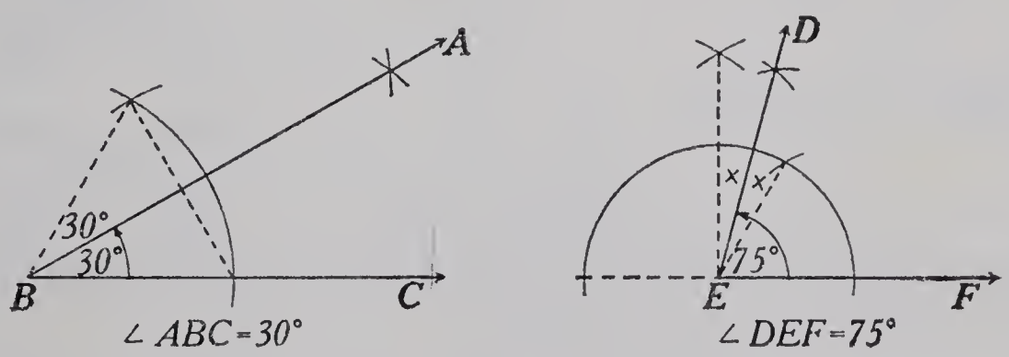


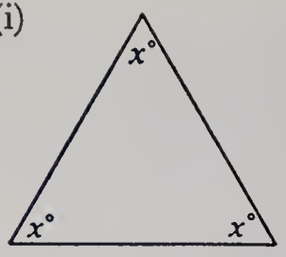
Fig. 13-28

Exercise 13-5

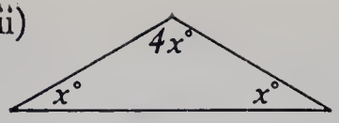
(A)

1. In the following diagrams, find x and y and the measurement of each angle:

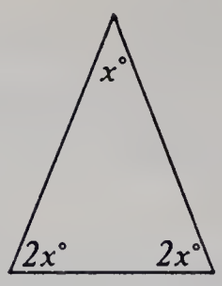
(i)



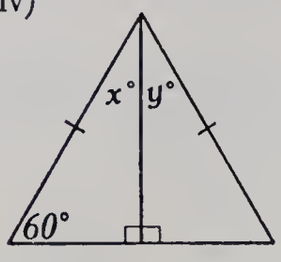
(ii)



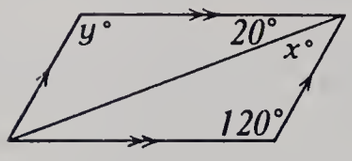
(iii)



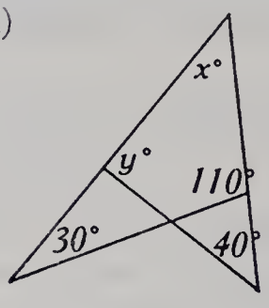
(iv)



(v)

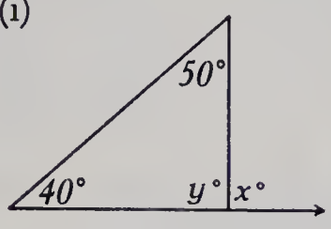


(vi)

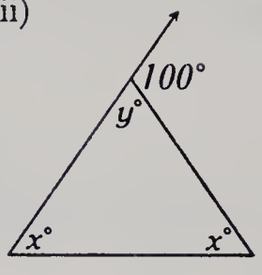


2. In each of the following diagrams, x and y represent the number of degrees in the angles indicated. Find x and y in each case.

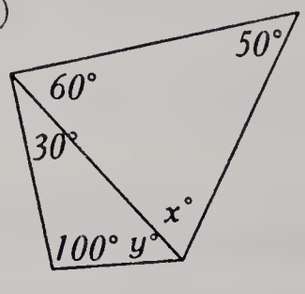
(i)



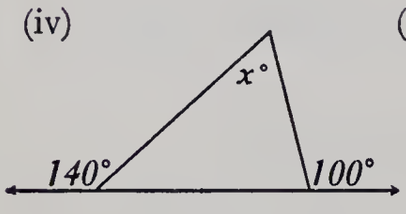
(ii)



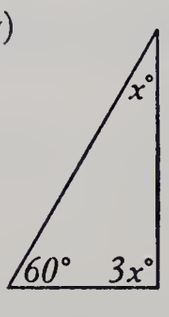
(iii)



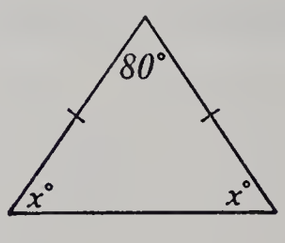
(iv)



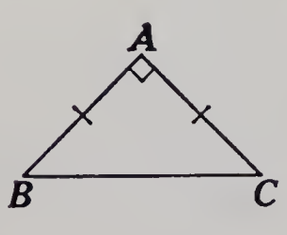
(v)

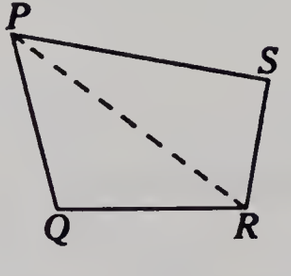


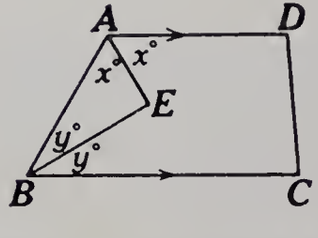
(vi)



3. In the diagram at the left below, ABC is an isosceles right triangle. Prove that $\angle B = \angle C = 45°$.







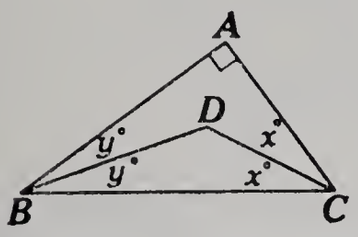
4. In the centre figure above, find the angle sum of the interior angles of quadrilateral $PQRS$.

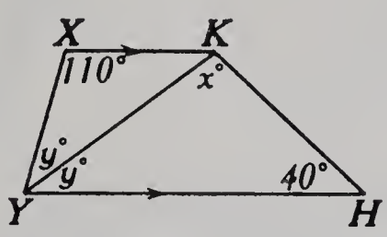
5. In the diagram at the right above, prove that $\angle AEB = 90°$.

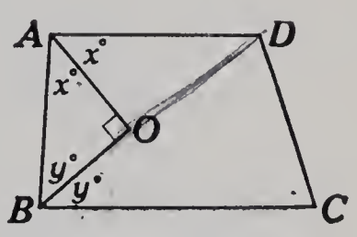
(B)

6. ABC is an equilateral triangle with AC extended to D so that $CD = AC$. Prove $\angle ABD = 90°$.

7. In the diagram left below, find the angle sum of $\angle ACB$ and $\angle ABC$, and calculate the number of degrees in $\angle BDC$.







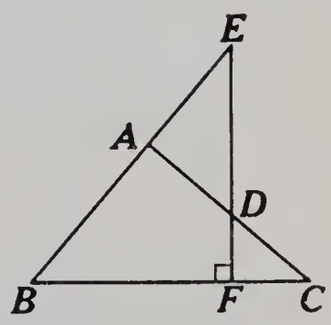
8. In the centre diagram above, $XK \parallel YH$ and KY bisects $\angle XYH$. Find x.

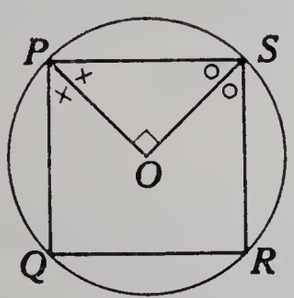
9. In the diagram at the right above, the bisectors of angles A and B meet on O to form right angle AOB. Prove $AD \parallel BC$.

Write an analysis of the deductions in 10, 11, 12 *below:*

10. ABC is an isosceles triangle in which $AB = AC$, and AC is extended to D so that $CD = CB$. Prove that $\angle ABD = 3\ \angle D$.

11. In the diagram at the right, $AB = AC$ and $EF \perp BC$.
Prove that $\angle ADE = \angle AED$.





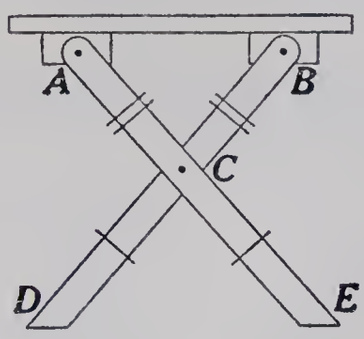
12. If the bisectors of angles P and S, in the diagram at the left, meet at the centre O of the circle, making $\angle POS = 90°$, prove $PQRS$ is a square.

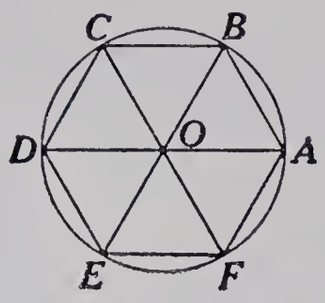
Write a complete proof for each of the following:

13. ABC is any triangle. Equilateral triangles ABD and ACE are drawn outward on AB and AC. Prove $\triangle DAC \cong \triangle ABE$ and hence $CD = BE$.

14. In $\triangle ABC$, $AB = AC$. D and E are points on AB and AC, respectively, such that $AD = AE$. Prove that $DE \parallel BC$.

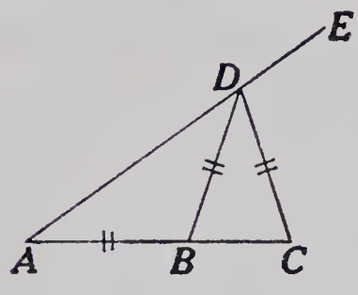
15. Explain why the table-top in the diagram left below, is always parallel to the floor, DE.



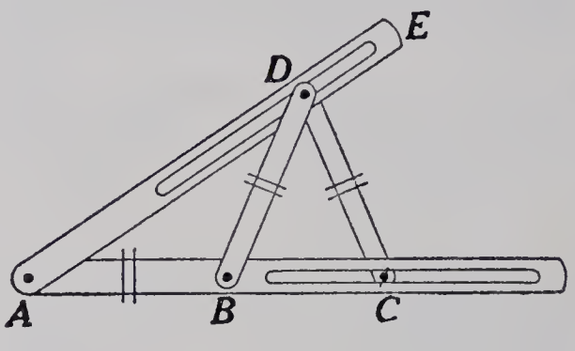


16. To make an equi-sided hexagon in a circle, centre O, choose any point A on the circumference. Starting with A as centre and radius AO, arcs are "stepped off around the circle" at B, C, D, E, and F. In the diagram, prove FA is equal to each of the other sides.

17. The Greeks, as well as scholars of the Middle Ages, spent much time trying to trisect an angle. It has been shown that this is impossible using only a pair of compasses and a straight edge. In (a) below,



(a)



(b)

prove that $\angle A = \frac{1}{2}\angle CBD = \frac{1}{3}\angle CDE$. Hence prove that the mechanical device shown in (b) may be used, within limits, to make an angle (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$ the measurement of a given angle.

(C)

18. In $\triangle ABC$ the bisectors of angles B and C meet at E. If the number of degrees in angle A is x, prove that $\angle BEC = (90 + \frac{1}{2}x)°$.

19. In $\triangle ABC$ the exterior angles at B and C are bisected. Prove that these bisectors meet at an angle with measurement $(90 - \frac{1}{2}x)°$ where x is the number of degrees in $\angle A$.

13·9 **The aas Congruence Theorem.** The congruence postulates (sas, sss, asa) specify three sets of sufficient conditions for congruence of triangles. Theorems 11 and 12 establish two additional sets of sufficient conditions for congruence of triangles.

THEOREM 11

The aas *Congruence Theorem*

If two angles and a side opposite one of them in one triangle are respectively equal to two angles and the corresponding side in a second triangle, then the two triangles are congruent.

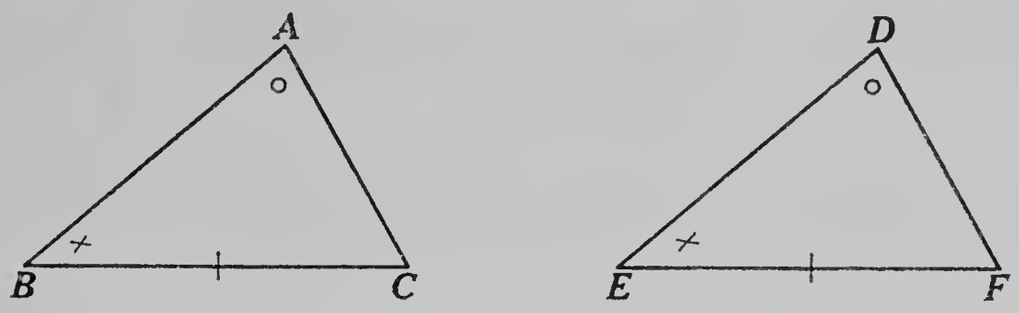


Fig. 13-29

Hypothesis: $\triangle ABC$ and $\triangle DEF$ such that $\angle A = \angle D$, $\angle B = \angle E$, $BC = EF$

Conclusion: $\triangle ABC \cong \triangle DEF$

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\triangle ABC \cong \triangle DEF$	1. The conditions for sas, sss, or asa.
2. The conditions for asa	2. $\angle C = \angle F$.
3. $\angle C = \angle F$ by the Angle Sum Theorem.	

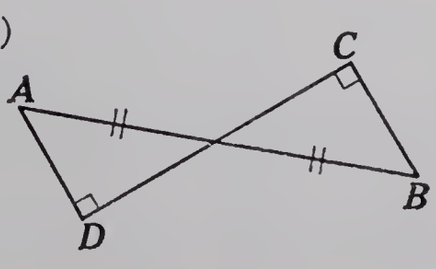
Proof: Write a complete proof; compare your proof with that on page 495.

Exercise 13-6

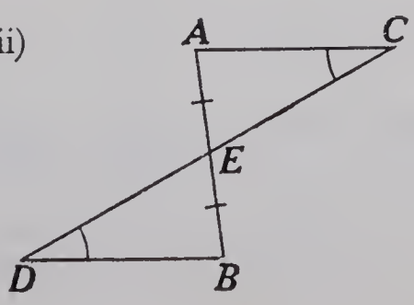
(A)

1. In each of the following diagrams, AB and CD are line segments. State all the facts you can about each diagram.

(i)

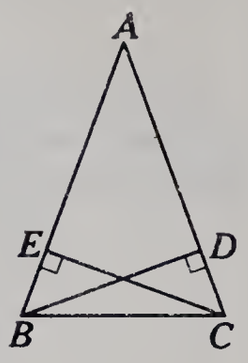


(ii)



(B)

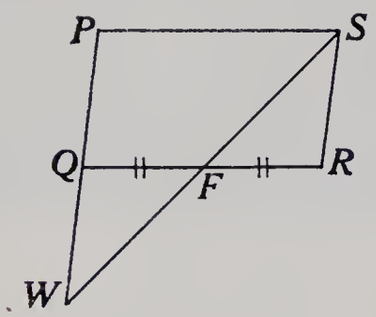
Write a complete proof of each of the following deductions:

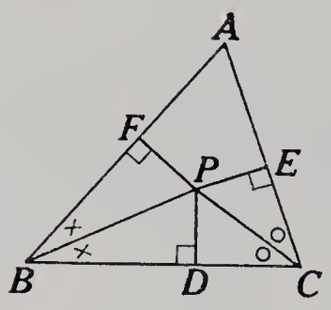


2. If the bisector of $\angle A$ in $\triangle ABC$ is perpendicular to BC, prove $\triangle ABC$ is isosceles.

3. In the diagram at the right $AB = AC$, $BD \perp AC$, $CE \perp AB$. Prove $\triangle EBC \cong \triangle DCB$.

4. In the diagram left below, F is the midpoint of QR, one side of parallelogram $PQRS$. SF is extended to intersect PQ extended on W. Prove that $SF = FW$.





5. In the diagram at the right above, PB and PC are the bisectors of angles B and C. $PD \perp BC$, $PF \perp AB$, $PE \perp AC$. Prove that the point P is equidistant from the three sides of the triangle. The circle drawn with centre P and radius PD is called the *inscribed circle* of the triangle; P is the *incentre* of the triangle.

13·10 The Right Triangle Congruence Theorem.

THEOREM 12

Right Triangle Congruence Theorem

If the hypotenuse and one side of a right triangle are respectively equal to the hypotenuse and the corresponding side of another right triangle, then the triangles are congruent.

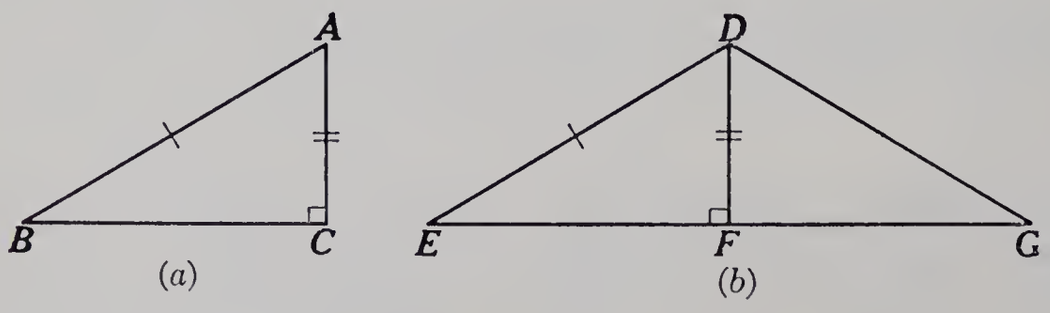


Fig. 13-30

Hypothesis: $\triangle ABC$ and $\triangle DEF$ such that $AB = DE$, $AC = DF$, $\angle ACB = \angle DFE = 90°$

Conclusion: $\triangle ABC \cong \triangle DEF$

Proof: Copy and complete the following proof; compare your proof with that on page 496.

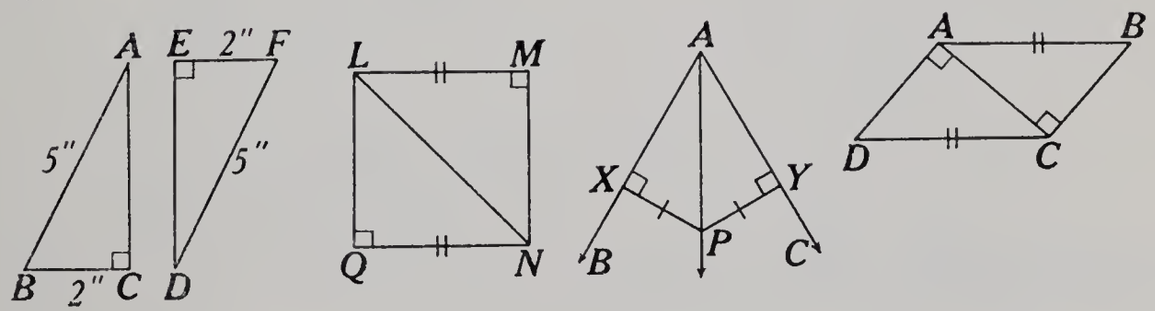
STATEMENTS	AUTHORITIES
1. On the ray opposite FE, G is the point such that $FG = CB$.	1.
2. In $\triangle$'s ABC, DGF $\left\{ \begin{array}{l} AC = DF \\ BC = GF \\ \angle ACB = \angle DFG = 90° \end{array} \right.$	2.
3.	3.
4.	4.
5. $\triangle ABC \cong \triangle DGF$	5.
6. $DG = AB = DE$	6.
7. $\angle DGF = \angle ABC = \angle DEF$	7.
8. $\triangle ABC \cong \triangle DEF$	8.

Exercise 13-7

(A)

1. State all the facts you can about each of the following diagrams:

(i) (ii) (iii) (iv)



(B)

Write a complete proof for each of the following:

2. A ray from the centre of a circle perpendicular to a chord bisects the chord.

3. A ray from the vertex of an isosceles triangle, perpendicular to the base, bisects the base.

4. If a point within a triangle is equidistant from the three sides, prove that the point is on the bisector of each angle.

5. If the line segments from the midpoint of the base of a triangle perpendicular to the sides are equal, the triangle is isosceles.

6. A, B, D, C are points on a circle, centre O. $OX \perp AB, OY \perp CD$, and $OX = OY$. Prove $AX = CY$ and $BX = DY$, and hence $AB = CD$.

13·11 **The Angle Bisector Theorem.** Write a complete proof of the following theorem; compare your proof with that on page 496.

THEOREM 13

Angle Bisector Theorem

A point is equidistant from the sides of an angle if and only if it is on the bisector of the angle.

13·12 **Planes.** Although a plane is unlimited in extent in space, we usually picture it as a quadrilateral drawn obliquely. In *Fig. 13-31*(*a*), quadrilateral $ABCD$ represents a plane. This plane may be referred to as plane AC or as plane p. The lines BC and CD are drawn heavier to make the plane appear horizontal. *Fig. 13-31*(*b*) illustrates a point M, a line, and a line segment OR, in a plane AC.

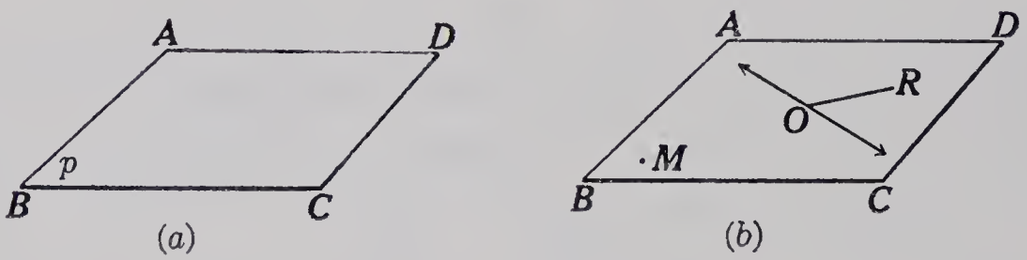


Fig. 13-31

a. *Fundamental properties of planes.*

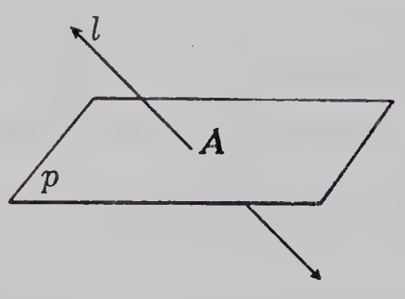


Fig. 13-32

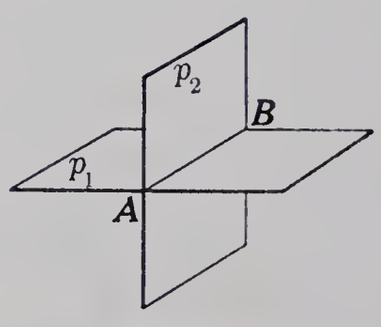


Fig. 13-33

(i) The intersection of a line and a plane is a point, *Fig. 13-32*.

(ii) The intersection of two planes is a line, *Fig. 13-33*.

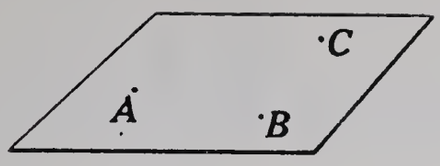


Fig. 13-34

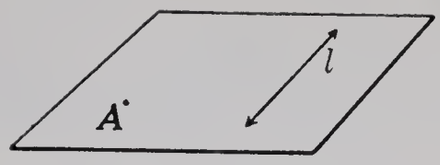


Fig. 13-35

(iii) A plane is determined by three points not in the same line, *Fig. 13-34.*

(iv) A plane is determined by a line and a point not on the line, *Fig. 13-35.*

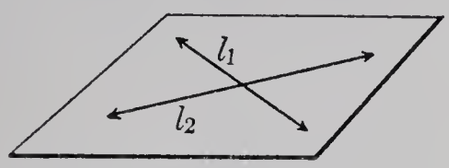


Fig. 13-36

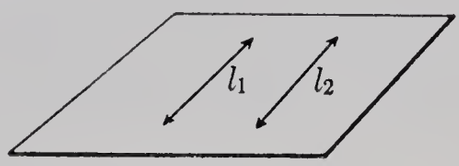


Fig. 13-37

(v) A plane is determined by two intersecting lines, *Fig. 13-36.*

(vi) A plane is determined by two parallel lines, *Fig. 13-37.*

b. *Parallel and perpendicular planes and lines.*

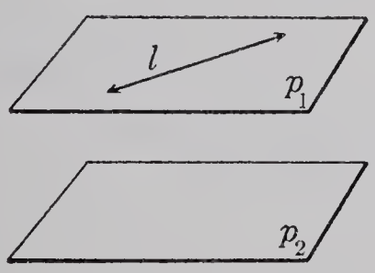


Fig. 13-38

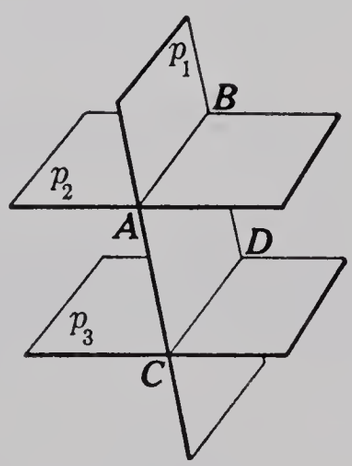


Fig. 13-39

(i) If two planes have no point in common, then they are parallel. If a straight line and a plane have no common point, then they are parallel, *Fig. 13-38.*

(ii) If two parallel planes are cut by a third plane, then the lines of intersection are parallel, *Fig. 13-39.*

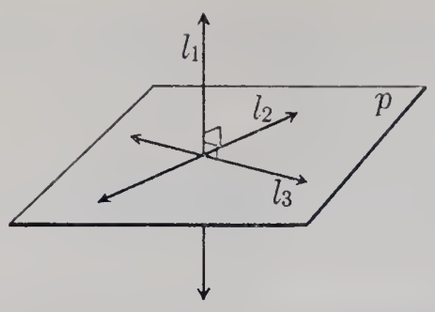


Fig. 13-40

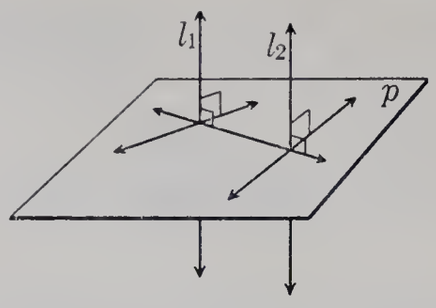


Fig. 13-41

(iii) If a line is perpendicular to each of two intersecting lines on their point of intersection, it is perpendicular to the plane of the two lines, *Fig. 13-40.*

(iv) If two lines are perpendicular to the same plane, then they are parallel, *Fig. 13-41.*

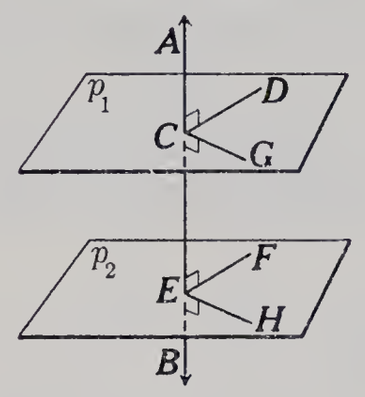


Fig. 13-42

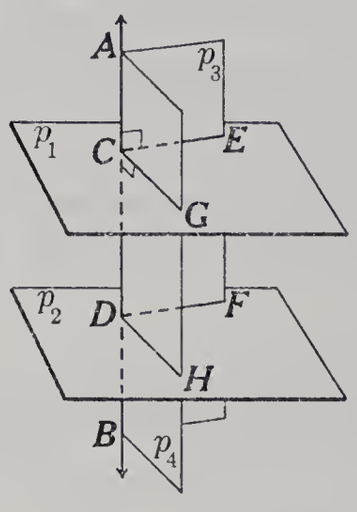


Fig. 13-43

(v) If two planes are perpendicular to the same line, then they are parallel, *Fig. 13-42.*
Through a point external to a plane, there is one and only one plane parallel to the given plane.

(vi) If a line is perpendicular to one of two parallel planes, then it is perpendicular to the other, *Fig. 13-43.*

Exercise 13-8

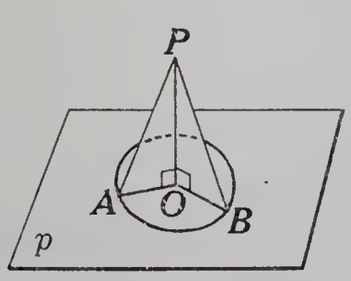
(A)

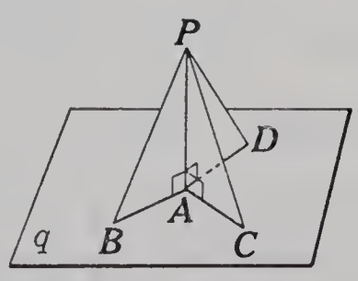
1. In *Fig. 13-32* what is the intersection set of line l and plane p?
2. In *Fig. 13-33*, name the intersection set of planes p_1 and p_2.
3. Are all triangles plane figures? Why?
4. Name the lines which determine the plane in *Fig. 13-36.*

5. If l_1 and l_2 determine the plane in *Fig. 13-37*, how are l_1 and l_2 related?
6. AB, CD, and EF are three parallel lines, but not in the same plane. How many planes are determined by these lines if they are taken in pairs?
7. AB, AC, and AD are three lines on point A, but not in the same plane. How many planes are determined by these lines if they are taken in pairs?
8. (i) In *Fig. 13-39*, what is the intersection set of planes p_1 and p_2?
   (ii) Name a pair of parallel lines.
9. In *Fig. 13-40*, how is it indicated that l_1 is perpendicular to plane p?
10. In *Fig. 13-41*, justify $l_1 \parallel l_2$.
11. In *Fig. 13-42*, how is it indicated that:
    (i) plane p_1 is perpendicular to line AB;
    (ii) plane p_2 is perpendicular to line AB?
12. In *Fig. 13-43*, justify that AB is perpendicular to plane p_2.

(B)

13. (i) Draw a quadrilateral to represent a horizontal plane.
    (ii) Draw a pair of intersecting lines in the plane.
    (iii) Draw a line perpendicular to the plane at their point of intersection.
14. (i) Draw a quadrilateral $ABCD$ to represent a horizontal plane.
    (ii) At the vertices, draw equal line segments, AE, BF, CG, DH, perpendicular to the plane.
    (iii) Draw a quadrilateral to represent the plane determined by E, F, G, H.
    (iv) What relation exists between plane AC and plane EG?
15. In the diagram at the left below, PO is perpendicular to plane p at O, the centre of a circle in the plane. If A and B are on the circle, prove that $PA = PB$ and $\angle PAO = \angle PBO$.





16. In the diagram at the right above, PA represents a flagpole perpendicular to plane q. PB, PC, PD are guy wires. If $AD = AC$, prove $PD = PC$.

17. (i) Draw a quadrilateral to represent a horizontal plane.

(ii) Draw a triangle ABC in the plane.

(iii) At the vertices, draw equal line segments AD, BE, CF, perpendicular to the plane.

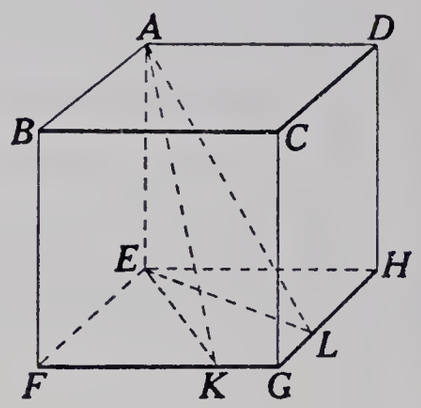
(iv) Draw a quadrilateral to represent the plane determined by D, E, and F.

18. (i) Draw a diagram to represent a horizontal plane p.

(ii) Draw a right circular cone with base in the plane.

(iii) Draw a quadrilateral to represent a plane q which is parallel to plane p and intersects the cone.

19. (i) Draw the diagram of a cube shown at the right.

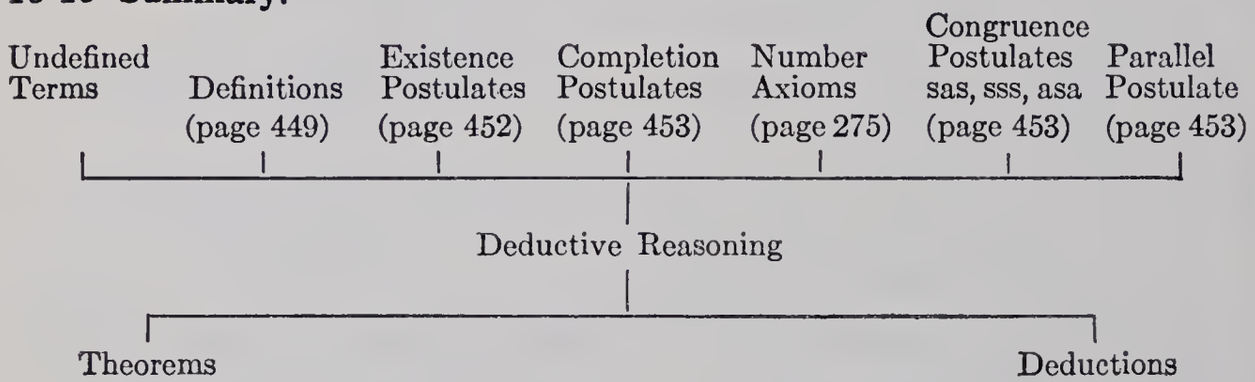


(ii) List all pairs of parallel planes.

(iii) List all pairs of perpendicular lines.

(iv) If $KG = GL$, prove $AK = AL$.

13·13 Summary.



Angle
1. Complementary Angle Theorem
2. Supplementary Angle Theorem
3. Vertical Angle Theorem

Basic Biconditional
4. Isosceles Triangle Theorem
5. Right Bisector Theorem
13. Angle Bisector Theorem

Inequality
6. Exterior Angle Theorem

Parallel
7. Parallel Line Theorem
8. Quadrilateral-Parallelogram Theorem
9. Parallelogram Theorem
10. Triangle Angle Sum Theorem

Congruence
11. aas Congruence Theorem
12. Right Triangle Congruence Theorem

Review Exercise 13-9

(B)

1. If AM, a median of $\triangle ABC$, is extended to N so that $AM = MN$, then $BA \parallel NC$.
2. If the bisector of $\angle R$ in $\triangle TRS$ intersects TS at B and the right bisector of RB intersects RT at A, then $AB \parallel RS$.
3. In $\triangle PQR$, if S is a point on PQ such that $PS = QS = RS$, then $\angle PRQ$ is a right angle.
4. Corresponding altitudes of congruent triangles are equal.
5. If the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.
6. If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
7. In $\triangle ABC$, BD is the median drawn from B to AC. Perpendiculars DE and DF are drawn to BA and BC, respectively. If $DE = DF$, prove $BA = BC$.
8. F is the midpoint of side AB in rectangle $ABCD$. G and H are points in AD and BC, respectively, so that $FG = FH$. Prove $GD = HC$.
9. ABC is an isosceles triangle with $AB = AC$. BM and CN are the altitudes from B and C. Prove $MN \parallel BC$.
10. In $\triangle ABC$, sides AB and AC are extended to D and E, respectively. If the bisectors of $\angle$'s BCE and CBD intersect at O, prove that the perpendiculars from O to BD, BC, and CE are equal.

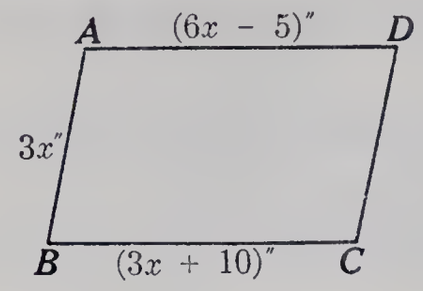
Review Exercise 13-10

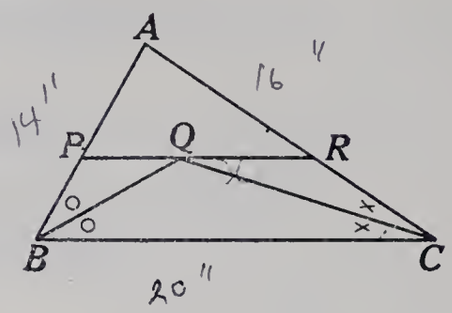
(*Multiple Choice Questions*)

(B)

1. In $\triangle ABC$, $\angle B = \angle C$ and $\angle B = 2\ \angle A$.
   $\angle A$ is: (a) 90° (b) 36° (c) 30° (d) 60° (e) 144°
2. In $\triangle LMN$, $\angle L = 2\ \angle M$, $\angle L + \angle M = 120°$.
   $\angle N$ is: (a) 80° (b) 70° (c) 60° (d) 50° (e) 40°
3. In $\triangle PQR$, $\angle P = 3\angle R$ and $\angle Q = 2\angle P$. The measurement of the largest angle is: (a) 18° (b) 90° (c) 120° (d) 108° (e) 20°
4. $\triangle ABC$ is right-angled at A. The bisectors of angles C and B intersect at D. $\angle BDC$ is:
   (a) 135° (b) 45° (c) $67\frac{1}{2}°$ (d) $122\frac{1}{2}°$ (e) $112\frac{1}{2}°$

5. In the diagram at the left below, $ABCD$ is a parallelogram. The length of DC is: (a) 5 in. (b) 9 in. (c) 12 in. (d) 15 in. (e) 18 in.





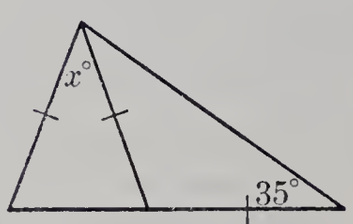
6. In the diagram at the right above, $AB = 14$ in., $AC = 16$ in., and $BC = 20$ in. $PQR \,||BC$; BQ bisects $\angle ABC$; CQ bisects $\angle ACB$. The sum, $AP + PR + RA$ is:
(a) 30 in. (b) 36 in. (c) 34 in. (d) 15 in. (e) 18 in.

7. In a parallelogram with its diagonals drawn, the number of pairs of congruent triangles formed is:
(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

8. $ABCD$ is a rhombus with $AB = AC$. $\angle BAD$ is:
(a) 60° (b) 75° (c) 90° (d) 120° (e) 135°

9. In the diagram below, x is: (a) 140 (b) 70 (c) 40 (d) 35 (e) 20

10. If a point is equidistant from the sides of a triangle, it is the intersection of the:
(a) bisectors of the angles (b) medians
(c) right bisectors of the sides (d) altitudes
(e) none of these



Exercise 13-11

(*Constructions*)

(B)

Draw the figures using only ruler and compasses; show all construction lines but do not give proofs or descriptions of constructions:

1. Construct $\triangle ABC$ with $BC = 2$ inches, $\angle B = 75°$, and $\angle C = 60°$. Draw a perpendicular from A to BC, meeting it at D. Measure AD.

2. Construct a quadrilateral $ABCD$, having $AD = 3$ inches, $AB = 2\frac{1}{2}$ inches, $BC = 4$ inches, $CD = 4$ inches, and $\angle BAD = 90°$. Bisect $\angle BCD$ and let the bisector meet AD at E; measure CE. Draw DK perpendicular to BC, meeting BC at K; measure DK. Join BD. Draw the median of $\triangle ABD$ from D and measure it.

3. Construct a quadrilateral $ABCD$ with base $AB = 4$ inches, $\angle DAB = 52\frac{1}{2}°$, diagonal $DB = 4\frac{1}{2}$ inches, side $BC = 3\frac{3}{4}$ inches, and side $DC = 2\frac{5}{8}$ inches. Inscribe a circle in $\triangle DBC$ and measure its radius.

4. Construct a quadrilateral $ABCD$, having $AB = 2\frac{1}{4}$ inches, $\angle ABC = 112\frac{1}{2}°$, $BC = 1\frac{3}{8}$ inches, $\angle BCD = 90°$, and $CD = DA$. Circumscribe a circle about $\triangle CAB$ and measure its radius.

5. Draw a line segment $AB = 3$ inches. On one side of AB draw $\triangle ABC$ having $\angle CAB = 30°$ and $BC = 3\frac{1}{4}$ inches. On the opposite side of AB construct a square $ABDE$. Join CE and measure it.

6. Draw a line segment $AB = 4$ inches. On AB construct $\triangle ABC$, having $\angle ABC = 45°$ and $\angle BAC = 60°$. With centre C and radius $1\frac{1}{2}$ inches, draw a circle. On CB as diameter, draw a circle cutting the first circle at D and E. Join BD and BE and measure these line segments.

7. Construct $\triangle ABC$, having base $BC = 6$ cm., $\angle A = 75°$, and $\angle B = 45°$. Draw and measure the altitude from B. Calculate the area of the triangle.

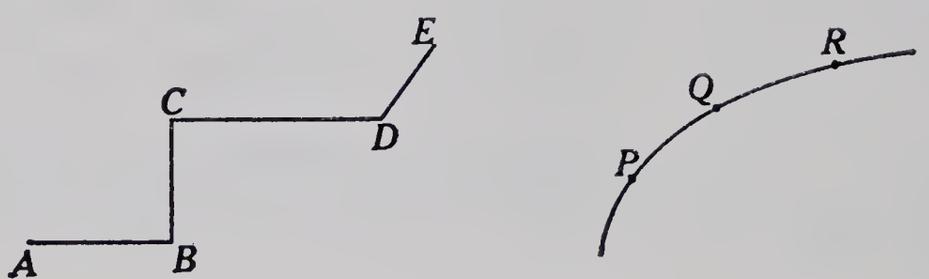
8. Construct a quadrilateral $ABCD$, having $AB = 10$ cm., $BC = 4$ cm., $AD = 9$ cm., $DB = 7$ cm., and $DC = 6$ cm. Draw and measure the diagonal AC. Draw the perpendicular from D to AC, meeting it at F; measure DF. Bisect $\angle ABC$ and let the bisector meet AC at K; measure BK. Draw the median of $\triangle DAC$ from C and measure it.

9. Construct $\angle BAC = 75°$. Show how to locate within this angle a point P which is 75 mm. from A and 38 mm. from AC. How far is P from AB?

Chapter XIV

AREAS OF POLYGONS

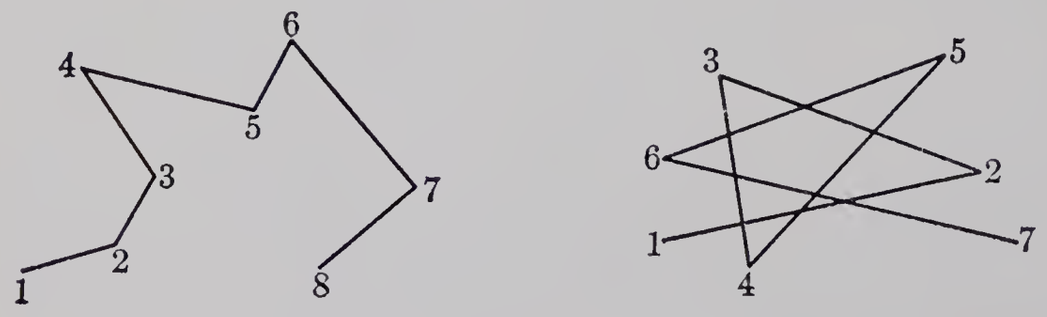
***14·1* Polygons.** The figures in *Fig. 14-1* are called *curves* in geometry.



(*a*) (*b*)

Fig. 14-1

In *Fig. 14-1* (*a*) points A, B, C, D, E are points on the curve or the curve is on these points. This figure consists of all points on the line segments AB, BC, CD, DE. Since it is made up of a succession of line segments, it



(*a*) (*b*)

Fig. 14-2

is called a *broken line figure* or a *polygonal path.* Point A is called the *beginning* of the path, and E is the *end.* The line segments are called the *sides* of the path. The points A, B, C, D, E are the *vertices* of the path.

The sides of a polygonal path may intersect, as in *Fig. 14-2*(*b*). If they do not intersect, *Fig. 14-2*(*a*), the path is called a *simple curve* or *simple polygonal path.*

a. *Definition of a polygon.*

(i) *Polygon. If the beginning and end of a polygonal path consisting of three or more line segments coincide, then the path is called a polygon.*

(ii) *Simple polygon. If the beginning and end of a polygonal path consisting of three or more line segments coincide and no two sides intersect, then the path is called a simple polygon* (*Fig. 14-3*).

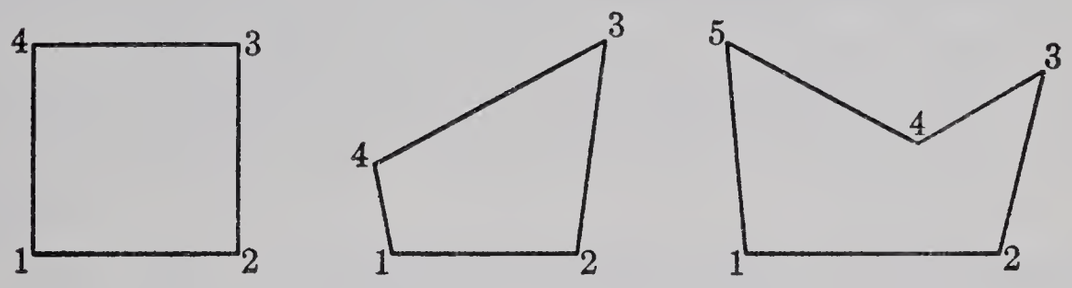


Fig. 14-3

b. *Kinds of polygons.* Polygons are named according to the number of sides, as shown in *Fig. 14-4.* It should be noted that the number of vertices of a given polygon is the same as the number of sides.

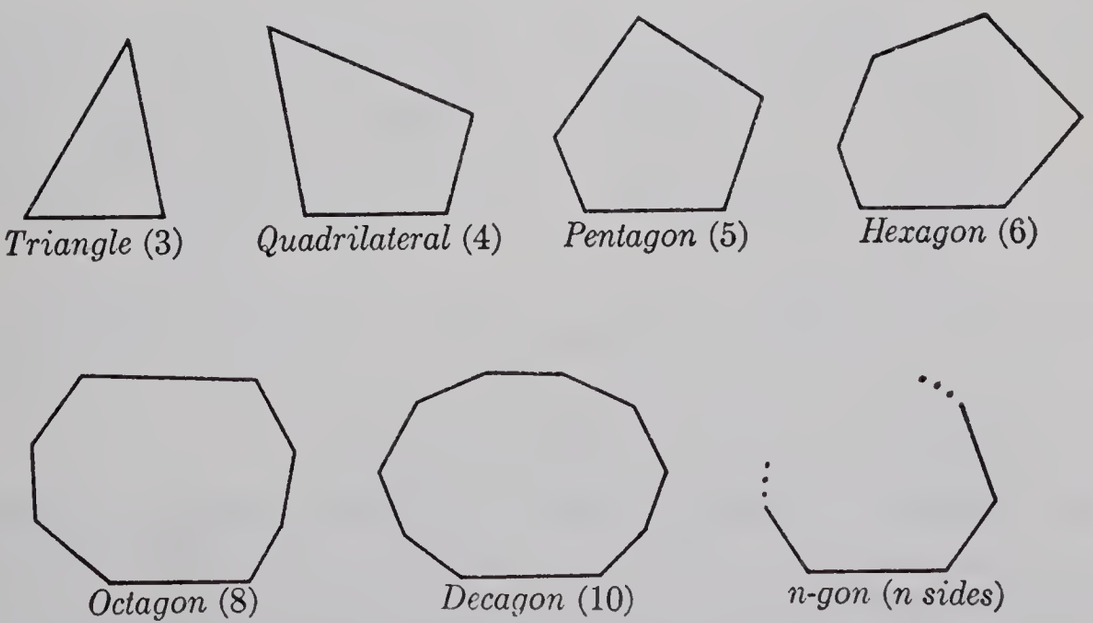


Fig. 14-4

14·2 Polygonal regions. A triangular region is shown in *Fig 14-5.*

DEFINITION: *A triangular region is the union of a triangle and its interior.*

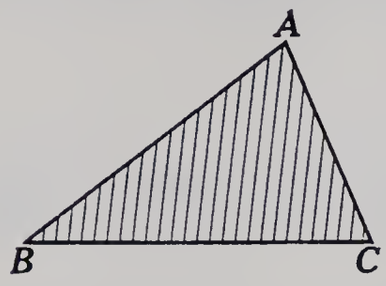


Fig. 14-5

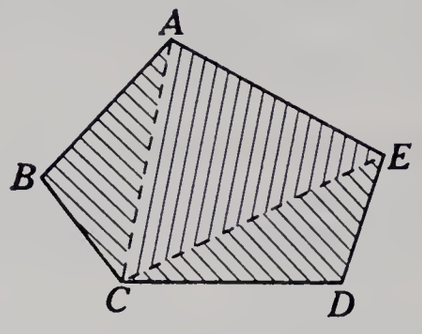


Fig. 14-6

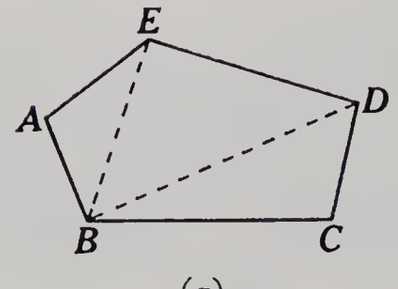
The simple polygonal region *ABCDE* in *Fig. 14-6* is separated into three non-overlapping triangular regions called *component triangular regions.* The polygonal region is the union of these component regions.

With a line segment we associate the notion of length. In a similar manner, we associate the idea of *area* with a region and make the following area postulates.

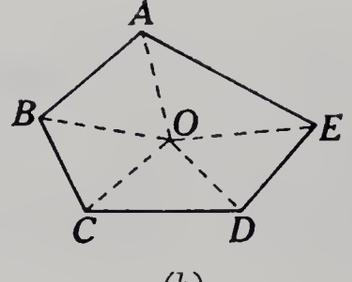
AREA POSTULATE 1. (Area Completion Postulate)

The area of a polygonal region is the sum of the areas of the component regions whose union is the polygonal region.

We use the symbols $\triangle ABC$, *Fig. DEFG*, and so on, to represent either the geometric figure or the area of the region as indicated by the context.



(*a*)



(*b*)

Fig. 14-7

(*a*) *Fig.* $ABCDE = \triangle ABE + \triangle EBD + \triangle DBC$

(*b*) *Fig.* $ABCDE = \triangle AOB + \triangle BOC + \triangle COD + \triangle DOE + \triangle EOA$

AREA POSTULATE 2.

If two polygonal regions are congruent, then they have the same area.

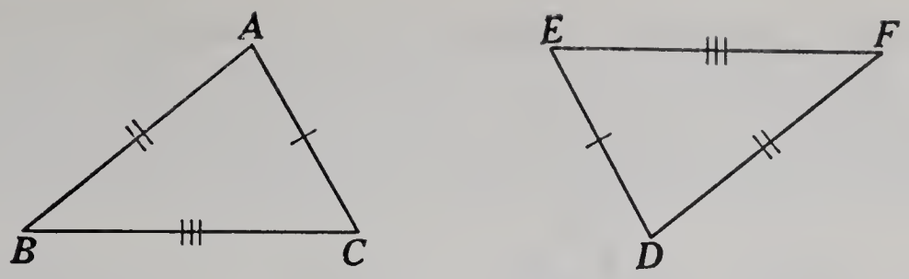


Fig. 14-8

In *Fig. 14-8*, $\triangle ABC \cong \triangle DFE$
$\therefore \triangle ABC = \triangle DFE$

AREA POSTULATE 3. (Equal Area Postulate)

Two polygonal regions are equal in area if they can be separated into a finite number of polygons which are congruent in pairs.

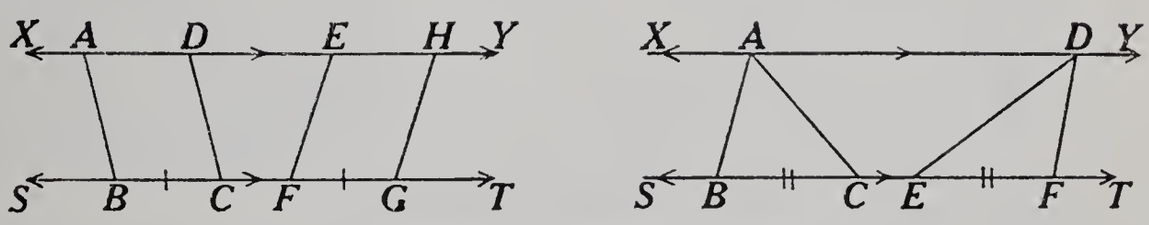
14·3 Figures between the same parallel lines.

A parallelogram is said to be between the pair of parallel lines determined by a pair of opposite sides.

A triangle is said to be between the pair of parallel lines determined by a side and the opposite vertex.

Example.

(i)



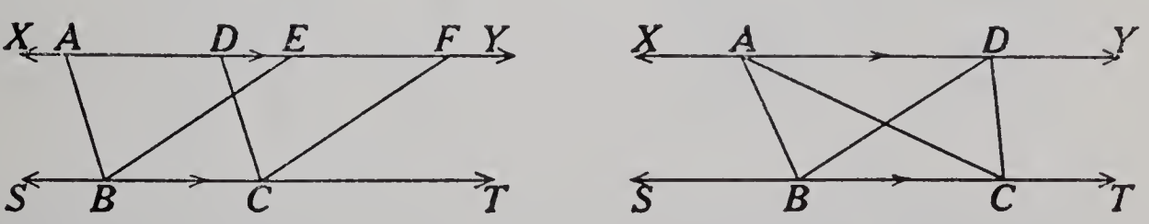
(*a*) (*b*)

Fig. 14-9

In *Fig. 14-9*(*a*), *ABCD* and *EFGH* are $||^{gms}$ between the same parallel lines *XY* and *ST*.

In *Fig. 14-9*(*b*), *ABC* and *DEF* are triangles between the same parallel lines *XY* and *ST*.

(ii)



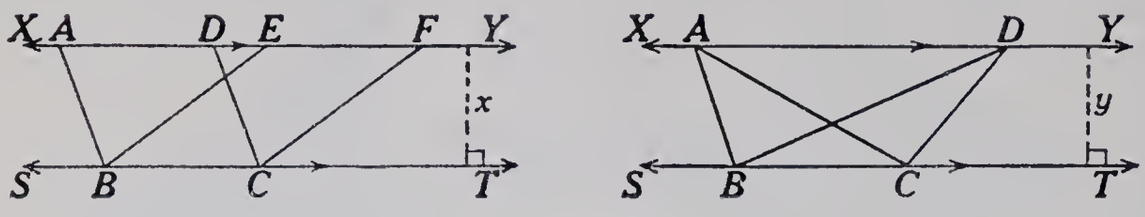
(*a*) (*b*)

Fig. 14-10

In *Fig. 14-10*(*a*), $||^{gms}$ *ABCD* and *EBCF* are on the same base *BC* and between the same parallel lines *XY* and *ST*.

In *Fig. 14-10*(*b*), triangles *ABC* and *DBC* are on the same base *BC* and between the same parallel lines *XY* and *ST*.

(iii) Since the perpendicular distance between two parallel lines is constant, the phrase *between the same parallel lines* also means that the parallelograms $ABCD$ and $EBCF$, *Fig. 14-11* (*a*), have the *same altitude*, x, and the triangles ABC and DBC, *Fig. 14-11* (*b*), have the *same altitude*, y.



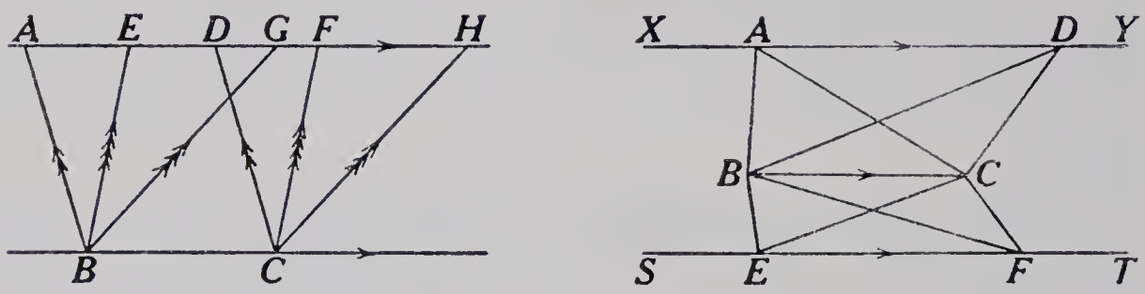
(*a*) (*b*)

Fig. 14-11

Exercise 14-1

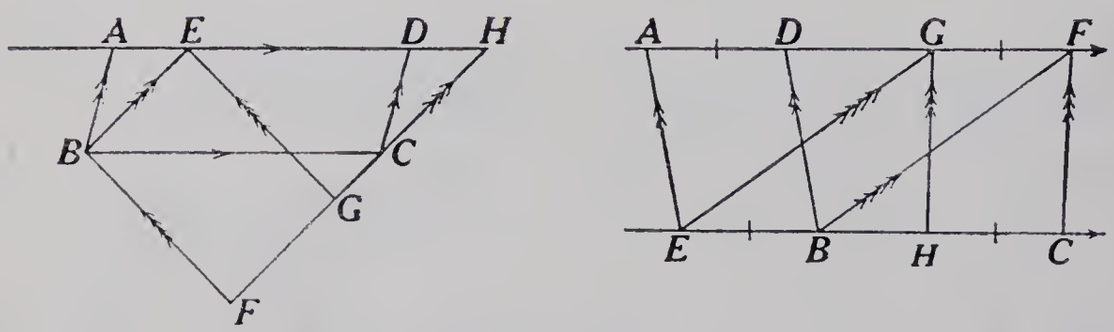
(A)

1. In the diagram at the left below, name three parallelograms on the same base and between the same parallel lines.



2. In the diagram at the right above, name pairs of triangles on the same base and between the same parallel lines.

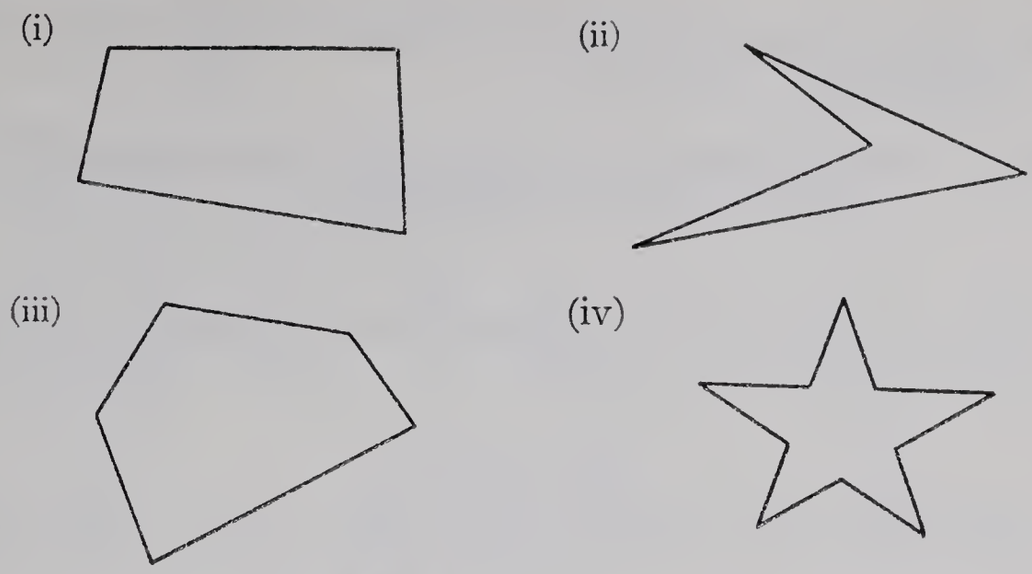
3. In the diagram at the left below, name pairs of parallelograms on the same base and between the same parallel lines.



4. In the diagram at the right above, name pairs of parallelograms on equal bases and between the same parallel lines.

(B)

5. Copy the following polygons and show how each may be separated into triangular regions such that the intersection of any pair of triangular regions is either a point or a line segment.



14·4 **Parallelogram Area Theorem.**

THEOREM 14

Parallelogram Area Theorem

Parallelograms on the same base and between the same parallel lines have equal areas.

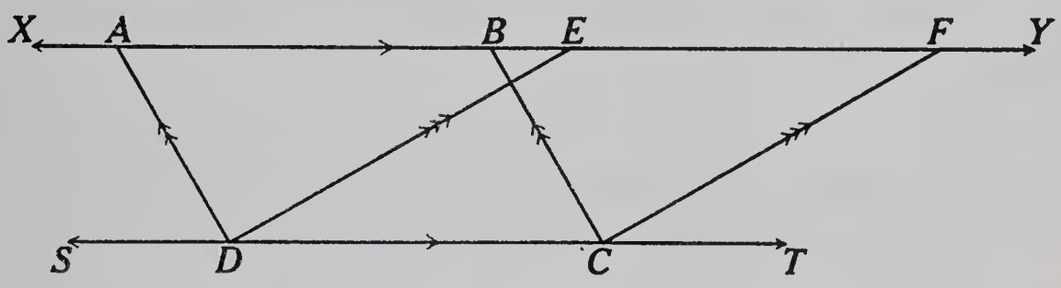


Fig. 14-12

Hypothesis: Parallelograms $ABCD$ and $EFCD$ on the same base DC and between the same parallel lines ST and XY (*Fig. 14-12*)

Conclusion: $||^{gm}\ ABCD = ||^{gm}\ EFCD$

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $||^{gm}\ ABCD = ||^{gm}\ EFCD$	1. $Fig.\ ADCF - \triangle BCF = Fig.\ ADCF - \triangle ADE.$
2. $Fig.\ ADCF - \triangle BCF = Fig.\ ADCF - \triangle ADE$	2. $\triangle BCF = \triangle ADE.$
3. $\triangle BCF = \triangle ADE$	3. $\triangle BCF \cong \triangle ADE.$
4. $\triangle BCF \cong \triangle ADE$	4. $\angle DAE = \angle CBF,$ $\angle AED = \angle BFC,\ AD = BC.$
5. $\angle DAE = \angle CBF$ and $\angle AED = \angle BFC$ by the Parallel Line Th., and $AD = BC$ by the Parallelogram Th.	

Proof: Write a complete proof for this theorem and compare it with that on page 496.

Corollary 1: **Parallelograms on equal bases and between the same parallel lines have equal areas.**

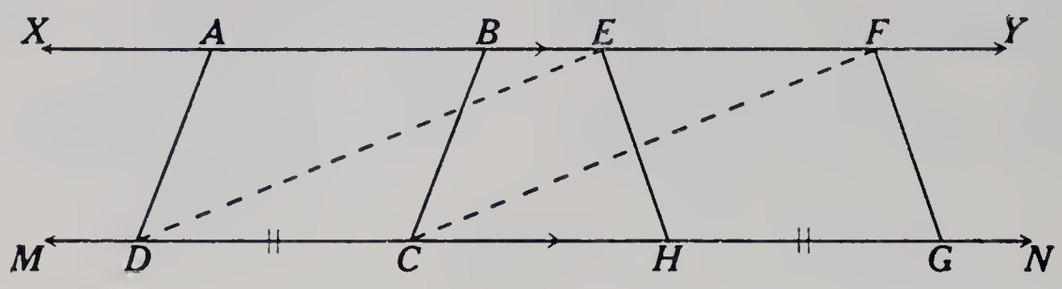


Fig. 14-13

Hypothesis: Parallelograms $ABCD$ and $EFGH$ between the same parallel lines XY and MN, such that $DC = HG$ (*Fig. 14-13*)

Conclusion: $\|^{gm}\ ABCD = \|^{gm}\ EFGH$

Write a proof for this corollary. Compare your solution with that on page 497.

Corollary 2: **A parallelogram and a rectangle on equal bases and between the same parallel lines have equal areas.**

Example.

Hypothesis: $\|^{gms}\ ABCD$ and $BEFC$, such that $AD \| BC \| EF$ (*Fig. 14-14*)

Conclusion: (i) $AEFD$ is a parallelogram.
(ii) $\|^{gm}\ ABCD + \|^{gm}\ BEFC = \|^{gm}\ AEFD$

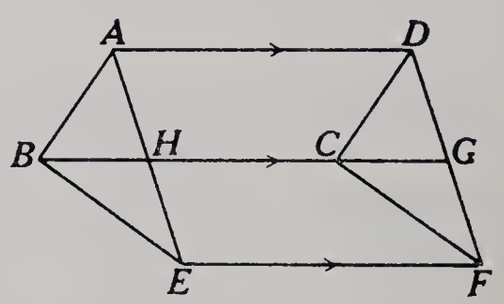


Fig. 14-14

Analysis:

I CAN PROVE	IF I CAN PROVE
(i) 1. $AEFD$ is a parallelogram	1. $AD = EF$ and $AD \| EF$.
2. $AD = EF$ and $AD \| EF$	2. $AD = BC$, $BC = EF$ and $AD \| BC$, $BC \| EF$.
3. $AD = BC$, $BC = EF$ and $AD \| BC$, $BC \| EF$ by the Parallelogram Theorem.	

(ii) 4. $\|\|^{gm} ABCD + \|\|^{gm} BEFC = \|\|^{gm} AEFD$	4. $\|\|^{gm} ABCD = \|\|^{gm} AHGD$, and $\|\|^{gm} BEFC = \|\|^{gm} HEFG$.
5. $\|\|^{gm} ABCD = \|\|^{gm} AHGD$, and $\|\|^{gm} BEFC = \|\|^{gm} HEFG$ by the Parallelogram Area Th.	

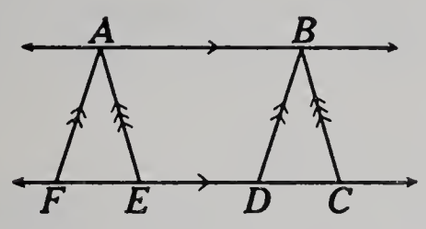
Proof: Write a complete proof and compare your proof for part (ii) with that on page 497.

Exercise 14-2

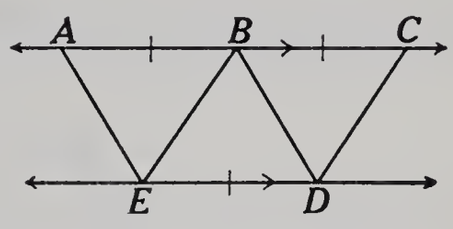
(A)

1. Name pairs of parallelograms which have equal areas in the following diagrams:

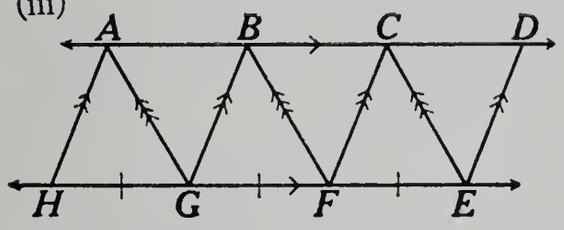
(i)



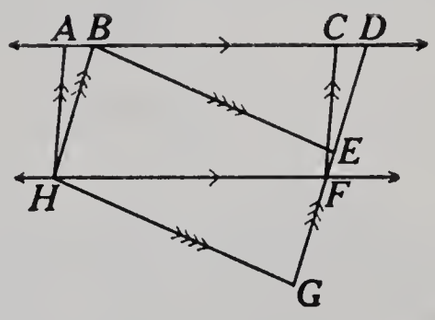
(ii)



(iii)



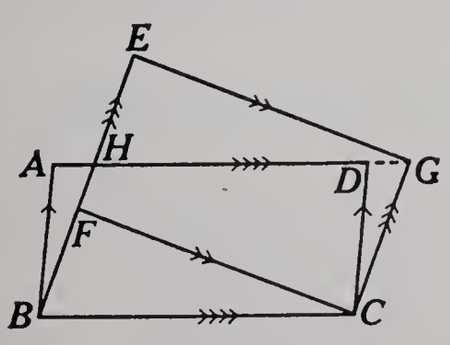
(iv)

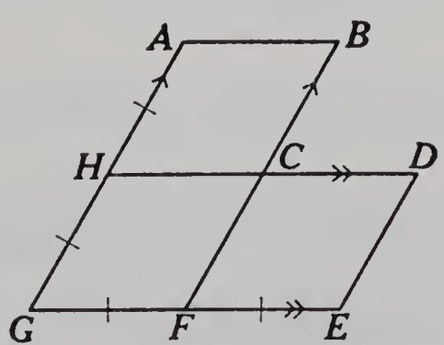


(B)

2. $ABCD$ is a parallelogram. AD is produced to E so that $DE = AD$. Prove that $DECB$ is a parallelogram and is equal to $\|\|^{gm} ABCD$.

3. In the diagram at the left below, $AB \| DC$, $AG \| BC$, $BE \| CG$, $FC \| EG$. Prove that $\|\|^{gm} ABCD = \|\|^{gm} FCGE$.





4. In the diagram at the right above, $AG \| BF$, $HD \| GE$, $AH = HG = GF = FE$. Prove that figure $AHFB$ = figure $HDEF$.

5. Construct a parallelogram equal to a given parallelogram and having a side equal to a given line segment which is longer than any side of the given parallelogram.
6. Construct a parallelogram equal to a given parallelogram and having a diagonal equal to a given line segment which is longer than any side of the given parallelogram.
7. Construct a parallelogram equal to a given parallelogram and having a side equal to a given line segment and an angle equal to a given angle.

14·5 Parallelogram-Diagonal Theorem.

THEOREM 15

Parallelogram-Diagonal Theorem

The diagonal of a parallelogram separates the parallelogram into two triangles that have equal areas.

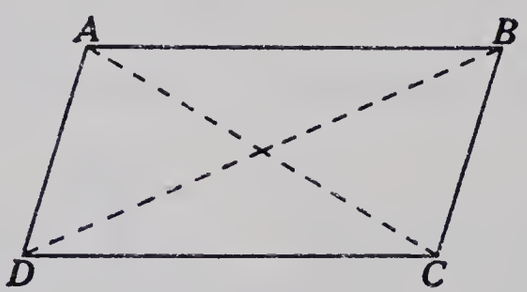


Fig. 14-15

Hypothesis: Parallelogram $ABCD$

Conclusion: (i) $\triangle ABC = \triangle CDA$
(ii) $\triangle ABD = \triangle CDB$

Analysis:

I CAN PROVE	IF I CAN PROVE
(i) 1. $\triangle ABC = \triangle CDA$	1. $\triangle ABC \cong \triangle CDA$.
2. $\triangle ABC \cong \triangle CDA$	2. The conditions for sas, sss, asa, or aas.
3. The conditions for sss	3. $AB = CD$, $BC = DA$, $CA = AC$.
4. $AB = CD$, $BC = DA$ by the Parallelogram Theorem and $CA = AC$ by the reflexive property.	
(ii) 5. $\triangle ABD = \triangle CDB$ in a manner similar to (i).	

Proof: Write a proof for this theorem and compare your proof with that on page 497.

14·6 **Parallelogram-Triangle Area Theorem.**

THEOREM 16

Parallelogram-Triangle Area Theorem

If a triangle and a parallelogram are on equal bases and between the same parallel lines, then the area of the triangle is one-half the area of the parallelogram.

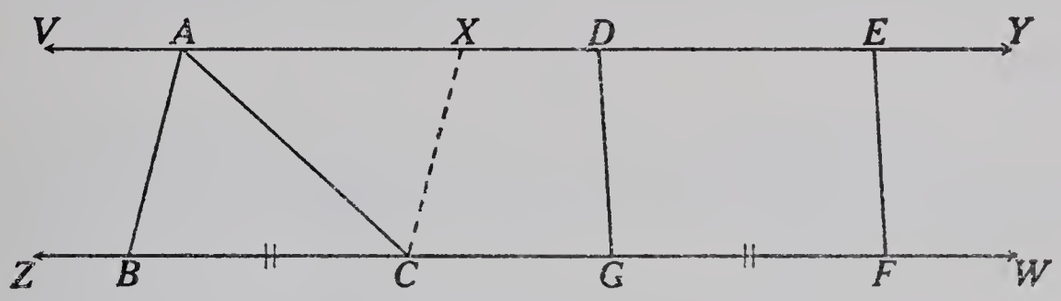


Fig. 14-16

Hypothesis: $\triangle ABC$ and $\|^{gm}$ $DGFE$ between the parallel lines VY and ZW, such that $BC = GF$ (*Fig. 14-16*)

Conclusion: $\triangle ABC = \frac{1}{2}\|^{gm}$ $DGFE$

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\triangle ABC = \frac{1}{2}\|^{gm}$ $DGFE$	1. $2\triangle ABC = \|^{gm}$ $DGFE$.
2. $2\triangle ABC = \|^{gm}$ $DGFE$	2. $2\triangle ABC = \|^{gm} ABCX$ and $\|^{gm} ABCX = \|^{gm}$ $DGFE$.
3. $2\triangle ABC = \|^{gm} ABCX$ by the Parallelogram-Diagonal Theorem, and $\|^{gm} ABCX = \|^{gm}$ $DGFE$ by the Parallelogram Area Theorem.	

Proof: Write a complete proof for this theorem and compare it with that on page 497.

Corollary 1: Triangles on equal bases and between the same parallel lines have equal areas.

Corollary 2: The median of a triangle separates the triangle into two triangles that have equal areas.

Write a proof for corollary 1 and compare it with that on page 498.

Example 1. If in $\triangle ABC$ (*Fig. 14-17*), $BD = DC$ and $AE = ED$, then $\triangle ABE = \frac{1}{4}\triangle ABC$.

Hypothesis: $\triangle ABC$ in which $BD = DC$ and $AE = ED$

Conclusion: $\triangle ABE = \frac{1}{4}\triangle ABC$

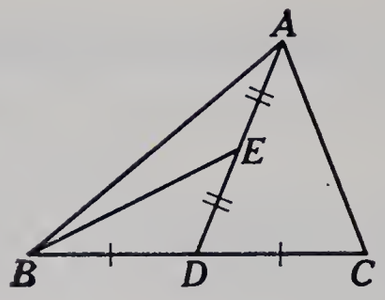


Fig. 14-17

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\triangle ABE = \frac{1}{4}\triangle ABC$	1. $4\triangle ABE = \triangle ABC$.
2. $4\triangle ABE = \triangle ABC$	2. $\triangle ABE = \triangle EBD$ and $2\triangle ABE = \triangle ADC$.
3. $\triangle ABE = \triangle EBD$ by the Parallelogram-Triangle Area Theorem.	
4. $2\triangle ABE = \triangle ADC$	4. $\triangle ABD = \triangle ADC$.
5. $\triangle ABD = \triangle ADC$ by the Parallelogram-Triangle Area Theorem.	

Proof: Write a complete proof for this theorem and compare it with that on page 498.

Example 2.

Hypothesis: $AE \parallel BD$, $BC = CD$ (*Fig. 14-18*)

Conclusion: $\triangle ABD = 2\triangle ECD$

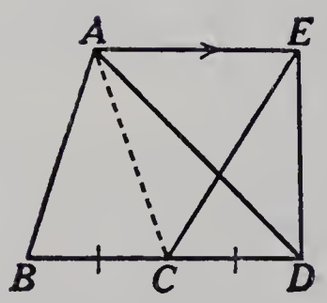


Fig. 14-18

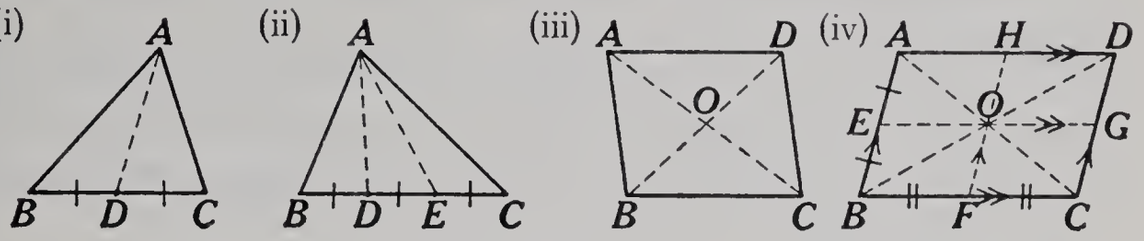
Write an analysis and proof for this deduction and compare your solution with that on page 498.

Exercise 14-3

(A)

1. Into what fractional parts do the dotted lines separate each of the following triangles and parallelograms?

(i) (ii) (iii) (iv)



(B)

2. $ABCD$ is a quadrilateral. A straight line is drawn through A parallel to DB and meets CB extended at E. Prove that $\triangle DEC$ = quadrilateral $ABCD$.

3. The side QR of $\triangle PQR$ is extended to S so that $RS = 2QR$. Prove that $\triangle PQR = \frac{1}{3}\triangle PQS$.

4. Medians BD and CE of $\triangle ABC$ intersect on F. Prove that:

(i) $\triangle DBC = \triangle EBC$ (ii) $\triangle EBF = \triangle DFC$

(iii) quadrilateral $AEFD = \triangle FBC$

5. In $\triangle ABC$, E is on AB and F on AC so that $EF \parallel BC$, BF and EC intersect on X. Prove $\triangle EBX = \triangle FXC$.

6. $ABCD$ is a parallelogram. The diagonal AC is extended to E so that $CE = AC$. Prove that quadrilateral $DCBE = \parallel^{gm} ABCD$.

7. $PQRS$ is a parallelogram whose diagonals intersect on X. PR is extended to Y so that $RY = XR$. Prove that $\triangle SQY = \parallel^{gm} PQRS$.

8. $ABCD$ is a parallelogram and P is a point on the diagonal AC. Prove that $\triangle PAB = \triangle PAD$.

9. The medians AD, BE of $\triangle ABC$ intersect on G. Prove that the three line segments AG, BG, CG trisect $\triangle ABC$.

(C)

10. Construct an isosceles triangle equal to a given triangle.

11. Construct an isosceles triangle equal to one-half a given triangle.

12. Construct a triangle equal to a given triangle and having an angle equal to a given angle.

13. $PQRS$ is a quadrilateral whose diagonals intersect on X. PR is extended to Y so that $RY = PX$. Prove that $\triangle SQY$ = quadrilateral $PQRS$.

14. D is any point on the side AB of $\triangle ABC$. DF is drawn parallel and equal to BC. If DF intersects AC on E, prove that $\triangle AEF = \triangle DBE$.

15. The side AB of $\|^{gm}$ $ABCD$ is extended to E. DE cuts BC on F. Prove that $\triangle AFE = \triangle CBE$.

14·7 **Measurement of area.** When we measure a line segment, we determine a number in terms of some unit which is called its *length.* This is done by choosing a unit length and counting the number of unit lengths in the line segment. This method is also followed in associating a real number with each angle. Similarly, by choosing a unit region, we can assign a positive real number to each region called the *area of the region.*

AREA POSTULATE 4.

For every polygonal region there exists a unique positive real number of units called the area of the region.

In order to find this real number associated with a polygonal region, it is necessary to agree on a unit area.

Since length is one-dimensional, deciding on a unit length is not difficult; the length of any line segment can be chosen as the unit. Because area is two-dimensional, any polygonal or curved region can be chosen to provide a unit area (*Fig. 14-19*).

Fig. 14-19

To find the area of a polygonal region, it is necessary to separate the region into non-overlapping regions which are congruent to, or known fractions of, the region defined as the unit area. This is most easily performed if the unit region chosen is a square region.

The unit area is defined to be the area of a square region whose sides are each the unit length. The area of this region is called *one square unit.*

It is now possible to find the area of a rectangular region as follows:

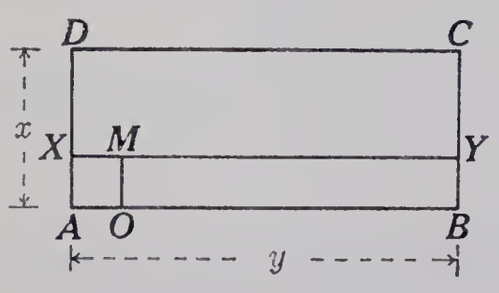


Fig. 14-20

If a point, X, is taken on side AD of rectangle $ABCD$ (*Fig. 14-20*) so that AX is the unit length, then rectangle $AXYB$ will contain as many unit areas as there are unit lengths in side AB. If the side AB has a length of y units ($y \in {}^{+}R$), then the rectangle $AXYB$ has an area of y square units.

Thus, if the length of AD is x units ($x \in {}^+R$), then

area rectangle $ABCD$ = x (area rectangle $AXYB$)
= xy square units.

DEFINITION: *The area of a rectangular region whose sides have lengths x units and y units is xy square units.*

Example. Rectangle $QRST$ (*Fig. 14-21*)
= (3.5×4.5) sq. cm.
= 15.75 sq. cm.

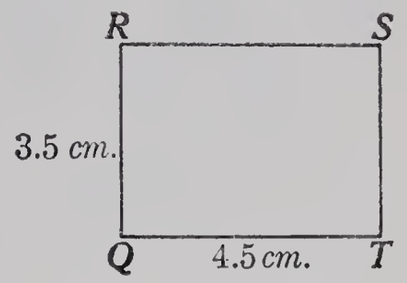


Fig. 14-21

***14·8* Area of a parallelogram.** If, in *Fig. 14-22*, DC and YC are b and h units, respectively, then

$||^{gm}$ $ADCB$ = rectangle $XDCY$ (Parallelogram Area Theorem)
= bh sq. units.

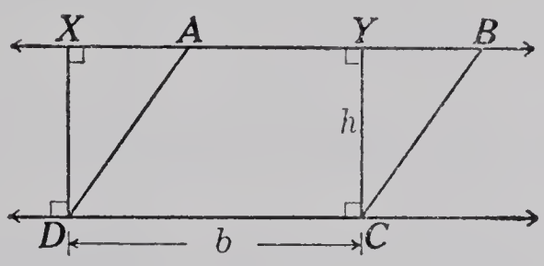


Fig. 14-22

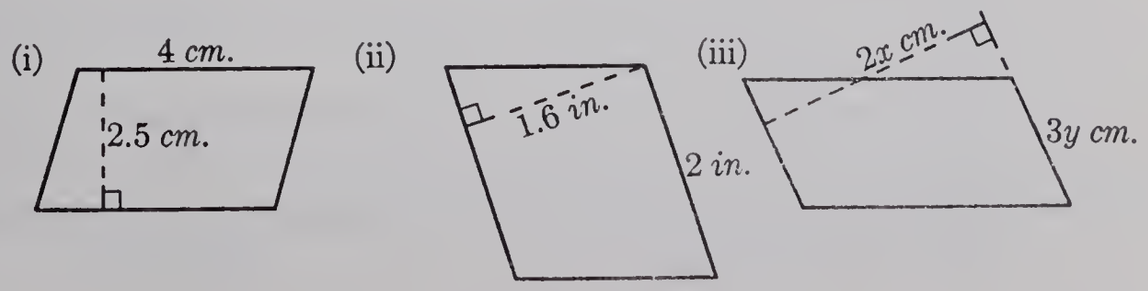
The area of a parallelogram with a side 6.5 in. long and an altitude to that side 2 in. long is (6.5×2) sq. in. = 13 sq. in.

For convenience, a positive real number associated with a line segment in a diagram represents the number of units of length in the line segment.

Exercise 14-4

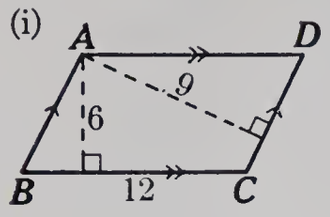
(A)

1. State the area of each of the following parallelograms:



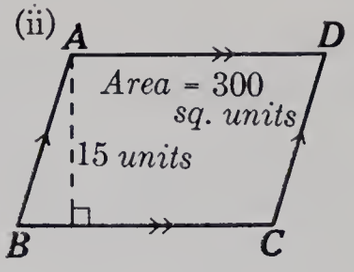
2. In each of the following diagrams, state the length of the line segment indicated:

(i)



Find CD.

(ii)



Find BC.

(B)

3. A lot has the shape of a parallelogram. The longer sides are each 120 feet, and the distance between them is 100 feet. Find the area of the lot.

4. Find the area of a rhombus with one side of length 60 cm., and the distance to the opposite side 40 cm.

5. $ABCD$ is a parallelogram, and AD is extended to E so that $DE = AD$. Prove that $DECB$ is a parallelogram and is equal to $\|^{gm}\ ABCD$.

6. Two parallelograms having equal areas lie between the same parallel lines which are h units apart. Show algebraically that their bases are equal.

7. In $\|^{gm}\ ABCD$, $AB = 10$ cm., AB and DC are 8 cm. apart. X and Y are the midpoints of AB and DC, respectively. XY is extended to E so that $YE = XY$. BC is extended to F so that $CF = BC$. Find the area of figure $XBFE$.

8. The dimensions of a rectangular metal box are 20″, 10″, and 5″. Calculate the area of the metal used in making the box.

9. For a rectangle whose base is m inches long and whose altitude is n inches long:

(i) Calculate the area.

(ii) What is the effect on the area of doubling the length of the base and leaving the altitude unchanged?

(iii) What is the effect on the area of tripling the length of the altitude and leaving the base unchanged?

(iv) What is the effect on the area of doubling the length of both the base and the altitude?

14·9 **Area of a triangle.**

If, in *Fig. 14-23*, DC and EM are b and h units, respectively, then

$\triangle EDC = \frac{1}{2} \|^{gm} ADCB$

$= \frac{1}{2} bh$ square units.

The area of a triangle with a side 6.5 inches long and an altitude to that side 2 inches long is $\frac{1}{2}(6.5 \times 2)$ square inches $= 6.5$ square inches.

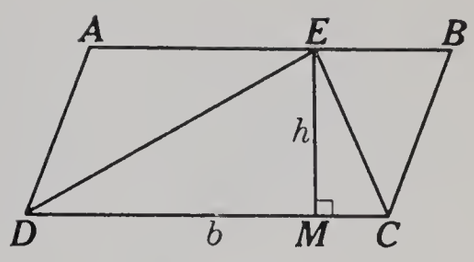


Fig. 14-23

14·10 **Equal Triangle-Parallel Line Theorem.**

THEOREM 17

Equal Triangle-Parallel Line Theorem

If two triangles having equal areas are on the same side of a common base, the straight line on their vertices is parallel to the common base.

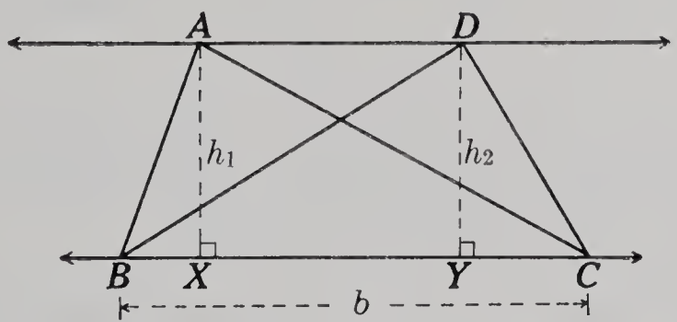


Fig. 14-24

Hypothesis: $\triangle ABC = \triangle DBC$, $BC = b$ units, $AX = h_1$ units, $DY = h_2$ units (*Fig. 14-24*)

Conclusion: $AD \| BC$

Analysis:

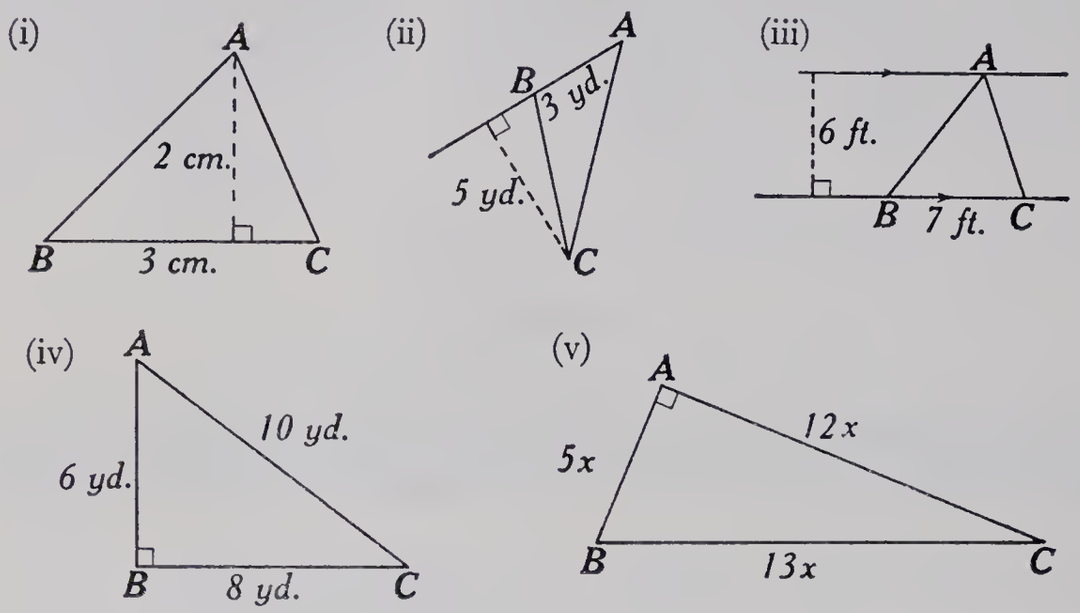
I CAN PROVE	IF I CAN PROVE
1. $AD \| BC$	1. $AX = DY$ $(AX \perp BC, DY \perp BC)$.
2. $AX = DY$	2. $h_1 = h_2$.
3. $h_1 = h_2$	3. $\frac{1}{2}bh_1 = \frac{1}{2}bh_2$.
4. $\frac{1}{2}bh_1 = \frac{1}{2}bh_2$ by hypothesis.	

Proof: Write a complete proof for this theorem and compare it with that on page 499.

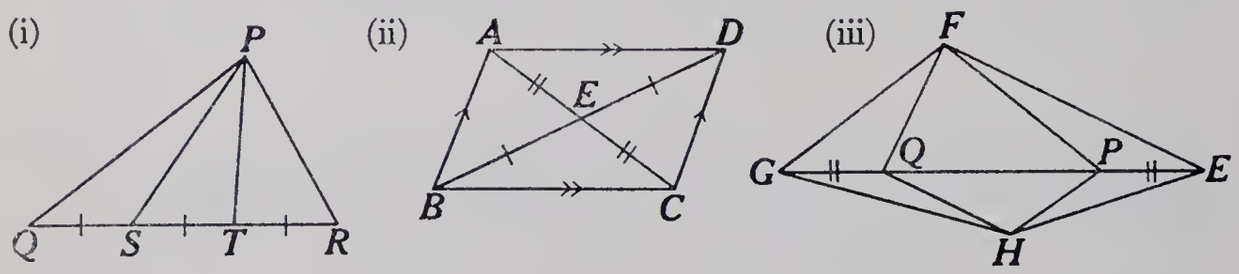
Exercise 14-5

(A)

1. State the area of $\triangle ABC$ for each of the following diagrams:



2. For each of the following diagrams, name triangles which have the same or equal bases and the same or equal altitudes and, hence, are equal in area. Give a reason in each case.



3. What is the effect on the area of a triangle if:

(i) the base is doubled and the altitude is constant;

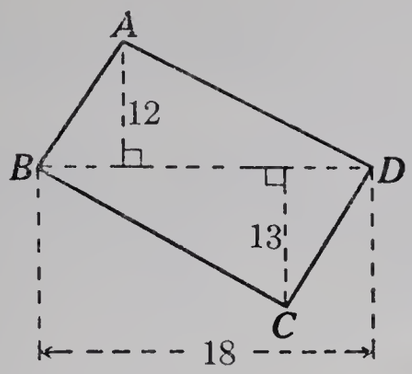
(ii) the altitude is trebled and the base is constant;

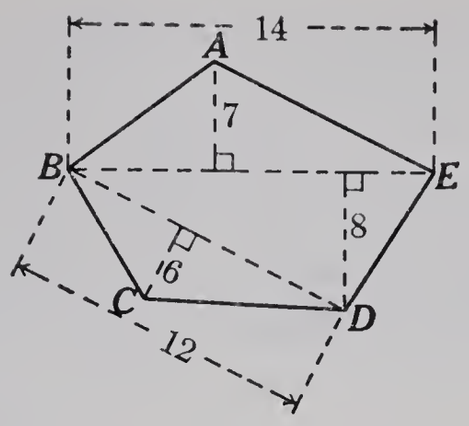
(iii) the base is doubled and the altitude is trebled?

(B)

4. Find the area of a triangle whose base has length 16 inches and whose altitude has length 12 inches.

5. Use the dimensions on the diagram (at the top left page 409) to find the area of quadrilateral $ABCD$.





6. Use the dimensions on the diagram, right above, to find the area of polygon $ABCDE$.

7. Construct a triangle with sides of length 6.1 cm., 7.9 cm., and 10.2 cm. Using the longest side as base, construct the corresponding altitude, measure it, and calculate the area correct to the nearest square centimetre.

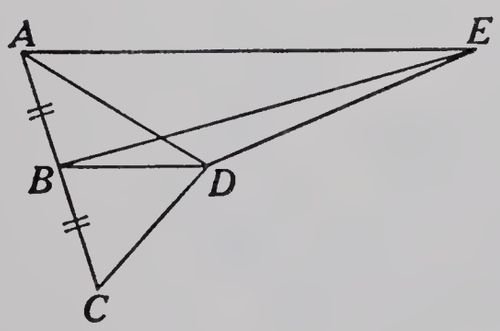
8. In quadrilateral $ABCD$, $\triangle ABC = \triangle DBC$. Prove that the quadrilateral is a trapezoid.

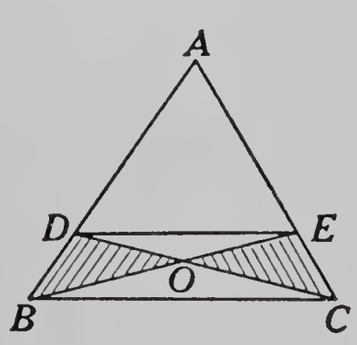
9. AOB and COD are two intersecting line segments. If $\triangle AOD = \triangle COB$, prove $AC \parallel DB$.

10. Prove the converse of the Parallelogram-Diagonal Theorem.

11. $ABCD$ is a line segment in which $AB = CD$. $\triangle$'s QAB and RCD with equal areas are on the same side of AD. Prove $QR \parallel AD$.

12. In the diagram at the left below, $AB = BC$, $\triangle EBD = \triangle BCD$. Prove: (i) $AE \parallel BD$ (ii) ray BD bisects CE





13. In the diagram at the right above, $AB = AC$ and $\triangle DBO = \triangle EOC$. Prove that the distance from D to AC is the same as that from E to AB.

14. Prove that the straight line on the midpoints of two sides of a triangle is parallel to the third side.

***14·11* The Pythagorean Theorem.** The Pythagorean Theorem states:

In a right triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.

Thus, in *Fig. 14-25*, square $ABYX$ = square $ACNM$ + square $CBSR$. If the length of AB is c units, AC is b units, BC is a units, then square $ABYX = c^2$ square units, square $ACNM = b^2$ square units, square $CBSR = a^2$ square units, and $c^2 = b^2 + a^2$.

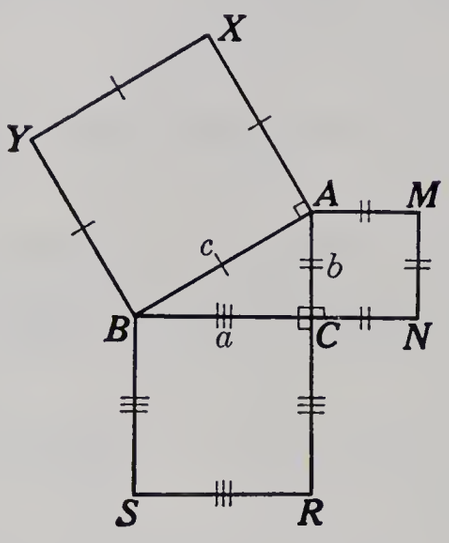


Fig. 14-25

Many ingenious models have been devised to demonstrate the Pythagorean Theorem; one of these is illustrated in *Fig. 14-26*. The square $ABKH$ is cut along AB, PQ, and RS to form parts 1, 2, 3, 4. These four parts, along with 5, can be fitted on square $BDEC$, as shown, to illustrate that:

Square $BDEC$ = Square $ACFG$ + Square $ABKH$

$$a^2 = b^2 + c^2$$

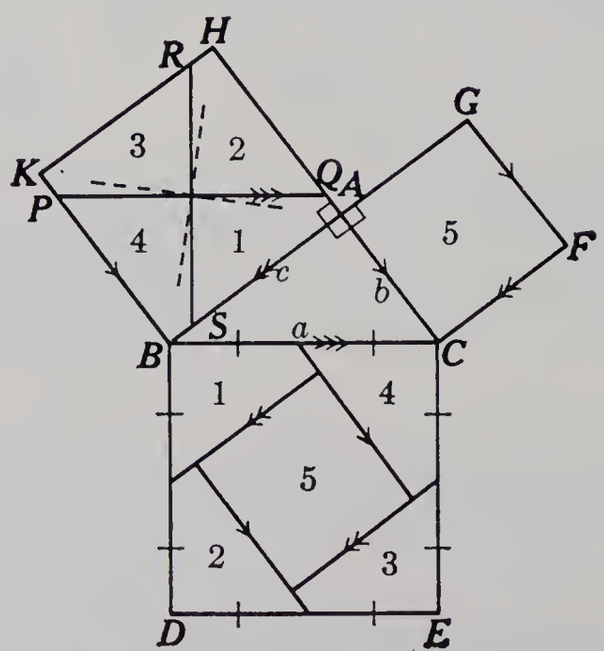


Fig. 14-26

Of course, this does not constitute a deductive proof of the theorem. The Greek mathematician, Pythagoras, is believed to have given the first deductive proof. The story is told that he was so elated over his discovery that he sacrificed an ox to the muses for his inspiration.

According to tradition, Pythagoras was the first European to insist that, in developing geometry, postulates must be set down first, so that the development thereafter would proceed by the application of deductive reasoning directly to the postulates. This introduction of proof into mathematics is considered to be Pythagoras' greatest contribution to learning.

Pythagoras was born at Samos about 580 B.C. It is believed that as a young student he visited many of the Mediterranean countries including Egypt, Gaul, Arabia, Persia, Palestine, Phoenicia, and India. Having absorbed much of the mathematical knowledge of his day, he eventually settled at Crotona, a city in southern Italy, about 529 B.C. There he established a new school which was similar to a secret society, or brotherhood, in which members were united for life. He gathered around him three hundred promising young men, most of whom came from prosperous families. Acting in the capacity of a moral reformer, Pythagoras believed that pure thinking, mathematics, philosophy, and natural science provided the best way to purify the soul. However, the religious mysticism, the secret observances, and the aristocratic tendencies of his society caused it to become an object of suspicion. A democratic group of lower Italy destroyed the school. Pythagoras fled to Metapontum where, at the age of almost eighty, he died.

Because of the secret nature of the Pythagorean Society, no record of Pythagoras' proof was ever discovered. Over the centuries many different deductive proofs have been given for this famous theorem.

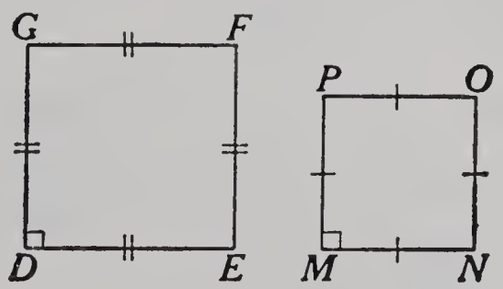


Fig. 14-27

A square is often named by referring to its side. Thus, the symbol AB^2 means the square with AB as one of its sides and is read, "the square on AB". In *Fig. 14-27*, the square $DEFG$ may be referred to as DE^2 (the square on DE) or EF^2 or FG^2 or DG^2; the square $MNOP$ as MN^2 or NO^2 or OP^2 or MP^2.

THEOREM 18

The Pythagorean Theorem

In a right triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.

a.

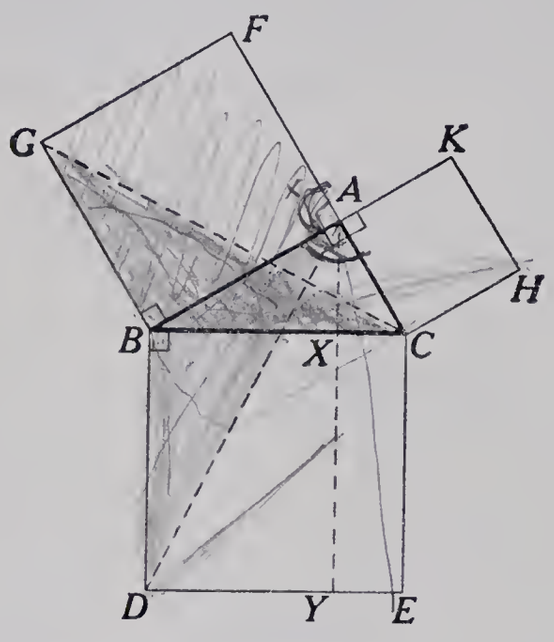


Fig. 14-28

Hypothesis: $\triangle ABC$ in which $\angle BAC$ is a right angle, $BDEC$, $AFGB$, and $ACHK$ are the squares on BC, AB, and AC, respectively.

Conclusion: $BC^2 = AB^2 + AC^2$

Proof:

STATEMENTS	AUTHORITIES
1. AXY is a line segment intersecting BC on X and DE on Y such that $AY \parallel BD$.	1. Existence
2. $\angle GBA = \angle CBD = 90°$	2. Definition
3. $\angle GBA + \angle ABC = \angle CBD + \angle ABC$	3. Addition
4. $\angle GBC = \angle ABD$	4. Completion
In $\triangle$'s GBC and ABD	
5. $GB = AB$	5. Definition
6. $BC = BD$	6. Definition
7. $\angle GBC = \angle ABD$	7. 4
8. $\triangle GBC \cong \triangle ABD$	8. sas
9. $\angle FAB = \angle BAC = 90°$	9. Definition, hypothesis
10. Square $AFGB = 2\ \triangle GBC$	10. Parallelogram-Triangle Area Theorem

11. Rectangle $BDYX = 2\triangle ABD$	11. Parallelogram-Triangle Area Theorem
12. $\triangle ABD = \triangle GBC$	12. Area postulate
13. Square $AFGB$ = rectangle $BDYX$	13. Replacement
14. Square $ACHK$ = rectangle $XYEC$	14. Similar to 2 to 13
15. Rectangle $BDYX$ + rectangle $XYEC$ = square $AFGB$ + square $ACHK$	15. Addition
16. $BC^2 = AB^2 + AC^2$	16. Completion

b. *Alternate proof of the Pythagorean Theorem.*

Hypothesis: Square $ADFH$ with side of length $(b + c)$ units; right $\triangle$'s ABC, DEB, FGE, HCG with sides about the right angles of lengths b units and c units, BC of length a units

Conclusion: $a^2 = b^2 + c^2$

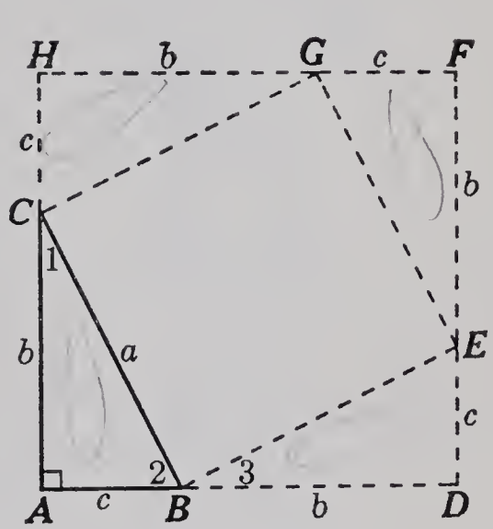


Fig. 14-29

Proof:

STATEMENTS	AUTHORITIES
1. $\triangle ABC \cong \triangle DEB \cong \triangle FGE \cong \triangle HCG$	1. sas
2. $BC = EB = GE = CG = a$	2. Definition
3. $\angle 1 + \angle 2 = 90°$	3. Triangle Angle Sum Theorem
4. $\angle 1 = \angle 3$	4. Definition
5. $\angle 3 + \angle 2 = 90°$	5. Replacement
6. $\angle CBE = 90°$	6. Completion, definition
7. $BEGC$ is a square.	7. Definition
8. $ADFH = BEGC + 4\triangle ABC$	8. Completion
9. $(b + c)^2 = a^2 + 4(\frac{1}{2}bc)$	9. Area of triangle
10. $b^2 + 2bc + c^2 = a^2 + 2bc$	10. Number axioms
11. $b^2 + c^2 = a^2$	11. Subtraction

Example 1. In *Fig. 14-30*, calculate the length of BC, if $b = 13$ in., $c = 5$ in.

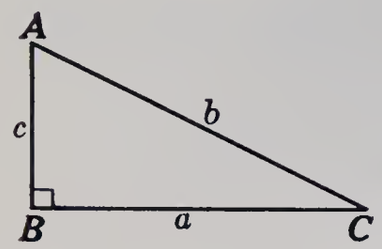


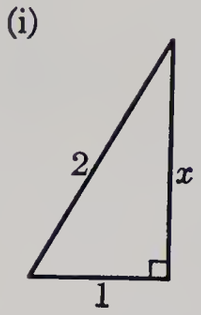
Fig. 14-30

Solution.

$$b^2 = a^2 + c^2$$
$$\therefore 13^2 = a^2 + 5^2$$
$$\therefore 169 = a^2 + 25$$
$$\therefore a^2 = 169 - 25$$
$$= 144$$
$$\therefore a = 12$$

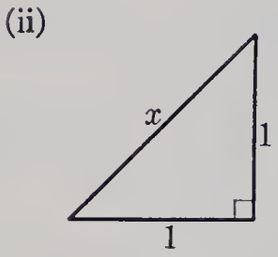
The length of BC is 12 in.

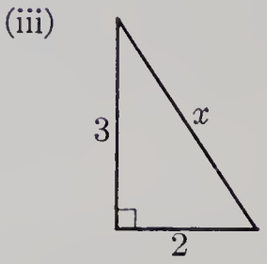
Example 2. In each of the following find x, the number of units in the length of the side indicated. The solution for (i) is given; compare your solutions to the others with those on page 499.



Solution. (i)

$$2^2 = x^2 + 1^2$$
$$\therefore 4 = x^2 + 1$$
$$\therefore x^2 = 3$$
$$\therefore x = \sqrt{3}$$





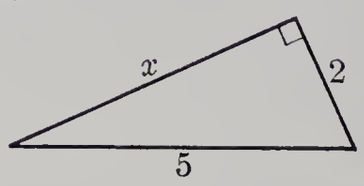


Fig. 14-31

Example 3. A rectangular lot, *Fig. 14-32*, has dimensions 100 ft. and 40 ft.

(i) Calculate, to the nearest foot, the length of a "short-cut" along a diagonal path.

(ii) Find by how much this path is shorter than the normal route around the corner.

Solution.

$$x^2 = 100^2 + 40^2$$
$$= 10{,}000 + 1{,}600$$
$$= 11{,}600$$
$$\therefore x = \sqrt{11{,}600}$$
$$\doteqdot 107.7$$

$\therefore$ the length of the "short-cut" is 108 feet; shorter by 32 feet.

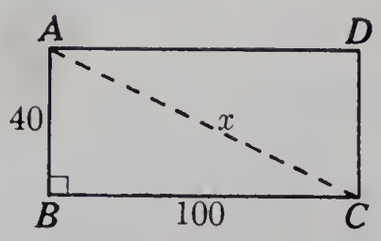


Fig. 14-32

Example 4. A building 36.0 feet wide has a gable roof with a rise of 10.0 feet, *Fig. 14-33*. Find the rafter length correct to the nearest tenth of a foot.

Solution.
$$x^2 = (18.0)^2 + (10.0)^2$$
$$= 324.0 + 100.0$$
$$= 424.0$$
$$\therefore x = \sqrt{424.0}$$
$$\doteqdot 20.59$$

$\therefore$ the rafter length is 20.6 feet.

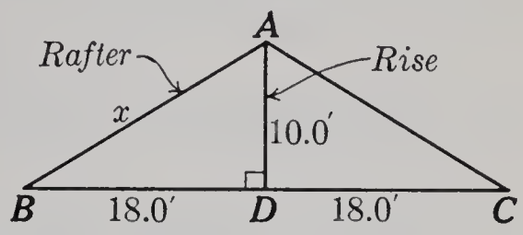


Fig. 14-33

Example 5. If, in *Fig. 14-34*, $PR \perp QS$, then $a^2 + b^2 = c^2 + d^2$.

Hypothesis: Quadrilateral $PQRS$ in which $PR \perp QS$

Conclusion: $a^2 + b^2 = c^2 + d^2$

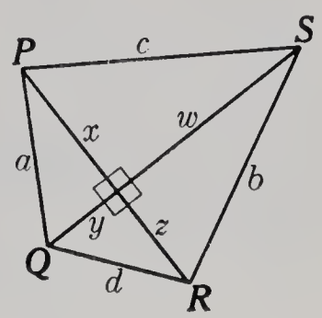


Fig. 14-34

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $a^2 + b^2 = c^2 + d^2$	1. $x^2 + y^2 + z^2 + w^2 = x^2 + w^2 + y^2 + z^2$.
2. $x^2 + y^2 + z^2 + w^2 = x^2 + w^2 + y^2 + z^2$ by the number axioms.	

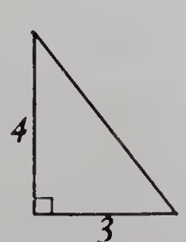
Proof: Write a complete proof; compare your proof with that on page 499.

Exercise 14-6

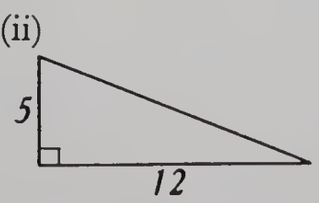
(A)

1. In each of the following, state the length of the hypotenuse:

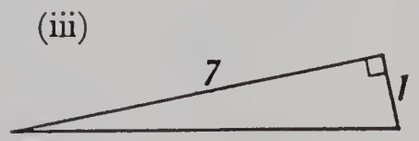
(i)



(ii)

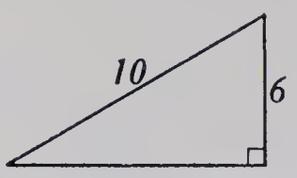


(iii)

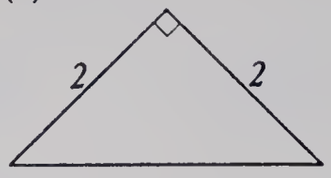


2. Find the length of the third side in each of the following:

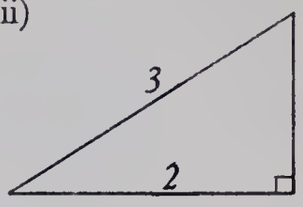
(i)



(ii)



(iii)



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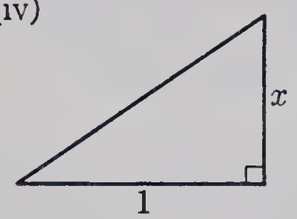
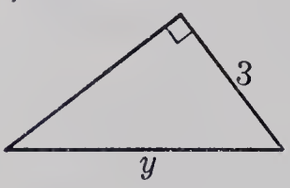


(v)



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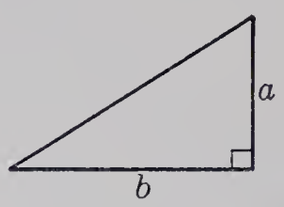
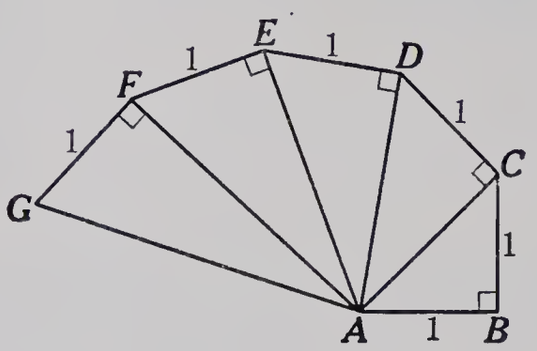


3. In the diagram below, $AB = BC = CD = DE = EF = FG = 1$ unit. State the lengths of AC, AD, AE, AF, AG.

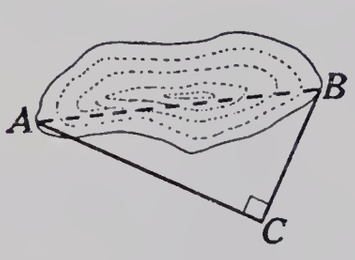


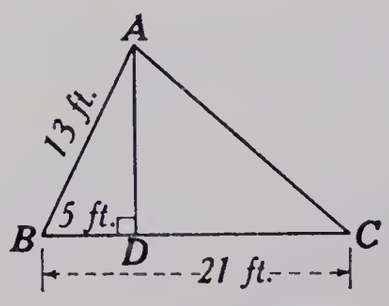
(B)

4. A ladder 25 feet long rests against a wall at a point 20 feet above the ground. How far is the foot of the ladder from the wall?

5. The diagonal of a rectangular field is 65 rods and the width is 25 rods. Find the length of the field.

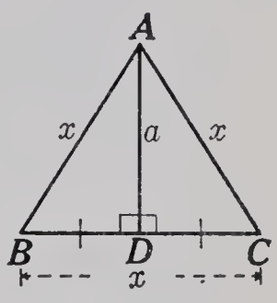
6. To find the length of the lake in the diagram at left below, stakes A and B were placed at each end of the lake. A third stake C was placed so that $\angle ACB = 90°$. $AC = 20$ rods, $BC = 21$ rods. Find the length of AB.





7. For the diagram, right above, calculate the length of AC.

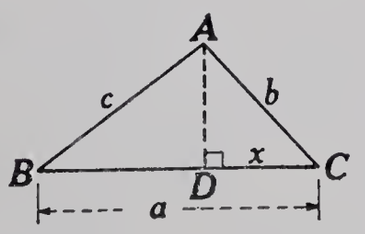
8. Two automobiles leave a highway intersection at the same time. One travels north at 36 miles per hour and the other travels west at 48 miles per hour. How far apart are they after $1\frac{1}{4}$ hours?
9. In $\triangle ABC$, $AB = 52$ in., $AC = 25$ in., and the altitude $AD = 20$ in. Find the length of BC.
10. The diagonals of a rhombus have lengths 48 cm. and 14 cm. Find the perimeter of the rhombus.
11. In an isosceles right triangle, prove that the square on the hypotenuse is double the square on either of the equal sides.
12. Construct a square whose area is double the area of a given square.
13. Draw a right triangle which has one of its sides $\sqrt{10}$ inches in length.
14. Draw a right triangle which has one of its sides $\sqrt{15}$ units in length.
15. For the diagram at the right, prove that $a^2 = \frac{3}{4}x^2$.



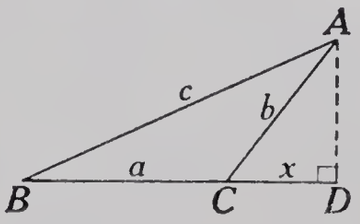
(C)

For each of the following, find x in terms of a, b, and c:

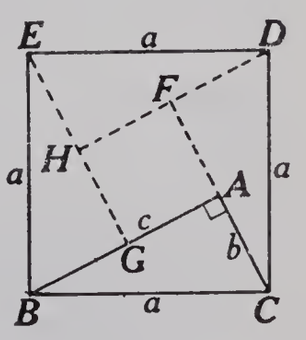
16.



17.



18. Construct a square whose area is three times the area of a given square.
19. Construct a square whose area is half the area of a given square.
20. In the diagram at the right, ABC is a right triangle with hypotenuse BC. $EBCD$ is a square. DF is parallel to AB, meeting CA extended on F. EG is parallel to AC meeting BA on G and DF extended on H. The lengths of BC, CA, AB are a, b, c units, respectively. Prove that $HGAF$ is a square of area $(c - b)^2$ square units, and hence obtain a proof of the Pythagorean Theorem.



21. XY is a given line segment. Find a point P on XY such that the sum of the areas of the squares on XP and PY is equal to the area of the square on a second given line segment.

22. Divide a given line segment into two parts such that the area of the square on one part is equal to twice the area of the square on the other part.

23. A room has a length of 24 feet, a width of 10 feet, and a height of 9 feet.

(i) Draw a diagram and calculate the length of a diagonal of the floor, correct to one decimal place.

(ii) Using the result of (i), calculate the length of a diagonal of the room, correct to one decimal place.

24. A flagpole AB is supported by two guy wires. One wire is fastened to the ground at a point C, 6 feet due south of B, the base of the flagpole. The other wire is fastened to the ground at a point D, 8 feet due west of C. The length of guy wire AD is 26 feet. Draw a diagram and calculate: (i) BD (ii) AB (iii) AC, to the nearest tenth of a foot.

14·12 Converse of the Pythagorean Theorem.

THEOREM 19

Pythagorean Converse

If the area of the square on one side of a triangle is equal to the sum of the areas of the squares on the other two sides, then the angle contained by these sides is a right angle.

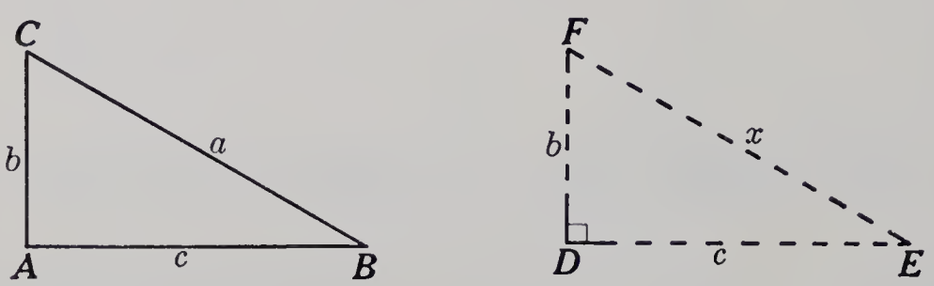


Fig. 14-35

Hypothesis: $\triangle ABC$ with lengths of sides a, b, and c units such that $a^2 = b^2 + c^2$

Conclusion: $\angle A = 90°$

Proof:

STATEMENTS	AUTHORITIES
1. DEF is a triangle such that $\angle D = 90°$ and the lengths of DF, DE, EF are b, c, x units respectively.	1. Existence postulate
2. $x^2 = b^2 + c^2$	2. Pythagorean Theorem

3. $a^2 = b^2 + c^2$	3. Hypothesis
4. $x^2 = a^2$	4. Replacement
5. $x = a$	5. Definition
6. $\triangle ABC \cong \triangle DEF$	6. sss
7. $\angle A = \angle D = 90°$	7. Definition

Example 1. Determine which of the following sets of numbers could be the number of units in the lengths of the sides of a right triangle. The solution of (i) is given; compare your solutions for the others with those on page 499.

(i) 6, 8, 10 (ii) 7, 9, 3 (iii) 1.5, 2, 2.5
(iv) 7, 24, 25 (v) 42, 40, 58 (vi) 2.5, 6, 6.5

Solution. (i) $6^2 = 36$, $8^2 = 64$, $10^2 = 100$
$\because\ 10^2 = 6^2 + 8^2$
$\therefore$ 6, 8, 10 units could be lengths of sides of a right triangle.

Example 2. If in $\triangle PQR$, *Fig. 14-36*, the number of units in the lengths of the sides are $2mn$, $m^2 + n^2$, $m^2 - n^2$, $(m, n \in {}^+I, m > n)$ as indicated, then $\angle Q = 90°$.

Hypothesis: $\triangle PQR$ with sides of lengths $2mn$, $m^2 + n^2$, $m^2 - n^2$ $(m, n \in {}^+I, m > n)$ units

Conclusion: $\angle Q = 90°$

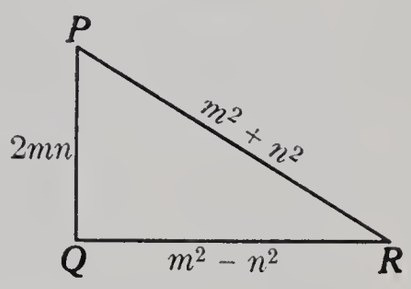


Fig. 14-36

Proof:

STATEMENTS	AUTHORITIES
1. $(m^2 + n^2)^2 = m^4 + 2m^2n^2 + n^4$	1. Number axioms
2. $(2mn)^2 = 4m^2n^2$	2. Definition
3. $(m^2 - n^2)^2 = m^4 - 2m^2n^2 + n^4$	3. Number axioms
4. $(2mn)^2 + (m^2 - n^2)^2 = m^4 + 2m^2n^2 + n^4$	4. Addition
5. $(m^2+n^2)^2=(2mn)^2 + (m^2-n^2)^2$	5. Replacement
6. $\angle Q = 90°$	6. Pythagorean Converse

Sets of positive integers like $\{3, 4, 5\}$, $\{6, 8, 10\}$, $\{5, 12, 13\}$ which satisfy the Pythagorean relation, $a^2 = b^2 + c^2$, are called *Pythagorean triples.*

From Example 2 above, it may be seen that Pythagorean triples may be determined by substituting particular positive integers for m and n. A few Pythagorean triples obtained in this way are listed in the table below.

m	n	PYTHAGOREAN TRIPLES		
		$m^2 - n^2$	$2mn$	$m^2 + n^2$
2	1	3	4	5
3	1	8	6	10
4	1	15	8	17
5	1	24	10	26
3	2	5	12	13
4	2	12	16	20
5	2	21	20	29
4	3	7	24	25

Exercise 14-7

(A)

The following sets of numbers represent the number of units in the lengths of the sides of a triangle. State whether or not the triangle is a right triangle; justify your answer:

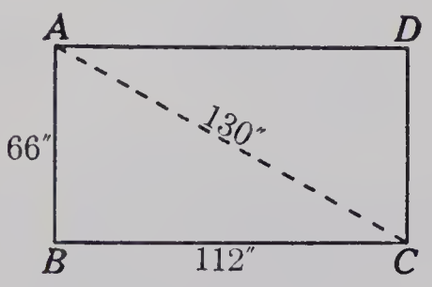
1. 6, 8, 10 **2.** 30, 40, 50 **3.** 2, 2, 3

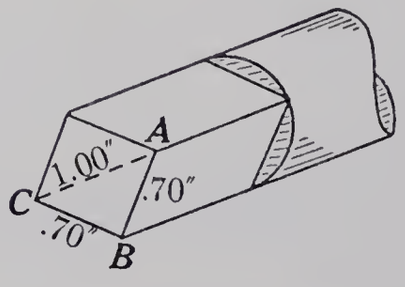
4. 2, 2, $\sqrt{8}$ **5.** 2, 3, $\sqrt{12}$ **6.** 1, 2, $\sqrt{5}$

7. 1, 3, $\sqrt{10}$ **8.** 5, 6, $\sqrt{60}$ **9.** 5, 12, 13

(B)

10. A concrete abutment for a bridge requires a base, $ABCD$, 66 in. by 112 in. (diagram at left below). The length of the diagonal of the base is 130 in. Show that $\angle B = 90°$.





11. A cylindrical shaft 1.00 in. in diameter, diagram right above, is to be milled so that the end is square. If $BC = .70$ in., $AB = .70$ in., and $AC = 1.00$ in., is $\angle ABC$ equal to, greater than, or less than 90°?

The following sets of numbers represent the number of units in the measurements of the sides of a triangle. In each case determine if the triangle is a right triangle; justify your answers:

12. 11, 60, 61 **13.** 1, 1, $\sqrt{2}$ **14.** 21, 20, 28

15. 3, 5, $\sqrt{34}$ **16.** 12, 13, 18 **17.** 2.1, 1.5, 2.5

18. In parallelogram $ABCD$, $BD = 48$ cm., $AC = 14$ cm., and $BC = 25$ cm. Prove that $ABCD$ is a rhombus.

19. In $\triangle PQR$, D is on QR so that $PD \perp QR$. $QD = 16$ units, $PD = 12$ units, $DR = 5$ units. Is $\angle QPR$ acute, obtuse, or a right angle?

14·13 Area of a trapezoid (supplementary).

DEFINITION: *A trapezoid is a quadrilateral with one pair of opposite sides parallel.*

THEOREM 20

Trapezoid Area Theorem

If the lengths of the parallel sides of a trapezoid are a units and b units and the perpendicular distance between these sides is h units, then the area of the trapezoid is $\frac{1}{2}(a + b)h$ square units.

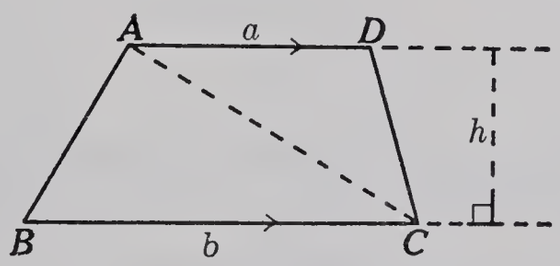


Fig. 14-37

Hypothesis: Trapezoid $ABCD$ having $AD \parallel BC$; the lengths of AD and BC, a, b units, respectively; the perpendicular distance between AD and BC, h units.

Conclusion: Area trapezoid $ABCD = \frac{1}{2}(a + b)\,h$ square units

Proof:

STATEMENTS	AUTHORITIES
1. Trapezoid $ABCD = \triangle ABC + \triangle ADC$	1. Area postulate
2. $\triangle ABC = \frac{1}{2}bh$ sq. units	2. Area of triangle
3. $\triangle ADC = \frac{1}{2}ah$ sq. units	3. Area of triangle
4. Trapezoid $ABCD = (\frac{1}{2}ah + \frac{1}{2}bh)$ sq. units	4. Replacement
5. Trapezoid $ABCD = \frac{1}{2}(a + b)h$ sq. units	5. Number axioms

Example 1. In *Fig. 14-38*, calculate the area of $\triangle ABC$. Compare your solution with that on page 500.

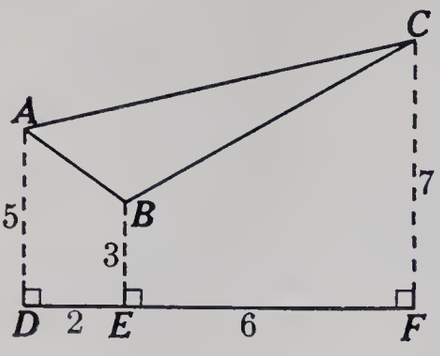


Fig. 14-38

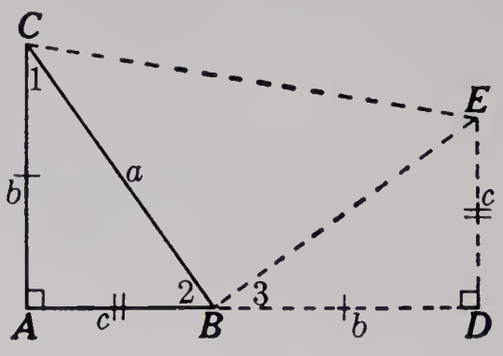


Fig. 14-39

Example 2. The Pythagorean Theorem.

Use the diagram of *Fig. 14-39* to prove the Pythagorean Theorem; compare your proof with that on page 500. This proof is accredited to President Garfield (President of the United States, 1881).

Hypothesis: Trapezoid $ADEC$ in which $\angle A = \angle D = 90°$, $AB = DE = c$ units, $AC = DB = b$ units

Conclusion: $a^2 = b^2 + c^2$

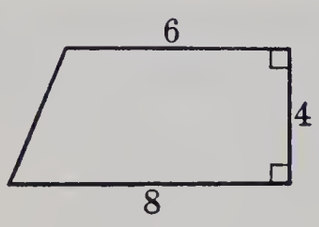
Exercise 14-8

(A)

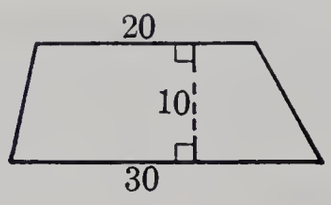
1. State the area of a trapezoid which has parallel sides of lengths l_1 and l_2 units, respectively, and distance between parallel sides, d units.

Find the area of each of the following trapezoids:

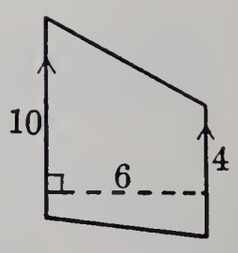
2.



3.



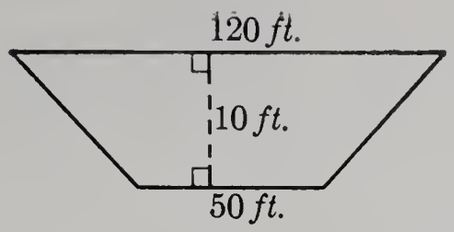
4.

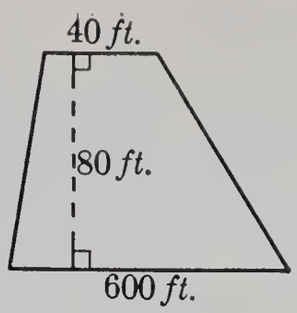


(B)

5. The diagram, on the left at the top of page 423, shows a cross-section of a highway open cut. Find the area of the cross-section.

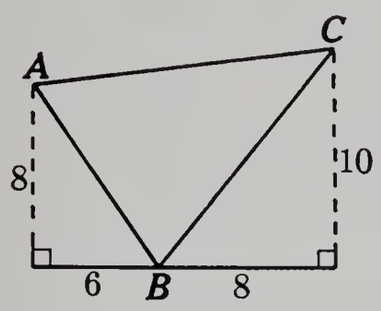
6. The cross-sectional dimensions of a concrete dam are as shown in the diagram on the right at the top of page 423. Find the area of this cross-section.



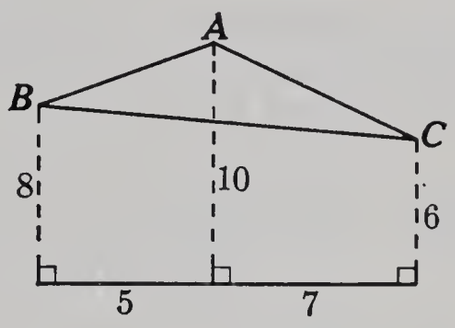


Find the area of $\triangle ABC$ in each of the following:

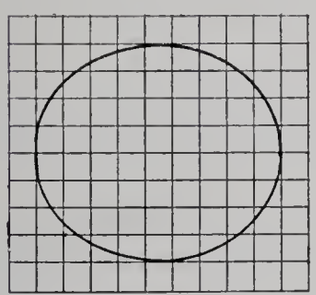
7.



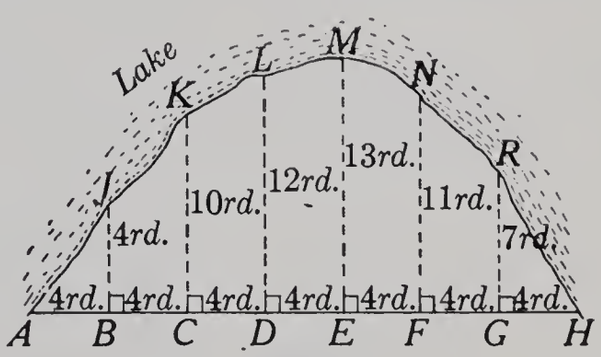
8.



9. A piece of sheet iron placed on squared paper produced the tracing shown in the diagram at the left below. Find its area by adding to the number of whole squares, one-half of the number of cut squares.



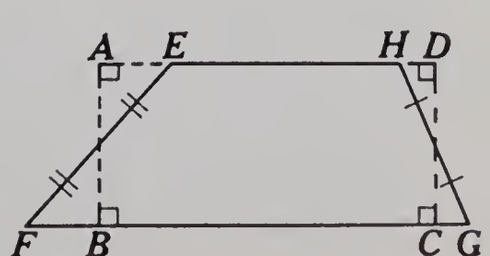
Each Square = 1 sq. in.



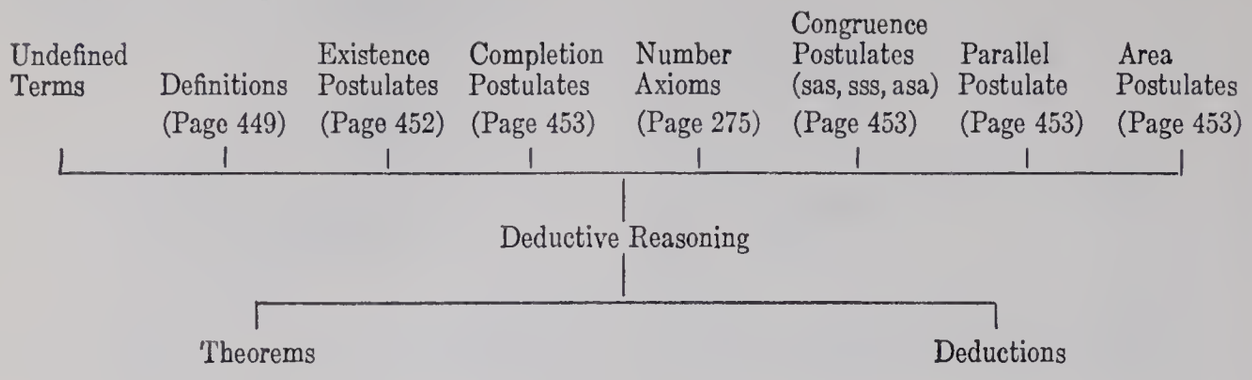
10. The diagram at the right above illustrates lake-front property on a peninsula having one boundary, AH, 28 rods long. At 4-rod intervals on AH, perpendicular lines drawn to the shore have lengths as shown. Regard AJ, JK, KL, etc., as line segments and find the area of the property to the nearest tenth of an acre (1 acre = 160 square rods).

11. What is the area of an equilateral triangle, if each side is 8.0 inches? Give the answer correct to the nearest square inch.

12. In the diagram at the right, show that the area of rectangle $ABCD$ is equal to the area of trapezoid $EFGH$ and hence deduce the formula for the area of a trapezoid.



14·14 Summary.



Angle
1. Complementary Angle Theorem
2. Supplementary Angle Theorem
3. Vertical Angle Theorem

Basic Biconditional
4. Isosceles Triangle Theorem
5. Right Bisector Theorem
13. Angle Bisector Theorem

Inequality
6. Exterior Angle Theorem

Parallel
7. Parallel Line Theorem
8. Quadrilateral-Parallelogram Theorem
9. Parallelogram Theorem
10. Triangle Angle Sum Theorem

Congruence
11. aas Congruence Theorem
12. Right Triangle Congruence Theorem

Area
14. Parallelogram Area Theorem
15. Parallelogram-Diagonal Theorem
16. Parallelogram-Triangle Area Theorem
17. Equal Triangle-Parallel Line Theorem
18. Pythagorean Theorem
19. Pythagorean Converse
20. Trapezoid Area Theorem

Exercise 14-9

(*Numerical Problems*)

Draw a representative sketch, mark on all given data, and complete the required calculations for each of the following:

1. In $\triangle ABC$, X and Y are on AC and AB, respectively, so that $BX \perp AC$ and $CY \perp AB$. If $AC = 16$ cm., $BX = 9$ cm., and $CY = 12$ cm., find AB.

2. The lengths of the diagonals of a rhombus are 20 and 24 units. What is the area of the rhombus?

3. In trapezoid $ABCD$, $AB \| CD$, $AB = 6$ in., $CD = 8$ in. If the distance between AB and CD is 2 in., find the area of $ABCD$.

4. The medians BD and CE of $\triangle ABC$ intersect at F. The area of $\triangle$'s DFC and FBC are 5 sq. cm. and 10 sq. cm., respectively. Find the area of (i) $\triangle EBF$ (ii) figure $AEFD$.

5. P is any point on AB of $\|^{gm}$ $ABCD$. The altitude of $ABCD$ drawn to BC is 6 in.; BC is 8 in. What is the area of $\triangle PCD$?

6. In $\|^{gm}$ $ABCD$, base $BC = 20$ in. and the length of the altitude to BC is 15 in. P is any point on AC so that $AP = \frac{1}{3}AC$. What is the area of $\triangle PBC$?

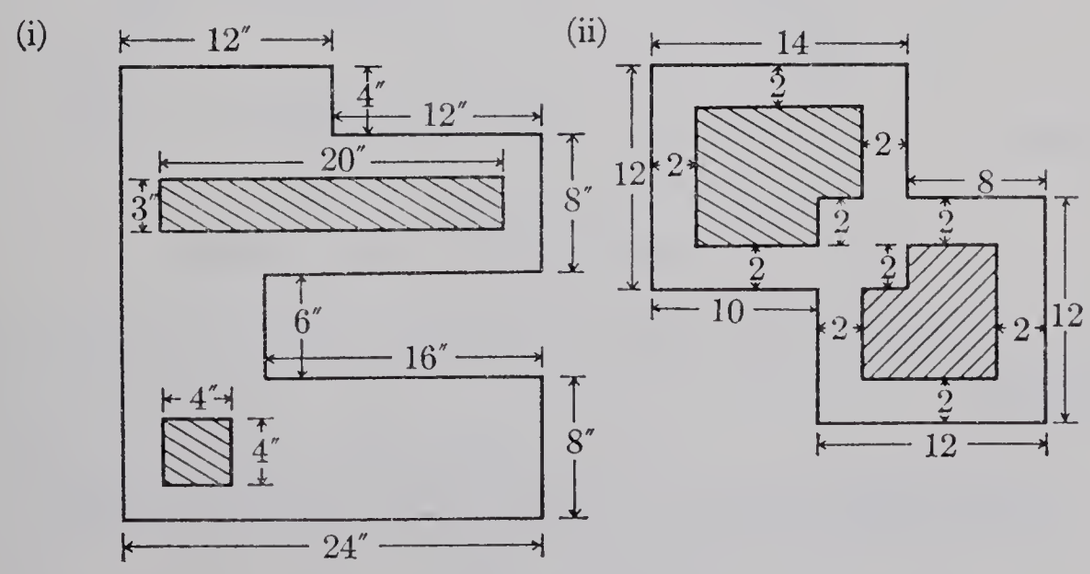
7. In $\|^{gm}$ $ABCD$ the lengths of the base and the altitude drawn to this base are 16 and 15 units, respectively. F is the midpoint of AD and E is the midpoint of AB. Find the area of $\triangle AEF$.

8. AD is a median of $\triangle ABC$. E is on AD so that $AE = \frac{1}{4}AD$. If $BC = 20$ in. and the area of $\triangle EDC$ is 30 sq. in., find the altitude of $\triangle ABC$ drawn from A to BC.

9. In $\triangle ABC$, $BC = 16$ in. and the altitude to BC is 8 in. D is any point on BC; E is the midpoint of AD. Find the area of ($\triangle AEC + \triangle BDE$).

10. The diagonal AC of quadrilateral $ABCD$ bisects the other diagonal DB. X is on AC so that $DX \perp AC$. DX is 3 in. and AC is 6 in. Find the area of $ABCD$.

11. Calculate the area of the unshaded portion of each of the following regions:



Chapter XV

RATIO AND PROPORTION

SIMILAR FIGURES

15·1 **Ratio and proportion.** The numbers 6, 3, 2 are said to be in the ratio 6 to 3 to 2 which is written $6 : 3 : 2$. The ratio $6 : 3 : 2$ is called a *three-term ratio.*

$$\text{Since} \quad \begin{aligned} 24 &= 6(4) \\ 12 &= 3(4) \\ 8 &= 2(4) \end{aligned}$$

the ratio $24 : 12 : 8$ is said to be *equivalent* to the ratio $6 : 3 : 2$; that is,

$$24 : 12 : 8 = 6 : 3 : 2 .$$

Also, $60 : 30 : 20 = 6(10) : 3(10) : 2(10)$

$$\therefore \quad 60 : 30 : 20 = 6 : 3 : 2 .$$

Similarly, $30 : 15 : 10 = 6(5) : 3(5) : 2(5)$

$$\therefore \quad 30 : 15 : 10 = 6 : 3 : 2$$

and $600 : 300 : 200 = 6(100) : 3(100) : 2(100)$

$$\therefore \quad 600 : 300 : 200 = 6 : 3 : 2 .$$

In general,

$$x : y : z = a : b : c$$

if and only if there is a real number, k $(k \neq 0)$ *such that*

$$\begin{aligned} x &= ka \\ y &= kb \\ z &= kc . \end{aligned}$$

The statement $x : y : z = a : b : c$ is called a *proportion.* Each of the

numbers x, y, z, a, b, c is called a *proportional*, and k is the *proportionality factor*.

DEFINITION: *A proportion is the statement of equality of two ratios.*

Example 1. Are 3, 8, 10 proportional to 12, 32, 40? Justify your answer.

Solution. Since $12 = 4(3)$, $32 = 4(8)$, $40 = 4(10)$, then

$$12 : 32 : 40 = 3 : 8 : 10 .$$

The proportionality factor is 4 .

a. *Two-term ratios.* The two-term ratio $60 : 30$ is equivalent to the two-term ratio $2 : 1$, since $60 : 30 = 30(2) : 30(1) = 2 : 1$.

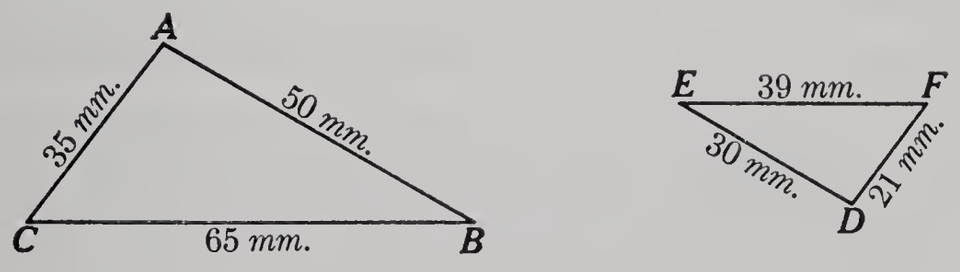
This proportion may also be written, $\dfrac{60}{30} = \dfrac{2}{1}$.

In general,

if $b \neq 0$, the ratio $a : b$ may be expressed in the form $\dfrac{a}{b}$.

15·2 **Ratios of line segments and areas.**

DEFINITION: *The ratio of line segments AB and CD means the ratio of the lengths of the line segments and is expressed $AB : CD$ or $\dfrac{AB}{CD}$.*



(*a*) (*b*)

Fig. 15-1

The ratio of the sides of $\triangle ABC$, *Fig. 15-1*(*a*), is $50 : 65 : 35$, or $10 : 13 : 7$.
The ratio of the sides of $\triangle DEF$, *Fig.15-1*(*b*), is $30 : 39 : 21$, or $10 : 13 : 7$.
Thus, $AB : BC : CA = DE : EF : FD$; that is, the sides of $\triangle ABC$ are proportional to the sides of $\triangle DEF$.

DEFINITION: *The ratio of the areas of two triangles ABC and DEF is expressed $\triangle ABC : \triangle DEF$ or $\dfrac{\triangle ABC}{\triangle DEF}$ and is the ratio of the number of square units in the areas of the triangles.*

Exercise 15-1

(A)

1. In the diagram below, AD is divided at B and C into segments with measurements as indicated.

State the ratio of:

(i) AC to CD (ii) BD to AB (iii) CD to AB
(iv) AB to AD (v) AB to CD (vi) AD to BC

2. State the numbers represented by a and b in each of the following proportions:

(i) $2 : 4 : 6 = 6 : a : b$ (ii) $\frac{1}{2} : \frac{1}{3} : \frac{1}{4} = a : b : 1$

(iii) $\frac{1}{2} : \frac{1}{3} : \frac{1}{4} = 1 : a : b$ (iv) $\frac{1}{2} : \frac{1}{3} : \frac{1}{4} = a : 1 : b$

(v) $x : y : z = x^2 : a : b$ (vi) $m : mn : n = mn^2 : a : b$

(B)

3. (a) List pairs of the following ratios which are equal:

(i) $5 : 7 : 9$ (ii) $12 : 18 : 24$ (iii) $2 : 3 : 4$

(iv) $\frac{10}{3} : \frac{14}{3} : 6$ (v) $\frac{8}{5} : 2 : \frac{12}{5}$ (vi) $2 : 0 : 0$

(b) What is the proportionality factor of each proportion in (a)?

4. $\triangle XYZ$ has a base whose length is 8 feet and an altitude whose length is 6 feet. $\triangle PQR$ has a base whose length is 4 feet and an altitude whose length is 15 feet. Find the ratio of $\triangle XYZ$ to $\triangle PQR$.

5. C is a point on line segment AB such that $AC : CB = 3 : 2$. If the length of AB is 30 inches, find the lengths of AC and CB.

6. Two triangles have equal altitudes, and their areas are in the ratio $5 : 3$. What is the ratio of the longer base to the shorter base?

7. Two circles have radii of lengths 8 cm. and 12 cm., respectively. Find (i) the ratio of their circumferences (ii) the ratio of their areas.

15·3 Properties of proportions involving two-term ratios.

In the proportion $a : b = c : d$, $a, b, c, d \in R$,

a and d are called the *extremes;*
b and c are called the *means;*
d is called the *fourth proportional to a, b, and c.*

a. THEOREM. If $a, b, c, d \neq 0$ and $\frac{a}{b} = \frac{c}{d}$,

then (i) $ad = bc$ (Cross multiplication)

(ii) $\frac{a}{c} = \frac{b}{d}$ (Alternation of the means)

(iii) $\frac{b}{a} = \frac{d}{c}$ (Inversion)

(iv) $\frac{a+b}{b} = \frac{c+d}{d}$ (Adding unity)

(v) $\frac{a-b}{b} = \frac{c-d}{d}$. (Subtracting unity)

(i) **Hypothesis:** $a, b, c, d \neq 0$ and $\frac{a}{b} = \frac{c}{d}$

Conclusion: $ad = bc$

Proof:

STATEMENTS	AUTHORITIES
1. $\frac{a}{b} = \frac{c}{d}$	1. Hypothesis
2. $\frac{a}{b}(bd) = \frac{c}{d}(bd)$	2. Multiplication
3. $ad = bc$	3. Definition, (CM)

Write a proof of the conclusions (ii), (iii), (iv), and (v) and compare your proofs with those on page 500.

b. If $\frac{a}{b} = \frac{b}{c}$ and $a, b, c \neq 0$, then a, b, c are in *continued proportion* and b is the *geometric mean* (*mean proportional*) of a and c, and c is the third proportional to a and b.

Example 1. Find the fourth proportional to 3, 5, and 8.

Solution. Represent the fourth proportional by x.

Then $\frac{3}{5} = \frac{8}{x}$ (Definition)

$\therefore\ 3x = 40$ (Cross multiplication)

$\therefore\ x = \frac{40}{3}$ (Division)

The fourth proportional to 3, 5, and 8 is $\frac{40}{3}$.

Example 2. Find the positive mean proportional of 4 and 64.

Solution. Represent the mean proportional by x.

Then $\frac{4}{x} = \frac{x}{64}$ (Definition)

$\therefore \quad x^2 = 256$ (Cross multiplication)

$\therefore \quad x = \pm 16$

The positive mean proportional of 4 and 64 is 16,

that is, $\frac{4}{16} = \frac{16}{64}$.

c. If $\frac{AB}{CD} = \frac{PQ}{XY}$, then $AB \cdot XY = CD \cdot PQ$ by cross multiplication. Geometrically, $AB \cdot XY$ represents the number of square units in the area of a rectangle with adjacent sides equal to AB and XY. $AB \cdot XY$ is read, "the rectangle AB times XY".

Exercise 15-2

(A)

State the equation obtained by (i) *cross multiplication* (ii) *alternation of the means* (iii) *inversion* (iv) *adding unity and* (v) *subtracting unity for each of the following proportions:*

1. $\frac{3}{5} = \frac{9}{15}$ **2.** $\frac{4}{1} = \frac{12}{3}$ **3.** $\frac{64}{32} = \frac{10}{5}$

4. $\frac{x}{2} = \frac{y}{3}$ **5.** $\frac{6}{x} = \frac{x}{150}$ **6.** $\frac{2}{5} = \frac{9}{x}$

7. $2 : 3 = 18 : 27$ **8.** $5 : x = x : 125$ **9.** $4 : 7 = 15 : x$

10. State the number (or numbers) represented by x in each of the following proportions:

(i) $\frac{x}{2} = \frac{3}{4}$ (ii) $\frac{3}{x} = \frac{21}{14}$ (iii) $x : 5 = 10 : 25$

(iv) $\frac{5}{2} = \frac{25}{x}$ (v) $\frac{a}{3} = \frac{x}{5}$ (vi) $12 : 3 = 44 : x$

(vii) $\frac{1}{x} = \frac{x}{16}$ (viii) $\frac{2}{3} = \frac{3}{x}$ (ix) $a : b = b : x$

11. What conclusion can be drawn in each of the following?

(i) $\frac{a}{b} = \frac{a}{c}$ (ii) $\frac{a}{b} = \frac{ma}{c}$ (iii) $\frac{a}{b} = \frac{c}{c}$

What conclusion can be drawn in each of the following?

(iv) $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AX}{AX}$ (v) $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{2BC}{BC}$

(vi) $\dfrac{AB}{CD} = \dfrac{2}{1}$ and $\dfrac{CD}{PQ} = \dfrac{3}{2}$

12. If $AB \cdot XY = CD \cdot PQ$, find:

(i) $\dfrac{AB}{CD}$ (ii) $\dfrac{XY}{CD}$ (iii) $\dfrac{PQ}{AB}$ (iv) $\dfrac{PQ}{XY}$

13. If $XY^2 = AB \cdot CD$, find: (i) $\dfrac{XY}{AB}$ (ii) $\dfrac{XY}{CD}$

(B)

14. If m represents the fourth proportional to 2, 6, 10, find m.

15. Find the fourth proportional to:

(i) $3, 2\frac{1}{3}, 9$ (ii) x, y, xy

(iii) $x + y, x - y, (x + y)^2$ (iv) $a^2, a + b, a$

16. If x represents the geometric means of 1 and 100, find x.

17. Find the geometric means of:

(i) 9, 16 (ii) $9, \dfrac{1}{4}$ (iii) $\dfrac{3}{4}, \dfrac{4}{3}$ (iv) a, ar^2

(v) $(x + y)^2, (x - y)^2$

18. If x represents the third proportional to 1 and 2, find x.

19. Find the third proportional to:

(i) 3, 9 (ii) 3, 2 (iii) a, ar (iv) x^2, xy

20. Find the ratio $x : y$ from each of the following:

(i) $5x = 7y$ (ii) $2x = 3y$

(iii) $x - 2y = 0$ (iv) $ax - by = 0$

(v) Evaluate $\dfrac{x + y}{y}$ for each of (i) to (iv).

(vi) Evaluate $\dfrac{x - y}{y}$ for each of (i) to (iv).

21. A line segment AB is extended to a point C beyond B, so that $BC = \frac{2}{3}AB$. Find each of the following ratios:

(i) $\dfrac{AB}{BC}$ (ii) $\dfrac{AC}{AB}$ (iii) $\dfrac{BC}{AC}$

22. A line segment PQ is extended beyond Q to R so that $\dfrac{PQ}{QR} = \dfrac{4}{3}$.

If $PQ = 24$ cm., find the length of PR.

S is a point between R and T in the line segment RT. Find what fraction (i) *RS is of RT* (ii) *ST is of RT, if RS : ST equals:*

23. 2 : 3 **24.** 1 : 4 **25.** 5 : 2

26. $m : n$ **27.** $2m : 3s$ **28.** $a + b : a - b$

15·4 **Triangle Area Ratio Theorem.**

THEOREM 21

Triangle Area Ratio Theorem

If two triangles have equal altitudes, then their areas are proportional to their bases.

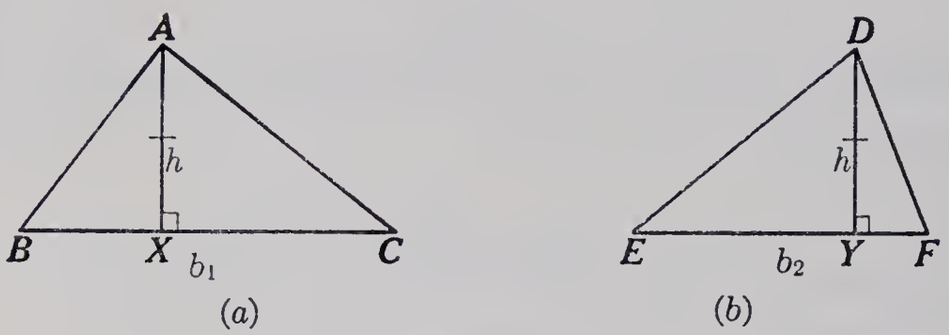


Fig. 15-2

Hypothesis: $\triangle ABC$ and $\triangle DEF$ with altitudes of length h units and bases of length b_1 and b_2 units, respectively

Conclusion: $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{b_1}{b_2}$

Proof:

STATEMENTS	AUTHORITIES
1. $\triangle ABC = \frac{1}{2}b_1h$ sq. units	1. Area of triangle
2. $\triangle DEF = \frac{1}{2}b_2h$ sq. units	2. Area of triangle
3. $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{\frac{1}{2}b_1h}{\frac{1}{2}b_2h}$	3. Division
4. $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{b_1}{b_2}$	4. Principle of equivalent fractions

Corollary: If two triangles have equal bases, then their areas are proportional to their altitudes.

Write a complete proof of this corollary and compare your proof with that on page 501.

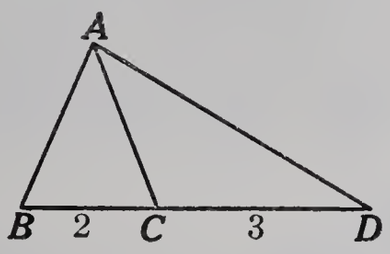
Exercise 15-3

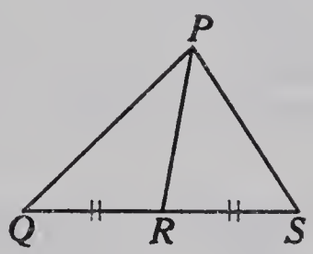
(A)

1. In the diagram at the left below, what is the ratio of:

(i) $\triangle ABC$ to $\triangle ACD$ (ii) $\triangle ABC$ to $\triangle ABD$

(iii) $\triangle ACD$ to $\triangle ABD$



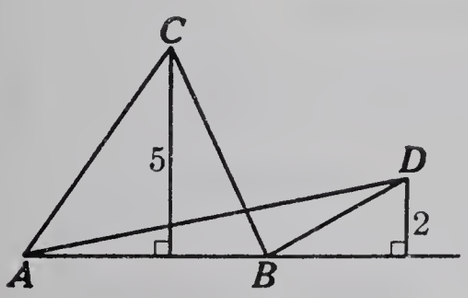


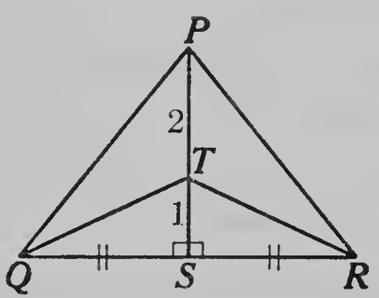
2. In the diagram at the right above, what is the ratio of:

(i) $\triangle PQR$ to $\triangle PRS$ (ii) $\triangle PQS$ to $\triangle PRS$

(iii) $\triangle PQR$ to $\triangle PQS$

3. In the diagram at the left below, what is the ratio $\triangle ABC$ $\triangle ABD$?





4. In the diagram at the right above, what is the ratio of:

(i) $\triangle QTS$ to $\triangle TSR$ (ii) $\triangle QTS$ to $\triangle QPS$

(iii) $\triangle TRS$ to $\triangle PSR$ (iv) $\triangle TQR$ to $\triangle PQS$

(v) $\triangle QSP$ to $\triangle SRP$ (vi) $\triangle PQR$ to $\triangle PQS$

(vii) $\triangle PTQ$ to $\triangle TQS$ (viii) $\triangle TSR$ to $\triangle PTR$

(ix) $\triangle PRT$ to $\triangle PQT$ (x) $\triangle PTR$ to $\triangle QTS$

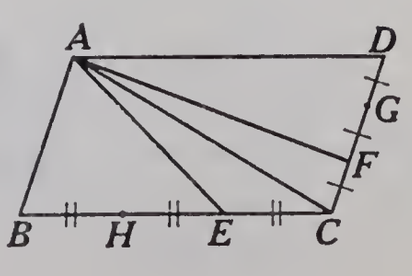
(xi) $\triangle TSR$ to $\triangle PQT$ (xii) $\triangle QTS$ to $\triangle PSR$

5. In the diagram at the right, $ABCD$ is a parallelogram.

What fraction is (i) $\triangle ABE$ of $\triangle ABC$

(ii) $\triangle ABE$ of parallelogram $ABCD$

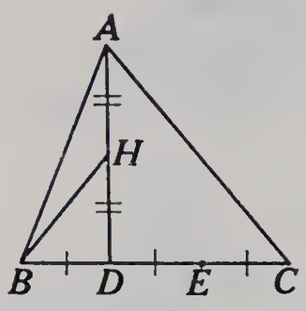
(iii) quadrilateral $AECF$ of parallelogram $ABCD$?

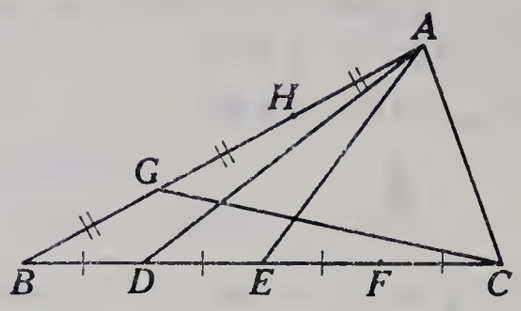


6. In the diagram at the left below, what is the ratio of:

(i) $\triangle ABH$ to $\triangle ABD$ (ii) $\triangle ABD$ to $\triangle ABC$

(iii) $\triangle ABH$ to $\triangle ABC$





7. In the diagram at the right above, find:

(i) $\dfrac{\triangle ABD}{\triangle ABC}$ (ii) $\dfrac{\triangle ADC}{\triangle ABE}$ (iii) $\dfrac{\triangle AGC}{\triangle ABC}$ (iv) $\dfrac{\triangle ABD}{\triangle BGC}$

(B)

8. $PQRS$ is a parallelogram and X is a point on QS such that $\dfrac{QX}{XS} = \dfrac{1}{4}$.

Prove that $\triangle PXS = \frac{2}{5}$ parallelogram $PQRS$.

9. $ABCD$ is a quadrilateral. The diagonals AC and BD intersect on E such that $AE = EC$. Prove that $\triangle ACD = \triangle BCD$.

10. Prove that the areas of the four non-overlapping triangles into which a quadrilateral is divided by its diagonals are proportional.

11. State and prove the converse of the Triangle Area Ratio Theorem.

15·5 Triangle Proportionality Theorem.

THEOREM 22

Triangle Proportionality Theorem

If a line parallel to one side of a triangle intersects the other two sides at distinct points, then the segments of these sides are proportional.

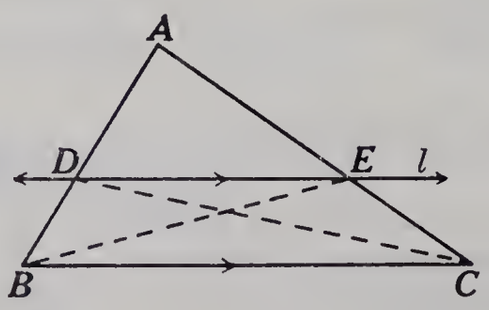


Fig. 15-3

Hypothesis: $\triangle ABC$ and a line l parallel to BC, such that l intersects AB on D and AC on E

Conclusion: $\dfrac{AD}{DB} = \dfrac{AE}{EC}$

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\dfrac{AD}{DB} = \dfrac{AE}{EC}$	1. The areas of two triangles with AD and DB as bases and with equal altitudes are proportional to the areas of two triangles with AE and EC as bases and with equal altitudes.
2. The areas of two triangles with AD and DB as bases and with equal altitudes are proportional to the areas of two triangles with AE and EC as bases and with equal altitudes	2. $\dfrac{\triangle AED}{\triangle DEB} = \dfrac{\triangle AED}{\triangle DEC}$.
3. $\dfrac{\triangle AED}{\triangle DEB} = \dfrac{\triangle AED}{\triangle DEC}$	3. $\triangle DEB = \triangle DEC$.
4. $\triangle DEB = \triangle DEC$	4. $BC \parallel DE$.
5. $BC \parallel DE$ by hypothesis.	

Proof: Write a complete proof of this theorem and compare your proof with that on page 502.

Corollary: **If a straight line parallel to one side of a triangle intersects the other two sides at distinct points, then the sides are proportional to corresponding segments.**

Write a complete proof of this corollary and compare your proof with that on page 502. (HINT: Refer to section 15·3, **a.** (iv) page 429.)

Example. In $\triangle PQR$ (*Fig. 15-4*), S is on PQ and T on PR so that $ST \parallel QR$. V is on QR so that $TV \parallel PQ$. Prove that $\dfrac{PQ}{QS} = \dfrac{RQ}{RV}$.

Hypothesis: In the diagram (*Fig. 15-4*) $ST \parallel QR$, $TV \parallel PQ$.

Conclusion: $\dfrac{PQ}{QS} = \dfrac{RQ}{RV}$

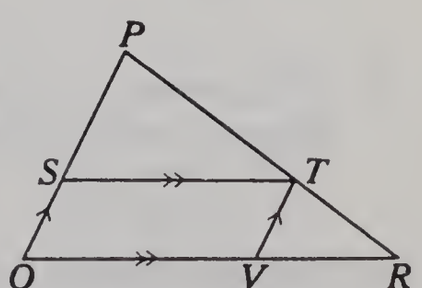


Fig. 15-4

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\frac{PQ}{QS} = \frac{RQ}{RV}$	1. $\frac{PQ}{QS} = \frac{PR}{RT}$ and $\frac{RQ}{RV} = \frac{RP}{RT}$.
2. $\frac{PQ}{QS} = \frac{PR}{RT}$ and $\frac{RQ}{RV} = \frac{RP}{RT}$ by the Triangle Proportionality Theorem.	

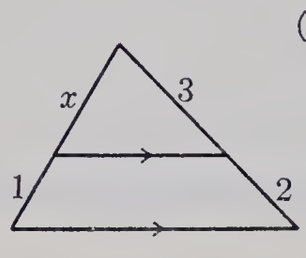
Proof: Write a complete proof and compare it with that on page 503.

Exercise 15-4

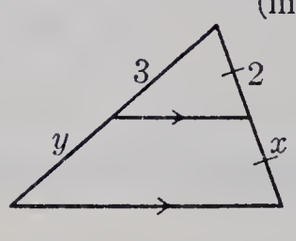
(A)

1. For each of the following diagrams, find the numbers represented by x and y:

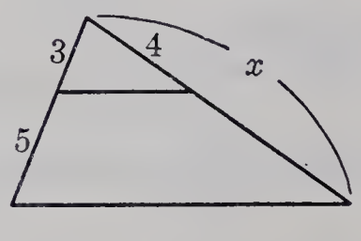
(i)



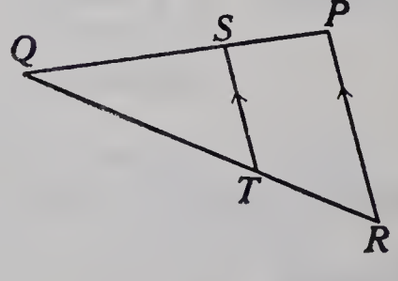
(ii)



(iii)



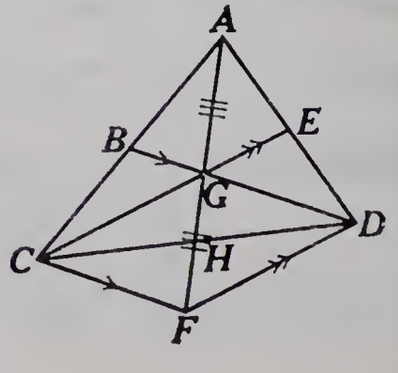
2. With reference to the accompanying diagram, complete the proportions:



(i) $\frac{PS}{SQ} =$ —— (ii) $\frac{QR}{QT} =$ ——

(iii) $\frac{RT}{TQ} =$ —— (iv) $\frac{QP}{SP} =$ ——

3. With reference to the accompanying diagram, what is the ratio of:

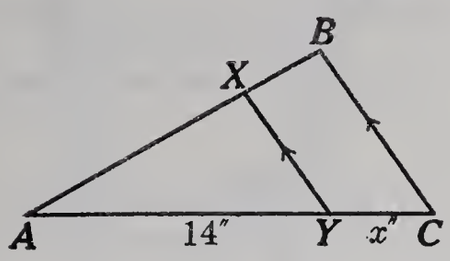


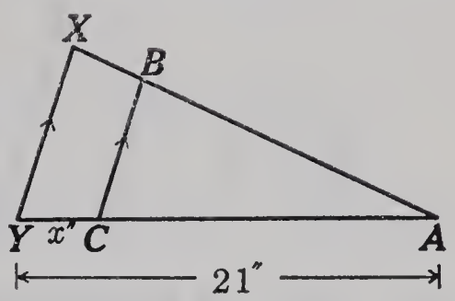
(i) AB to BC (ii) AE to ED

(iii) GH to HF (iv) AG to GH

(v) CH to HD

4. In each of the diagrams below, $\frac{AX}{XB} = \frac{7}{2}$. Find the number represented by x in each case:





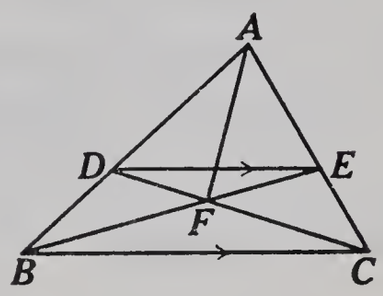
(B)

5. Copy the following proof and complete the authorities column:

Hypothesis: In the diagram at the right $DE \parallel BC$.

Conclusion: $\triangle ADF = \triangle AEF$

Proof:

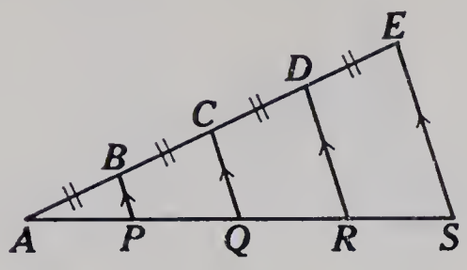


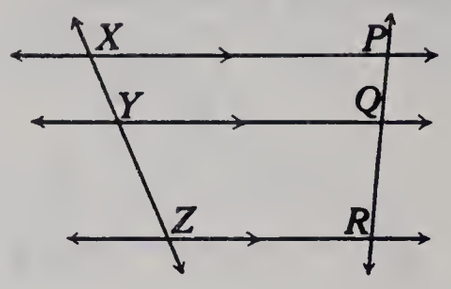
STATEMENTS	AUTHORITIES
1. $\frac{\triangle ADF}{\triangle DBF} = \frac{AD}{DB}$	1.
2. $\frac{\triangle AEF}{\triangle ECF} = \frac{AE}{EC}$	2.
3. $DE \parallel BC$	3.
4. $\frac{AD}{DB} = \frac{AE}{EC}$	4.
5. $\frac{\triangle ADF}{\triangle DBF} = \frac{\triangle AEF}{\triangle ECF}$	5.
6. $\triangle DBC = \triangle EBC$	6.
7. $\triangle DBC - \triangle FBC = \triangle BEC - \triangle FBC$	7.
8. $\triangle DBF = \triangle ECF$	8.
9. $\frac{\triangle ADF}{\triangle DBF} = \frac{\triangle AEF}{\triangle ECF}$	9.
10. $\triangle ADF = \triangle AEF$	10.

6. Use the Triangle Proportionality Theorem to prove that the line on the midpoint of one side of a triangle, parallel to a second side, bisects the third side.

7. In the diagram at the left below, $AB = BC = CD = DE$ and $BP \parallel CQ \parallel DR \parallel ES$.

Prove $AP = PQ = QR = RS$.





8. In the diagram at the right above, three parallel lines are intersected by two transversals at X, Y, Z, and P, Q, R, respectively.

Prove $\dfrac{XY}{YZ} = \dfrac{PQ}{QR}$.

9. In parallelogram $ABCD$, X and Y are the midpoints of AD and BC, respectively. BX and DY intersect AC on M and N, respectively. Prove $AM = MN = NC$, that is, BX and DY *trisect* AC.

15·6 **Similar figures.**

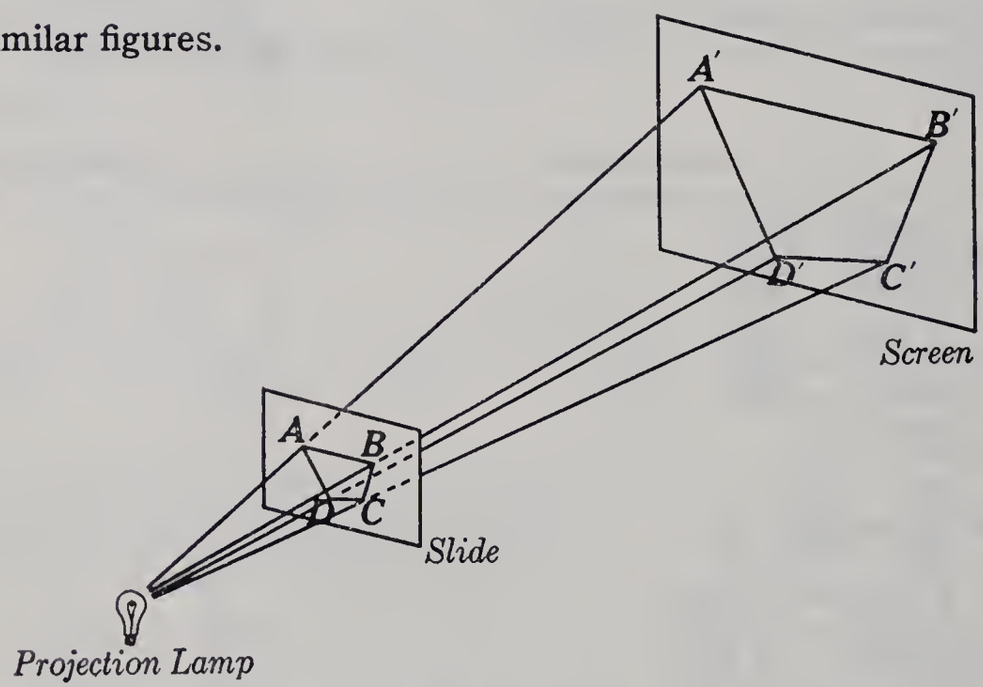


Fig. 15-5

If a picture of a quadrilateral $ABCD$ (*Fig. 15-5*) is projected on a screen, in a plane parallel to the plane of the slide, the image $A'\ B'\ C'\ D'$ will be a uniform enlargement of the original quadrilateral. Quadrilateral $ABCD$ can be considered a uniform reduction of quadrilateral $A'\ B'\ C'\ D'$. Two such geometric figures are said to be *similar*, that is, they have the same shape but not the same size. This means that the angles of one figure are respectively equal to the corresponding angles of the other, and the lengths of the sides of one are proportional to the lengths of the corresponding sides of the other.

DEFINITION: *Geometric figures are similar if:*

(i) *corresponding angles are equal,*

and (ii) *corresponding sides are proportional.*

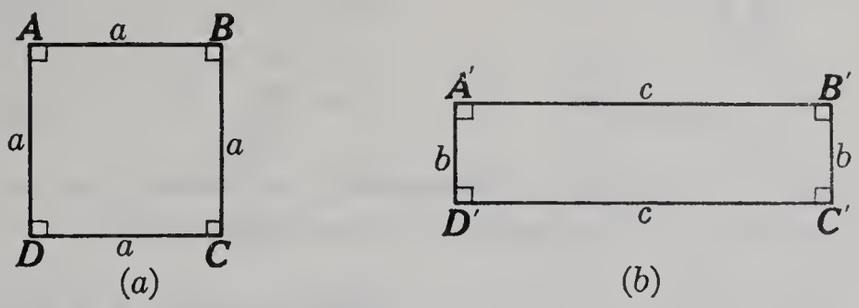


Fig. 15-6

Corresponding angles of square $ABCD$ and rectangle $A'\ B'\ C'\ D'$ (*Fig. 15-6*) are equal, but corresponding sides are not proportional, since $a : a : a : a \neq b : c : b : c$. Thus, quadrilaterals $ABCD$ and $A'\ B'\ C'\ D'$ are not similar. In *Fig. 15-7*, quadrilaterals $PQRS$ and $P'Q'R'S'$ do not have corresponding angles equal, but corresponding sides are proportional, since $a : a : a : a = b : b : b : b = 1 : 1 : 1 : 1$. Thus, quadrilaterals $PQRS$ and $P'\ Q'\ R'\ S'$ are not similar.

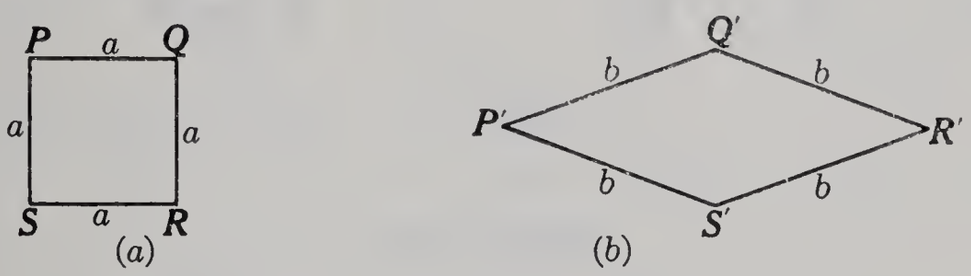


Fig. 15-7

In *Fig. 15-8*, corresponding angles of quadrilaterals $WXYZ$ and $W'\ X'\ Y'\ Z'$ are equal, and corresponding sides are proportional, since $a : b : c : d = 2a : 2b : 2c : 2d$. Thus, quadrilateral $WXYZ$ is similar to quadrilateral $W'\ X'\ Y'\ Z'$. The symbol $\sim$ means *is similar to*. Thus, we write quadrilateral $WXYZ \sim$ quadrilateral $W'\ X'\ Y'\ Z'$.

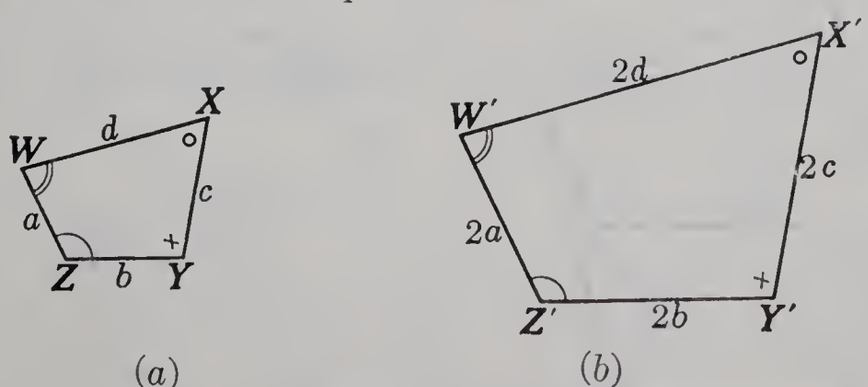


Fig. 15-8

The correspondences between the parts of the similar quadrilaterals in *Fig. 15-9* are as follows:

$\angle A = \angle Q$	$A \leftrightarrow Q$	$AB \leftrightarrow QP$
$\angle B = \angle P$	$B \leftrightarrow P$	$BC \leftrightarrow PS$
$\angle C = \angle S$	$C \leftrightarrow S$	$CD \leftrightarrow SR$
$\angle D = \angle R$	$D \leftrightarrow R$	$DA \leftrightarrow RQ$

If care is taken in listing corresponding vertices in order, all of the above correspondences are contained in the statement:

Quadrilateral $A\ B\ C\ D \sim$ quadrilateral $Q\ P\ S\ R$

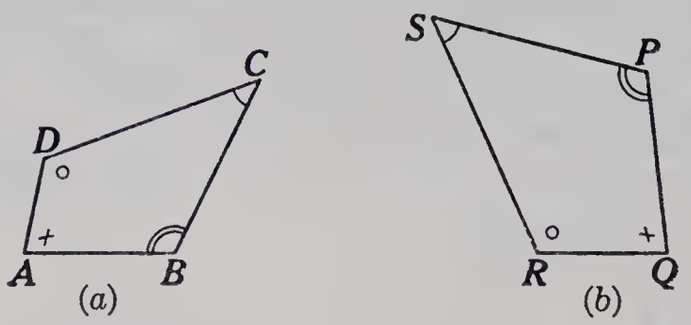


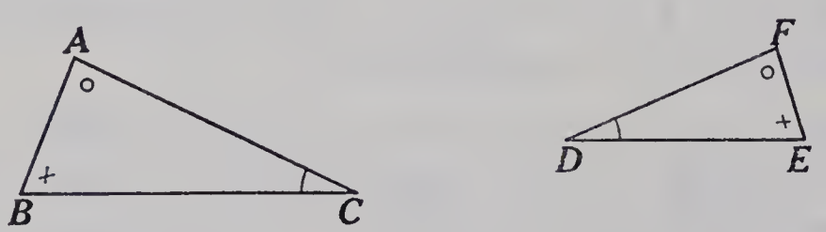
Fig. 15-9

Exercise 15-5

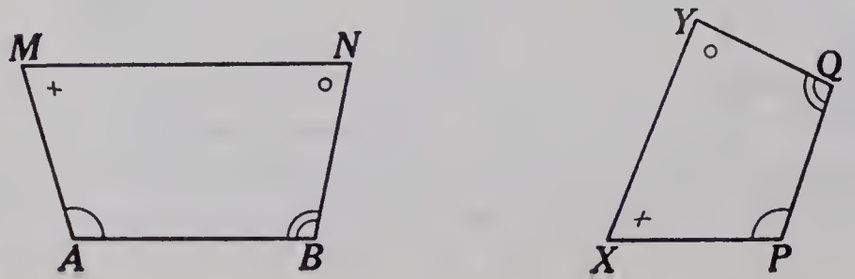
(A)

1. Name corresponding vertices and corresponding sides in each of the following:

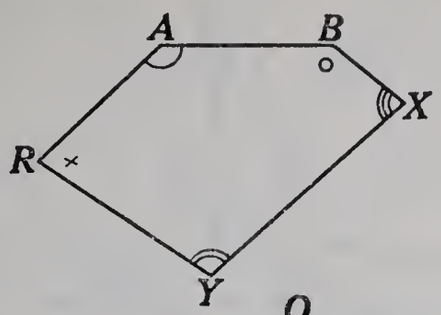
(i)

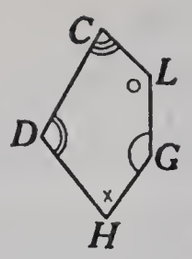


(ii)

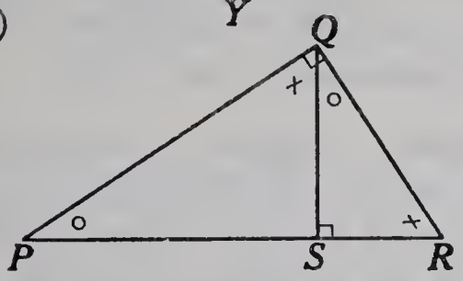


(iii)

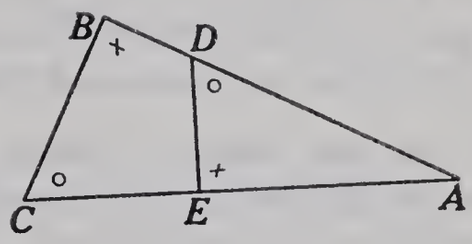




(iv)



(v)

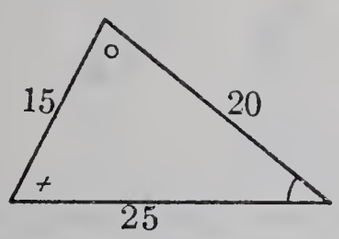


2. Quadrilateral $DGHM \sim$ quadrilateral $PRTS$. Without referring to a diagram, name (i) corresponding sides (ii) corresponding angles.

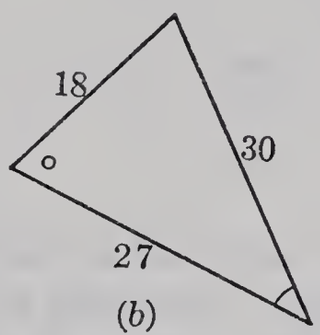
(B)

3. In each of the following, which figure is not similar to the other two? Give reasons.

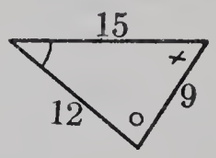
(i)



(*a*)

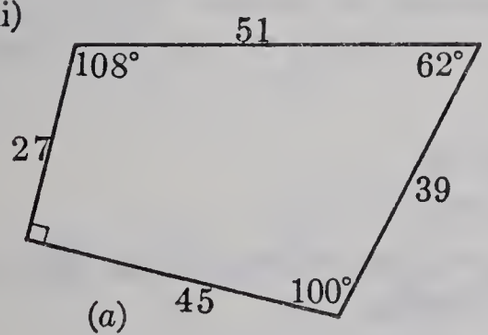


(*b*)

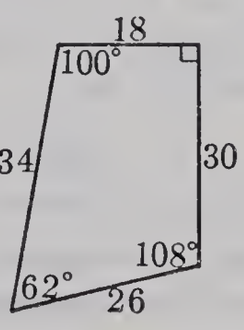


(c)

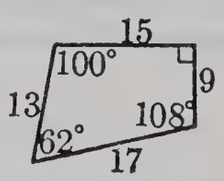
(ii)



(*a*)



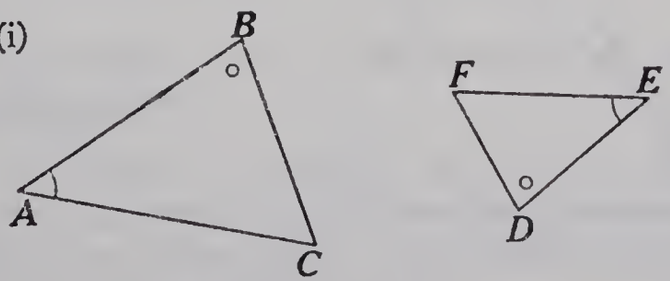
(*b*)



(c)

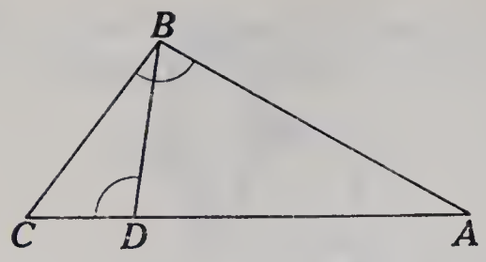
4. In each of the following, write the proportion obtained from the lengths of corresponding sides of the similar triangles:

(i)



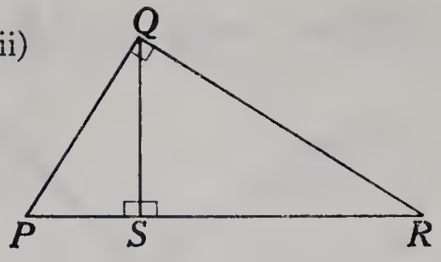
$\triangle ABC \sim \triangle EDF$

(ii)



$\triangle ABD \sim \triangle ACB$

(iii)



$\triangle PQR \sim \triangle PSQ \sim \triangle QSR$

5. Prove that all equilateral triangles are similar.

***15·7* The aaa Similar Triangle Theorem.** Suppose a correspondence exists between two sets of non-collinear points $\{A, B, C\}$ and $\{D, E, F\}$ so that $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$. Under what conditions are the triangles determined by these points similar? By the definition of similar figures it is necessary that corresponding angles be equal and corresponding sides be proportional.

In *Fig. 15-10*, the lengths of the sides are represented by a, b, c, d, e, f units. Then $\triangle ABC \sim \triangle DEF$ if (i) $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and (ii) $\dfrac{a}{d} = \dfrac{b}{e} = \dfrac{c}{f}$.

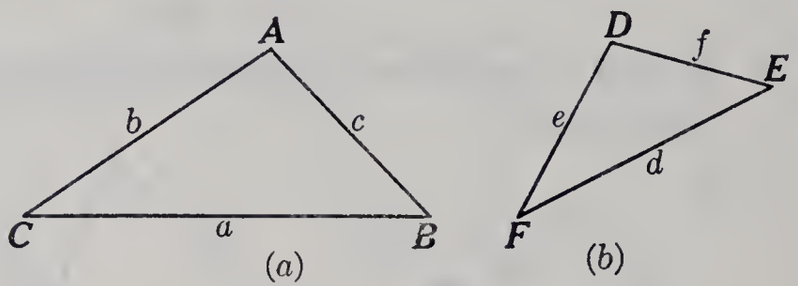


Fig. 15-10

However, we are now interested in the conditions that are *sufficient* to determine similar triangles. It has been proved that equilateral triangles are equiangular, and since corresponding sides of equilateral triangles are proportional, then all equilateral triangles are similar. This suggests that equiangular triangles are similar and leads to the aaa *Similar Triangle Theorem* which follows.

THEOREM 23

The aaa *Similar Triangle Theorem*

If two triangles are equiangular, then the corresponding sides are proportional and hence the triangles are similar.

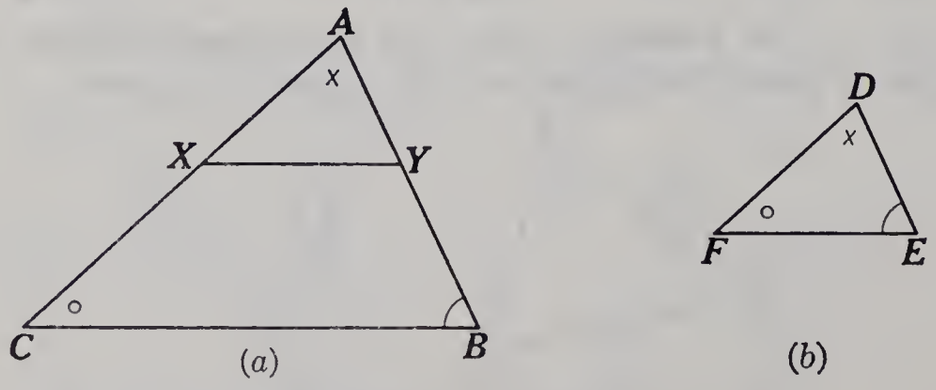


Fig. 15-11

Hypothesis: $\triangle$'s ABC and DEF such that $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ (*Fig. 15-11*)

Conclusion: (i) $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$ (ii) $\triangle ABC \sim \triangle DEF$

Analysis: Recall that the Triangle Proportionality Theorem and its corollary are the only authorities available for proving line segments proportional.

I CAN PROVE	IF I CAN PROVE
(i) 1. $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$	1. $\dfrac{AB}{DE} = \dfrac{AC}{DF}$ and $\dfrac{AB}{DE} = \dfrac{BC}{EF}$.
2. $\dfrac{AB}{DE} = \dfrac{AC}{DF}$	2. $\dfrac{AC}{AX} = \dfrac{AB}{AY}$ where $AX = DF$ and $AY = DE$.
3. $\dfrac{AC}{AX} = \dfrac{AB}{AY}$	3. $XY \parallel BC$.
4. $XY \parallel BC$	4. $\angle AXY = \angle ACB$.
5. $\angle AXY = \angle ACB$	5. $\angle AXY = \angle DFE$.
6. $\angle AXY = \angle DFE$	6. $\triangle AXY \cong \triangle DFE$.
7. $\triangle AXY \cong \triangle DFE$ by the sas postulate.	
8. $\dfrac{AB}{DE} = \dfrac{BC}{EF}$ in a similar manner.	
(ii) 9. $\triangle ABC \sim \triangle DEF$	9. $\dfrac{AB}{DE} = \dfrac{CA}{FD} = \dfrac{BC}{EF}$ and $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.
10. $\dfrac{AB}{DE} = \dfrac{CA}{FD} = \dfrac{BC}{EF}$ by part (i); $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ by hypothesis.	

Proof: Write a complete proof for this theorem and compare it with that on page 503.

Corollary: State and prove a corollary of the aaa Similar Triangle Theorem which makes use of Corollary 3 of the Triangle Angle Sum Theorem (page 376). Compare your statement and proof with that on page 503.

Example 1.

Hypothesis: In *Fig. 15-12*, $\angle ABD = \angle BDC$.

Conclusion: $\triangle EBA \sim \triangle EDC$

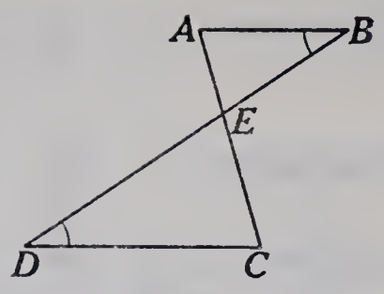


Fig. 15-12

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\triangle EBA \sim \triangle EDC$	1. Two angles of $\triangle EBA$ equal to two angles of $\triangle EDC$.
2. $\angle AEB = \angle CED$ by the Vertical Angle Theorem and $\angle EDC = \angle EBA$ by hypothesis.	

Proof: Write a complete proof for this deduction and compare it with that on page 504.

Example 2.

Hypothesis: In *Fig. 15-13*, BX and CY are altitudes of $\triangle ABC$.

Conclusion: $AB \cdot AY = AC \cdot AX$

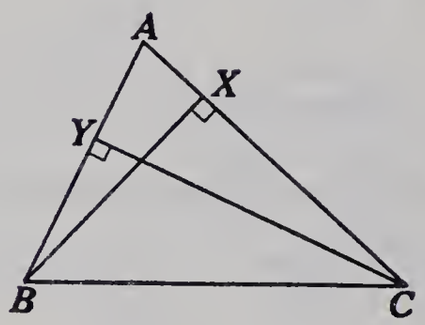


Fig. 15-13

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $AB \cdot AY = AC \cdot AX$	1. (i) $\frac{AB}{AX} = \frac{AC}{AY}$ or
	(ii) $\frac{AB}{AC} = \frac{AX}{AY}$.
(i) 2. $\frac{AB}{AX} = \frac{AC}{AY}$	2. $\triangle ABC$ equiangular to $\triangle AXY$.
3. $\triangle ABC$ equiangular to $\triangle AXY$ (insufficient information).	

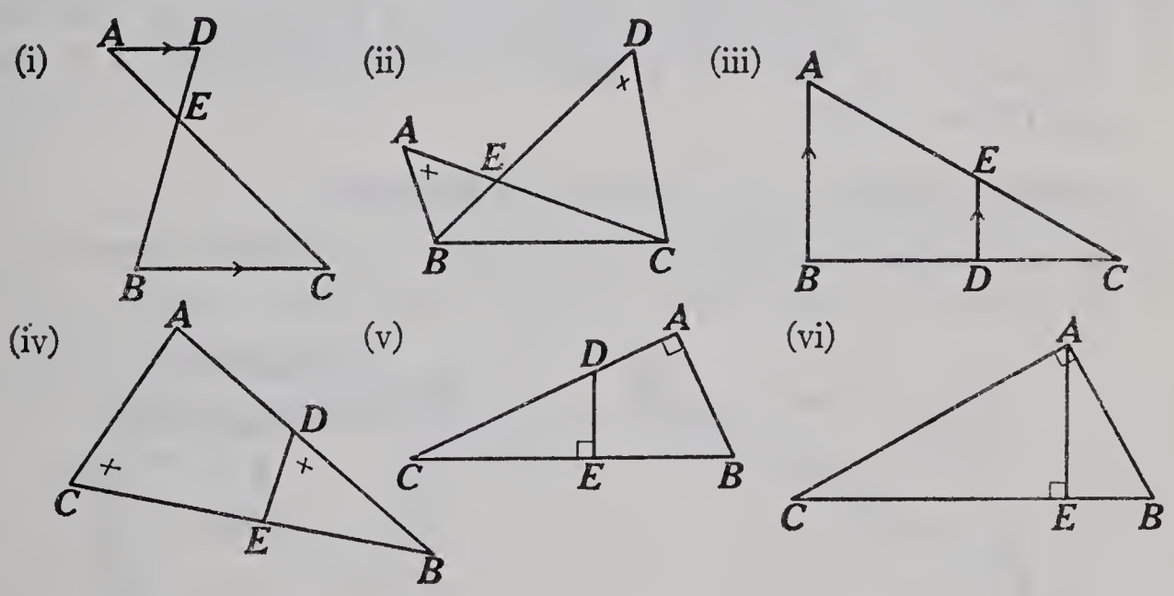
(ii) 4. $\frac{AB}{AC} = \frac{AX}{AY}$	4. $\triangle ABX$ equiangular to $\triangle ACY$.
5. $\angle BAX = \angle CAY$ by the reflexive property and $\angle AXB = \angle AYC$ by hypothesis.	

Proof: Write a complete proof for this deduction and compare your proof with that on page 504.

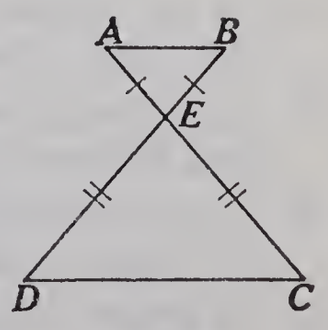
Exercise 15-6

(A)

1. Name pairs of similar triangles in each of the following diagrams; give reasons in each case:



2. For each diagram in question 1, state the proportion obtained from corresponding sides of similar triangles.



3. In the following, state the authorities required to complete the proof.

Hypothesis: In the diagram at the right,
$AE = BE$, $EC = ED$

Conclusion: $\triangle ABE \sim \triangle DCE$

Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle ABE$, $AE = BE$	1.
2. $\angle EAB = \angle EBA$	2.
3. In $\triangle EDC$, $ED = EC$	3.
4. $\angle EDC = \angle ECD$	4.
5. $\angle AEB = \angle DEC$	5.
6. $\angle AEB + \angle EBA + \angle EAB = 180°$	6.
7. $\angle DEC + \angle ECD + \angle EDC = 180°$	7.
8. $\angle EBA + \angle EAB = \angle ECD + \angle EDC$	8.
9. $\angle EBA = \angle ECD$	9.
10. $\triangle ABE \sim \triangle EDC$	10.

(B)

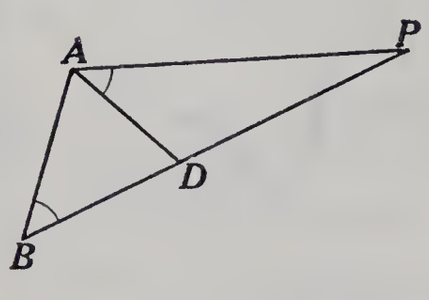
Write an analysis for the following deductions:

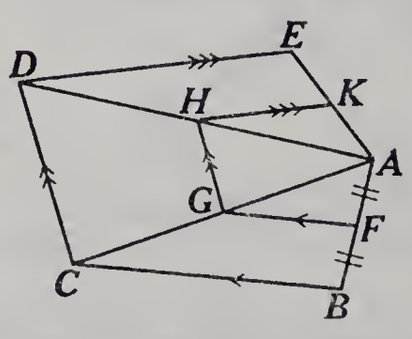
4. $ABCD$ is a trapezoid with $AB \parallel CD$. The diagonals AC and DB meet at E. Prove that $\triangle ABE \sim \triangle CDE$.

5. D is a point on side AB in $\triangle ABC$. The line on D parallel to BC meets AC at E; the line on D parallel to AC meets BC at F. Prove $\triangle ADE \sim \triangle DBF$.

Write a complete proof for each of the following:

6. In the diagram at the left below, $\angle ABP = \angle PAD$. Prove $AP^2 = BP \cdot PD$.





7. In the diagram at the right above, $FG \parallel BC$, $GH \parallel CD$, $HK \parallel DE$, $AF = FB$. Prove figure $ABCDE \sim$ figure $AFGHK$.

8. $\triangle PQR$ is right-angled at Q. QM is the altitude from Q to PR. Prove that QM is the geometric mean between PM and MR.

9. D is the midpoint of side AB in $\triangle ABC$. The line on D parallel to BC meets AC at E. Prove (i) $AE = EC$ (ii) $DE = \frac{1}{2}BC$.

10. In similar triangles, prove that corresponding altitudes are proportional to corresponding sides.

11. Any three straight lines are drawn on a point P. Two other parallel lines, neither of which lies on P nor is parallel to any of the lines on P, meet the lines on P at the points A, B, C and X, Y, Z, respectively.

Prove $\dfrac{AB}{XY} = \dfrac{BC}{YZ}$.

***15·8* An alternate proof of the Pythagorean Theorem (supplementary).** Copy the following proof and complete the authorities column; compare your solution with that on page 504.

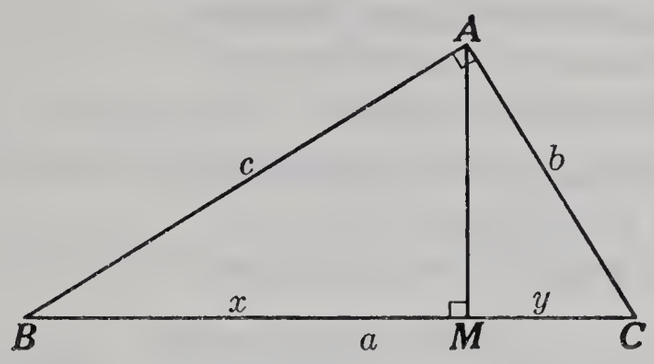


Fig. 5-14

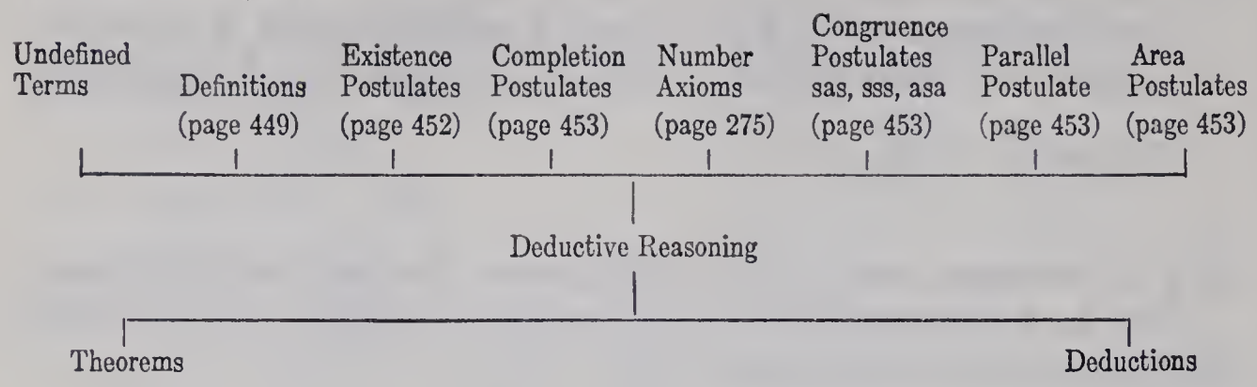
Hypothesis: $\triangle ABC$, right-angled at A, with the lengths of AB, BC, CA represented by c, a, b units, respectively

Conclusion: $b^2 + c^2 = a^2$

Proof:

STATEMENTS	AUTHORITIES
1. AM is the altitude from A to BC.	1.
2. Let x, y units represent the lengths of BM and MC, respectively.	2.
3. In $\triangle$'s AMC and BAC $\angle AMC = \angle BAC$	3.
4. $\angle MCA = \angle ACB$	4.
5. $\dfrac{y}{b} = \dfrac{b}{a}$	5.
6. $b^2 = ay$	6.
7. In $\triangle$'s AMB and CAB $\angle AMB = \angle CAB$	7.
8. $\angle MBA = \angle ABC$	8.
9. $\dfrac{x}{c} = \dfrac{c}{a}$	9.
10. $c^2 = ax$	10.
11. $b^2 + c^2 = ay + ax$	11.
12. $b^2 + c^2 = a(y + x)$	12.
13. $b^2 + c^2 = a^2$	13.

15·9 Summary.



Angles
1. Complementary Angle Theorem
2. Supplementary Angle Theorem
3. Vertical Angle Theorem

Basic Biconditional
4. Isosceles Triangle Theorem
5. Right Bisector Theorem
13. Angle Bisector Theorem

Inequality
6. Exterior Angle Theorem

Parallels
7. Parallel Line Theorem
8. Quadrilateral-Parallelogram Theorem
9. Parallelogram Theorem
10. Triangle Angle Sum Theorem

Congruence
11. aas Congruence Theorem
12. Right Triangle Congruence Theorem

Area
14. Parallelogram Area Theorem
15. Parallelogram-Diagonal Theorem
16. Parallelogram-Triangle Area Th.
17. Equal Triangle-Parallel Line Th.
18. Pythagorean Theorem
19. Pythagorean Converse
20. Trapezoid Area Theorem

Proportionality
21. Triangle Area Ratio Th.
22. Triangle Proportionality Th.

Similar Triangles
23. aaa Similar Triangle Th.

DEFINITIONS

Acute angle. An acute angle is an angle with degree measure less than 90.

Acute triangle. An acute triangle is a triangle determining three acute angles.

Altitude of a triangle. The altitude of a triangle is the line segment drawn from a vertex perpendicular to the opposite side and terminating on the opposite side.

Angle. An angle is the set of all points on two distinct rays with a common end point.

Angle sum of two angles. The angle sum of two angles is the sum of the measurements of the two angles.

Betweeness on a line. If A, B, C are three points on a line, B is between A and C if $AC = AB + BC$.

Bisect. To bisect is to divide into two equal parts.

Centroid. The centroid of a triangle is the point of intersection of the medians.

Circle. A circle is a point set such that each member of the set is the same distance (radius) from a fixed point (centre).

Circumcentre. The circumcentre of a triangle is the point of intersection of the right bisectors of the sides of a triangle.

Circumscribed circle of a triangle. The circumscribed circle of a triangle is the circle on the three vertices.

Collinear points. Collinear points are points on the same straight line.

Complementary angles. Two angles whose angle sum is 90° are complementary angles.

Concurrent lines. Concurrent lines are three or more lines on the same point.

Congruence of triangles. If a correspondence $ABC \leftrightarrow DEF$ between the vertices of two triangles is such that every pair of corresponding sides is congruent and every pair of corresponding angles is congruent, then this correspondence is a congruence matching and $\triangle ABC \cong \triangle DEF$.

Converse statements. When two conditional statements are so related that the hypothesis and conclusion consist of single statements which are

interchanged, then each conditional statement is said to be the converse of the other.

Converse theorem. A converse theorem is a theorem that is formed from a given theorem by interchanging any number of statements in the hypothesis of the given theorem with an equal number of statements in the conclusion.

Corollary. A corollary of a theorem is an implication which follows as a direct consequent of the theorem.

Degree. A degree is $\frac{1}{180}$ of a straight angle.

Dihedral angle. A dihedral angle is the point set consisting of the union of a line and two non-coplanar half-planes having the line as a common edge.

Equilateral triangle. An equilateral triangle is a triangle with all sides equal.

Exterior of an angle. The exterior of an angle is the point set not contained in an angle or its interior.

Exterior angle of a triangle. See page 351.

Half-line. A half-line is the point set consisting of all points on a line which are on one side of a given point on the line.

Half-plane. A half-plane is the point set consisting of all points in a given plane which are on one side of a given line.

Interior of an angle. A point D is in the interior of $\angle ABC$ if D is on the same side of line BA as C and on the same side of line BC as A.

Isosceles triangle. An isosceles triangle is a triangle with two sides equal.

Line segment. A line segment is the point set consisting of two distinct points on a line and all the points between them.

Median of a triangle. The median of a triangle is the line segment joining any vertex to the midpoint of the opposite side.

Midpoint of a line segment. If AB is a line segment and C is a point on the segment, then C is the midpoint if $AC = CB$.

Obtuse angle. An obtuse angle is an angle with degree measure greater than 90 and less than 180.

Obtuse triangle. An obtuse triangle is a triangle with one angle an obtuse angle.

Orthocentre. The orthocentre of a triangle is the point of intersection of the altitudes.

Parallel lines. If two distinct lines lie in the same plane and have no point in common, then the two lines are parallel.

Parallel line segments. If two distinct line segments are segments of parallel lines, then the two line segments are parallel.

Parallelogram. If a quadrilateral has both pairs of opposite sides parallel, then it is a parallelogram.

Perpendicular distance. Perpendicular distance is the length of a line segment from a point to a line (or segment) at right angles to the line.

Polygon. If the beginning and end of a polygonal path consisting of three or more line segments coincide, then the path is called a polygon.

Postulate. A postulate is an assumption.

Proportion. A proportion is the statement of equality of two ratios.

Radius of a circle. The radius of a circle is a line segment with end points, the centre, and a point on the circle.

Ratio of line segments. The ratio of line segments is the ratio of the number of units in the lengths of the line segments.

Ratio of areas of triangles. The ratio of the areas of triangles is the ratio of the number of square units in the areas of the triangles.

Ray. A ray is the point set consisting of a point on a line and all the points on the line on one side of the given point.

Remote interior angles. See page 351.

Right angle. A right angle is an angle whose degree measure is 90.

Right bisector of a line segment. The right bisector of a given line segment is the line which bisects the line segment at right angles.

Rhombus. A rhombus is a parallelogram with a pair of adjacent sides equal.

Scalene triangle. A scalene triangle is a triangle with no two sides of equal length.

Similar figures. Geometric figures are similar if:

(i) corresponding angles are equal, and

(ii) corresponding sides are proportional.

Simple polygon. If the beginning and end of a polygonal path consisting of three or more line segments coincide and no two sides intersect, then the path is called a simple polygon.

Simple polygonal region. A simple polygonal region is the union of triangular regions such that if any two of these intersect, the intersection is a line segment or a point.

Straight angle. A straight angle is an angle whose degree measure is 180.

Supplementary angles. Supplementary angles are angles whose angle sum is 180°.

Transversal. A transversal is a line intersecting two or more lines or line segments at distinct points.

Trapezoid. A trapezoid is a quadrilateral with one pair of opposite sides parallel.

Triangle. A triangle is the set of all points on the line segments determined by three points which are not all on one line.

Triangular region. A triangular region is the union of a triangle and its interior.

Vertical angles. Two angles are vertical angles if their sides form two pairs of opposite rays.

SUMMARY OF POSTULATES

In our study of geometry, we made many assumptions; this summary is a partial list of these.

Existence Postulates.

1. An unlimited number of lines in one plane may lie on one point in the plane.
2. There exists one and only one line which lies on two distinct points.
3. Two distinct lines intersect on, at most, one point.
4. A plane contains at least three non-collinear points.
5. Any three non-collinear points determine a plane.
6. If two distinct planes intersect, they intersect on a line.
7. If two distinct points lie in a plane, then the line containing these points lies in the same plane.
8. Two intersecting straight lines determine a plane.
9. A line intersects a plane not containing the line in, at most, one point.
10. There exists an unlimited number of angles equal to an angle.
11. There exists one and only one ray which bisects an angle.
12. There exists one and only one right bisector of a line segment.

Measurement Postulates.

13. There is a one-to-one correspondence between the points on a line and the real numbers.
14. To every line segment there corresponds a unique positive real number of units called the length of the line segment or the distance between its end points.
15. To each geometric angle there corresponds a unique positive real number of degrees; the degree measure of an angle is greater than zero and equal to or less than 180.

Betweeness Postulates.

16. If three distinct points lie on a line, one and only one of the points is between the other two.
17. Between the two end points on a line segment AB, there is one and only one point C such that $AC = CB$. (C is the midpoint of AB.)
18. Between any two distinct points on a line there exists at least one point.

Completion Postulates.

19. *Line segments.* See page 241.
20. *Angles.* See page 245.
21. *Areas.* See area postulate 28, below.

Congruence Postulates.

22. Equal line segments are congruent; congruent line segments are equal.
23. Equal angles are congruent; congruent angles are equal.
24. If two sides and the contained angle of one triangle are respectively equal to two sides and the contained angle of another triangle, the two triangles are congruent (sas).
25. If two triangles have the three sides of one respectively equal to the three sides of the other, then the triangles are congruent (sss).
26. If two triangles have two angles and the contained side of one respectively equal to two angles and the corresponding side of the other, then the two triangles are congruent (asa).

Parallel Postulate.

27. If l is any line and A is any point not on l, then there is one and only one line on A which is parallel to line l.

Area Postulates.

28. The area of a polygonal region is the sum of the areas of the component regions whose union is the polygonal region.
29. If two polygonal regions are congruent, then they have the same area.
30. Two polygonal regions are equal in area if they can be separated into a finite number of polygons which are congruent in pairs.
31. For every polygonal region, there exists a unique positive real number of units which is called the area of the region.

MISCELLANEOUS DEDUCTIONS

Chapters 8-13

1. Prove that the altitudes of an equilateral triangle are equal.
2. The equal sides QP and RP of an isosceles $\triangle PQR$ are extended to E and F, respectively, so that $PE = PF$. Join QF and RE and prove $QF = RE$.
3. In $\triangle ABC$, $AB = AC$. If AB and AC are extended, the exterior angles so formed are obtuse.
4. If the diagonals of a parallelogram are equal, all of its angles are right angles.
5. If two altitudes of a triangle are equal, then two of the medians are equal.
6. In $\triangle ABC$ the bisectors of angles B and C meet on O. A line on O parallel to BC meets AB on E and AC on F. Prove $EF = EB + FC$.
7. In $\triangle ABC$, D and E are the midpoints of AB and AC, respectively. A line on C parallel to BA meets DE produced at F. Prove that $DF \parallel BC$ and $DE = \frac{1}{2} BC$.
8. D is the midpoint of the side AB of $\triangle ABC$. Prove that a line on D parallel to BC bisects AC.
9. In $\triangle ABC$, $\angle A = \angle B + \angle C$. D is the midpoint of BC. Prove $BC = 2\,AD$.
10. AB is a given line, and P is any point not on line AB. Prove that not more than two line segments of a given length can be drawn from P to line AB.
11. If in $\triangle ABC$ perpendiculars drawn from B and C to the opposite sides meet at O, prove $\angle BOC = \angle ABC + \angle ACB$.
12. In any quadrilateral, (i) the line segments on the midpoints of the sides in order form a parallelogram and (ii) the line segments on the midpoints of opposite sides of the quadrilateral bisect each other.
13. Prove that the bisectors of the angles of a triangle are concurrent.
14. Prove that two medians of a triangle intersect at a point of trisection remote from a vertex; hence, prove that the medians are concurrent.
15. Prove that the right bisectors of the sides of a triangle are concurrent.
16. By drawing a line through each vertex parallel to the opposite side, prove that the altitudes of a triangle are concurrent.
17. The exterior angles at B and C of $\triangle ABC$ are bisected, and the bisectors meet on D. If the number of degrees in angle A is x, find the number of degrees in angle D in terms of x.

Chapters 8-15

1. If in $||^{gm} ABCD$, X and Y are any points on CD and DA, respectively, prove that triangles ABX and BCY are equal.
2. On the same base with a given triangle, construct a right triangle equal to the given triangle.
3. In the trapezoid $ABCD$, AB is parallel to CD, and P is the midpoint of BC. A line XPY is drawn on P parallel to AD, meeting AB extended on X and DC on Y. Prove trapezoid $ABCD = ||^{gm} AXYD$.
4. In $||^{gm} PQRS$, A is any point on PS extended. QA cuts RS on B. Prove $\triangle PBA = \triangle SRA$.
5. P is any point on the side AB of $||^{gm} ABCD$. Prove $\triangle PAD + \triangle PBC = \triangle PDC$.
6. P is any point on diagonal DB of $||^{gm} ABCD$. L is on AD, and M is on BC, so that $LPM \,||\, DC$. Q is on AB, and R is on DC, so that $QPR \,||\, BC$. Prove that figure $AQPL$ = figure $RPMC$.
7. E and F are the midpoints of AC and AB, respectively, in $\triangle ABC$. Prove $\triangle BEF = \triangle CEF$, and hence $EF \,||\, BC$.
8. Prove that a parallelogram is bisected by any straight line which is on the point of intersection of its diagonals.
9. In $\triangle ABC$, D is a point on BC. CE is drawn through C parallel to AD meeting BA extended at E. Prove $\triangle ABC = \triangle EBD$.
10. ABC is an equilateral triangle in which each side is 2 units. Prove that $\triangle ABC$ is $\sqrt{3}$ square units.
11. In $\triangle ABC$, $\angle A = 90°$. E is a point on AC, and D is a point on AB. Prove $BE^2 + CD^2 = ED^2 + BC^2$.
12. If from any point P in the interior of $\triangle ABC$ perpendiculars PX, PY, PZ are drawn to BC, CA, AB, respectively, prove that $AZ^2 + BX^2 + CY^2 = AY^2 + CX^2 + BZ^2$.
13. $\triangle ABC$ is right-angled at A. The altitude from A intersects BC on D.
    Prove: (i) $\dfrac{BD}{AD} = \dfrac{AD}{DC}$ (ii) $\dfrac{BD}{AB} = \dfrac{AB}{BC}$ (iii) $\dfrac{CD}{AC} = \dfrac{AC}{BC}$
14. In $\triangle ABC$, the bisector of $\angle A$ intersects BC on D. The line on C parallel to AD intersects BA extended on E. Prove: (i) $AE = AC$ (ii) $BD : DC = AB : AC$
15. $ABCD$ is a trapezoid with $AB \,||\, CD$. The diagonals AC and BD intersect on P. If $\triangle APD \sim \triangle CPB$, prove $AD = BC$.
16. $\triangle ABC$ and $\triangle DEF$ are similar triangles. AM and DN are corresponding altitudes.
    Prove: (i) $AM : DN = BC : EF$ (ii) $\triangle ABC : \triangle DEF = BC^2 : EF^2$

TEST PAPERS 1-4

Chapters 8-13

TEST PAPER 1

THEOREM

1. If two lines intersect, then the vertical angle pairs are equal.

CALCULATIONS

2. Lines AB and CD intersect on O. OX and OY are the bisectors of angles DOB and AOD, respectively. Calculate the measurement of $\angle XOY$.

3. ABC is an isosceles triangle with $AB = AC$ and $\angle ACB = 70°$. The exterior angles formed by extending AB and AC are bisected, the bisectors meeting at D. Find the measurement of $\angle BDC$.

4. In quadrilateral $ABCD$, BD bisects $\angle ABC$, $\angle C = 50°$, $\angle ADB = 40°$, and $\angle DBC = 60°$. Calculate the measurement of the exterior angle at A.

DEDUCTIONS

5. If ABC and DBC are two isosceles triangles on the same side of the same base BC, prove that AD extended is perpendicular to BC.

6. $ABCD$ is a square. Y is a point on AB, and X is a point on BC such that $AY = BX$. Prove that AX and DY intersect at right angles.

7. (a) Define: rhombus, supplementary angles, median, parallelogram.
(b) (i) Explain the method of proof in an indirect proof.
(ii) How is a converse theorem related to the original theorem?
(c) AD, BE, and CF are diameters of a circle. Prove $\triangle ABC \cong \triangle DEF$.

CONSTRUCTION

Draw the figure using only ruler and compasses; show all construction lines, but do not give proof or description of construction:

8. On a straight line AB two inches in length, construct a $\triangle ABC$ having $\angle A$ equal to 60° and $\angle B$ equal to 75°. Produce CA to D and CB to E. Bisect $\angle DAB$ and $\angle ABE$, and let the bisectors meet on F. From F draw $FG \perp AB$, meeting AB on G. With centre F and radius FG, draw a circle. Measure FG.

TEST PAPER 2

THEOREM

1. State and prove the Isosceles Triangle Theorem.

CALCULATIONS

2. AB and CD are two straight lines intersecting at E and such that $\angle CEB = 130°$. EF and EG are the bisectors of angles AEC and DEB, respectively. Calculate the measurement of $\angle FEG$.
3. Two circles with centres A and B intersect on C and D. AB intersects CD on E. If $AE = 2$ cm., calculate the measurement of BE.
4. In $\triangle ABC$, $\angle A = 90°$ and the bisectors of angles B and C meet on D. Calculate the measurement of $\angle BDC$.
5. In quadrilateral $ABCD$, $\angle B = \angle C = 140°$, $\angle A = 40°$. Calculate the measurement of $\angle D$.

DEDUCTIONS

6. Two isosceles triangles, ABC and DBC, are on opposite sides of the same base, BC. Prove that AD bisects BC at right angles.
7. BAC is an angle and P is a point on the bisector of $\angle BAC$. The right bisector of AP is drawn, cutting AB on X and AC on Y. Prove XP is parallel to AYC.
8. (a) Define a square.
   (b) $ABCD$ is a square, and E, F, G, H are points on AB, BC, CD, DA, respectively, such that $AE = BF = CG = DH$. Show that E, F, G, H are the vertices of a square.

TEST PAPER 3

THEOREM

1. State and prove the Exterior Angle Theorem.

CALCULATIONS

2. ABC is an isosceles triangle with $AB = AC$ and $\angle A = 60°$. AD is an altitude. If $BD = 3$ cm., calculate the measurement of AB.
3. The side BA of $\triangle ABC$ is extended to D. AE bisects $\angle DAC$ and is parallel to BC. If $\angle BAC = 70°$, calculate the measurements of $\angle B$ and $\angle C$.
4. D is the midpoint of the side YZ of $\triangle XYZ$. If $DX = DY$, calculate the measurement of $\angle YXZ$.
5. In $\triangle ABC$, the bisectors of angles A and B meet on D. If $\angle C = 50°$, calculate the measurement of $\angle ADB$.

DEDUCTIONS

6. ADE is an isosceles triangle having $AD = AE$. The base DE is extended to B, and ED is extended to C, such that $DB = EC$. Prove that ABC is an isosceles triangle.
7. $ABCD$ is a quadrilateral with $\angle A = \angle C$, and $\angle B = \angle D$. Prove $ABCD$ is a parallelogram.
8. ABC is a triangle. AB is extended to D so that $BD = BC$. A line BE is drawn through B parallel to DC and meeting AC on E. Prove $\angle BDC = \frac{1}{2}\ \angle ABC$, and also that BE bisects $\angle ABC$.

CONSTRUCTION

Draw the figure using only ruler and compasses; show all construction lines, but do not give proof or description of construction:

9. Construct a quadrilateral $ABCD$ from the following measurements: $AB = 1\frac{3}{4}$ inches, $BC = 2\frac{1}{4}$ inches, $AD = 2\frac{1}{2}$ inches, $\angle BAC = 60°$, and $\angle ABD = 22\frac{1}{2}°$. Circumscribe a circle about $\triangle ABD$ and measure its radius.

TEST PAPER 4

THEOREM

1. State and prove the Parallel Line Theorem.

CALCULATIONS

2. The side AB of $\triangle ABC$ is extended to D so that $BD = BC$. If $AB = AC$ and $\angle D = 25°$, calculate the measurement of $\angle A$.
3. AB, CD, EF are diameters of a circle. If $AC = 2$ cm., $CE = 3$ cm., and $AE = 6$ cm., calculate the measurements of BD, DF, and BF.
4. The bisectors of the exterior angles at A and B of $\triangle ABC$ meet on P. If $\angle C = 50°$, calculate the measurement of $\angle P$.
5. Transversal EH meets the parallel lines AB and CD at F and G, respectively. If $\angle AFG = (x + 15)°$, calculate the measurement of $\angle DGH$.

DEDUCTIONS

6. In any triangle, show that the ends of the base are equally distant from the median drawn to the base.
7. DE is parallel to base AB of isosceles triangle CAB and meets CA and CB on D and E, respectively. AE and BD intersect on F. Prove that AFB is an isosceles triangle.
8. If the diagonal AC of quadrilateral $ABCD$ bisects the angles at A and C, prove it is perpendicular to the other diagonal.

TEST PAPERS 5-8

Chapters 8-15

TEST PAPER 5

THEOREM

1. State and prove the Parallelogram-Triangle Area Theorem.

CALCULATIONS

2. BE and CF are altitudes of $\triangle ABC$. If $AC = 12$ cm., $BE = 8$ cm., and $CF = 10$ cm., calculate the length of AB.
3. The area of a right-angled isosceles triangle is 32 square inches. Calculate the length of each of the sides.
4. In $\triangle ABC$, $\angle B = \angle C = 2\angle A$. The bisector of $\angle B$ meets AC at D. If $BD = 4$ cm., calculate the length of BC.
5. Calculate the area of a rhombus, with length of diagonals 4 inches and 6 inches, respectively.

DEDUCTIONS

6. Construct a line segment of length $\sqrt{20}$ inches.
7. In $\triangle ABC$, $\angle B = 55°$, $\angle C = 45°$. Calculate the measurement of the acute angle between the bisector of $\angle BAC$ and the perpendicular from A to BC. Write a complete proof.
8. A straight line drawn perpendicular to BC, the base of an isosceles $\triangle ABC$, cuts AB at X and CA extended at Y. Prove that $\triangle AXY$ is isosceles.

CONSTRUCTION

Draw the figure using only ruler and compasses; show all construction lines, but do not give proof or description of construction:

9. Construct a rectangle with sides of lengths 8 cm. and 5 cm. Measure and record the length of a diagonal; check this measurement by calculation. Construct a parallelogram equal in area to the rectangle, having one pair of opposite angles 75°.

TEST PAPER 6

THEOREM

1. State and prove the Pythagorean Theorem.

CALCULATIONS

2. A ladder 25 ft. long has one end resting against the wall of a building and the other end on the ground a distance of 7 ft. from the wall. Calculate how high the upper end of the ladder is above the ground.
3. The distance between the parallel sides AD and BC of a trapezoid $ABCD$ is 8 cm. If $AD = 5$ cm. and $BC = 8$ cm., calculate the area of the trapezoid.
4. Find the diameter of the smallest log which will provide a square timber measuring 12 inches to a side.
5. $\triangle ABC$ is right-angled at B. E is the midpoint of the median AD. Calculate the area of $\triangle AEB$, if $AC = 13$ cm. and $AB = 12$ cm.

DEDUCTIONS

6. P is any point in the interior of rectangle $ABCD$. Prove that $PA^2 + PC^2 = PB^2 + PD^2$.
7. Any line through the point of intersection of the diagonals of a parallelogram bisects the parallelogram.
8. $ABCD$ is a quadrilateral in which $\angle ABC$ is equal to $\angle BCD$ and AD is parallel to BC. Prove that AB is equal to CD.

TEST PAPER 7

THEOREM

1. State and prove the Triangle Area Ratio Theorem.

CALCULATIONS

2. The medians BD and CE of $\triangle ABC$ intersect at F. If the area of $\triangle DFC$ is 3 sq. cm. and that of $\triangle FBC$ is 4 sq. cm., find the area of the triangle EBF and the quadrilateral $AEFD$.
3. A square has the same area as a rectangle with base 15 cm. and diagonal 17 cm. Calculate the length of the side of the square correct to one decimal place.
4. The diagonal BD of quadrilateral $ABCD$ bisects diagonal AC on M. $AX \perp BD$, and X is on BD. If $AX = 3$ cm. and $BD = 5$ cm., find the area of quadrilateral $ABCD$.
5. The $\triangle ABC$ is right-angled at A. $BC = 13$ in. and $AB = 5$ in. D is any point on BC, and E is the midpoint of AD. Calculate the area of $\triangle CEB$.

DEDUCTIONS

6. ABC is a right-angled triangle with right angle at B. P is any point on AB, and Q is any point on BC. Prove that $PC^2 + AQ^2 = PQ^2 + AC^2$.

7. Prove that the diagonals of a quadrilateral divide it into four triangles whose areas are proportional.
8. The straight line drawn from the midpoint of one side of a triangle parallel to another side bisects the third side.

CONSTRUCTION

Draw the figure using only ruler and compasses; show all construction lines, but do not give proof or description of construction:

9. Construct $\triangle ABC$ with $AB = 2\frac{3}{4}$ inches, $AC = 3\frac{1}{4}$ inches, and $BC = 3\frac{3}{4}$ inches. Circumscribe a circle about $\triangle ABC$. Take any point on the circle and draw perpendiculars from this point to the three sides of $\triangle ABC$. Join the three points where the perpendiculars meet the sides and measure the angle so formed.

TEST PAPER 8

THEOREM

1. State and prove the Triangle Proportionality Theorem.

CALCULATIONS

2. P is any point on the side AB of parallelogram $ABCD$. If $BC = 10$ cm. and the length of a perpendicular drawn from A to BC is 8 cm., calculate the area of $\triangle PCD$.
3. AX is an altitude of $\triangle ABC$, and X is between B and C. If $AB = 41$ cm., $BX = 9$ cm., and $XC = 30$ cm., calculate the length of AC.
4. The lengths of the diagonals of a rhombus are 16 cm. and 30 cm. Calculate the length of a side of the rhombus.
5. The lengths of base and altitude of parallelogram $ABCD$ are 12 cm. and 8 cm., respectively. P is any point in the interior of the parallelogram. Calculate the sum of the areas of $\triangle PAD$ and $\triangle PBC$.

DEDUCTIONS

6. Square $ABGE$ and $ACHF$ are drawn externally on two sides of a triangle ABC. Prove $EC = BF$.
7. $PQRS$ is a trapezoid with $PS \parallel QR$. A line on M, the midpoint of PQ, parallel to QR meets SR on N. Prove $SN = NR$.
8. Prove that corresponding altitudes of similar triangles are proportional to corresponding sides.

SOLUTIONS

Review A

Section A.1 (page 2)

1. (i) A is the set of even whole numbers from two to ten inclusive.
(ii) $6 \in A$

2. (i) The set of odd whole numbers from three to nine inclusive.
(ii) The set of fractions with denominators two and numerators the odd whole numbers from one to seven inclusive.

3. (i) 4 members $\{1, 2, 4, 9\}$ (ii) 4 members $\{a, b, c, d\}$
(iii) 2 members $\{1, 2\}$ (iv) 1 member $\{0\}$
(v) 4 members $\{0, 2, 3, 4\}$ (vi) 2 members $\{a, m\}$

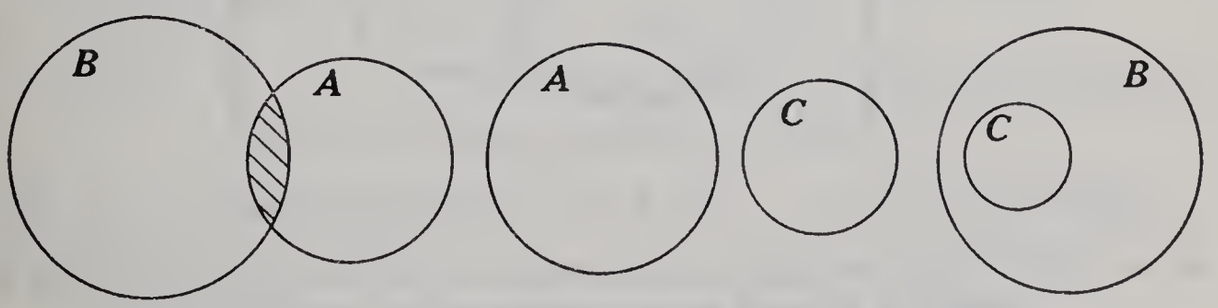
Section A.4 (page 5)

1. One member sets: $\{a\}, \{b\}, \{c\}, \{d\}$, _____ 4
Two member sets: $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$, _____ 6
Three member sets: $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$, _____ 4
Is $\{a, b, c, d\}$ a subset of A? Since every member of $\{a, b, c, d\}$ is also a member of A, the definition allows $\{a, b, c, d\}$ to be considered a subset of A, but it is not a *proper subset* of A. The term *proper subset* is used to refer to a subset which does not contain all the elements of the original set.
The null set, $\emptyset$, is considered to be a subset of every set. Thus to complete the list of subsets the following must be included:
no member subsets: $\emptyset$, _____ 1
four member subsets: $\{a, b, c, d\}$, _____ 1
Total 16

By counting it is observed that the four member set, A, has 16 or 2^4 subsets including the null set and A itself. It may be shown, in general, that if a set, A, has n elements, where n represents a counting number, that the number of subsets of A is 2^n.

Section A.5 (page 6)

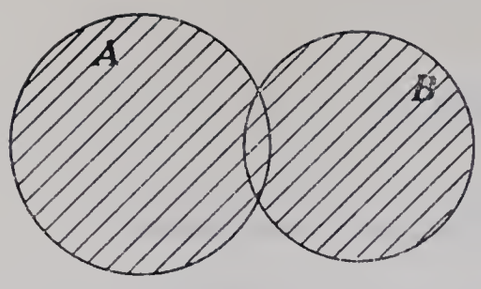
1.



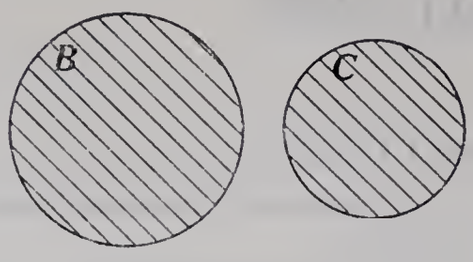
(i) (iii) (vi)

(i) Since some members are common to A and to B, the circles overlap.
(ii) $A \cap B = \{c, d, e\}$; c, d, and e are the members which belong to both sets A and B.
(iii) Since no members belong to *both* sets, the circles do not overlap.
(iv) The intersection set of A and B.
(v) $B \cap C = \{f, g, h\}$
(vi) $C \subset B$, or C is a proper subset of B because all the members of C are also members of B, but C does not include all the members of B. Hence the circle which represents the set C must be entirely inside the circle representing set B.

2.



(i)



(iv)

(i), (iii) Since the members 3 and 4 belong to both A and B, the circles overlap.

(ii) $A \cup B = \{1, 2, 3, 4, 5, 6\}$; each member of A and each member of B appears once in the list.

(iv) There is no member common to B and C; hence the sets are *disjoint*, and the circles do not overlap.

(v) B or $C = \{3, 4, 5, 6, 7, 8, 9\}$

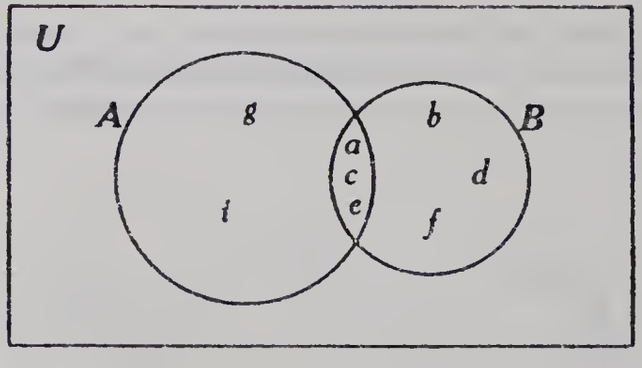
(vii) $n(B \cup C) = n(B) + n(C)$ if B and C are disjoint sets.

(viii) $n(A \cup B) = 6$

It should be noted that when the intersection of two sets is the null set, the number of members in the union of the sets is equal to the sum of the number of members in each set, but if the intersection of two sets is not the null set, the number of members in the union of the sets is less than the sum of the number of members in each set. In the example in this problem:

$$B \cap C = \emptyset,\ n(B \cup C) = n(B) + n(C)$$
$$A \cap B \neq \emptyset,\ n(A \cup B) < n(A) + n(B)$$
$$\text{in fact, } n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

3.



(i) $A \cap B = \{a, c, e\}$ (ii) $A \cup B = \{a, c, e, g, i, b, d, f\}$

4. (a) (i) $B \subset A$ is true because each of the members of B, i.e., 1, 3, 5, 7, 9, is a member of A, but B does not contain all the members of A.

(ii) $C \subseteq A$ is true because 2, 4, 6, and 8 are members of A.

(iii) $D \subseteq A$ is true because 2, 3, 5, and 7 are members of A.

(iv) $\emptyset \subseteq A$ is true by agreement.

(b) (i) $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A \cap B = \{1, 3, 5, 7, 9\}$

(ii) $A \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A \cap C = \{2, 4, 6, 8\}$

(iii) $B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $B \cap C = \emptyset$

(iv) $A \cup D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A \cap D = \{2, 3, 5, 7\}$

(v) $C \cup D = \{2, 3, 4, 5, 6, 7, 8\}$, $C \cap D = \{2\}$

(vi) $(B \cup C) \cup D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

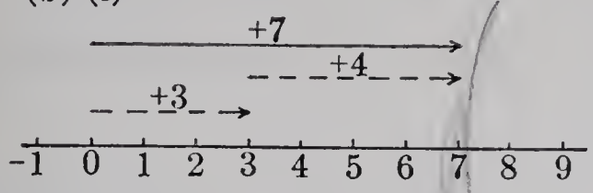
(c) (i) $2^9 = 512$ (ii) $2^5 = 32$ (iii) $2^4 = 16$ (iv) $2^4 = 16$

Review B

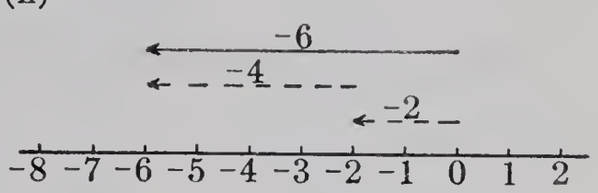
Section B.5 (page 16)

1. (a) (i) $(+3)+(+4) = +(3+4) = +7$ (ii) $(-2)+(-4) = -(2+4) = -6$
$(+4)+(+3) = +(4+3) = +7$ $(-4)+(-2) = -(4+2) = -6$
(iii) $(-6)+(+3) = -(6-3) = -3$
$(+3)+(-6) = -(6-3) = -3$

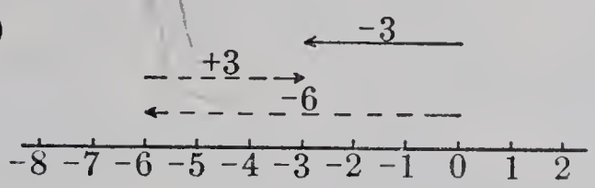
(b) (i)



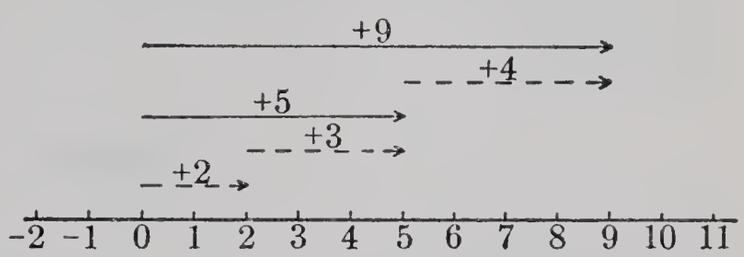
(ii)



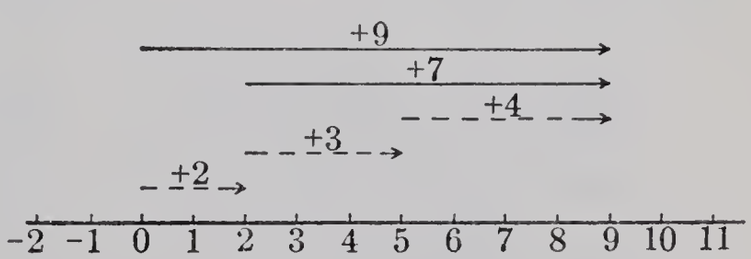
(iii)



2. (i) $[(+2)+(+3)]+(+4)$
$= [+(2+3)]+(+4)$
$= (+5)+(+4)$
$= +(5+4)$
$= +9$

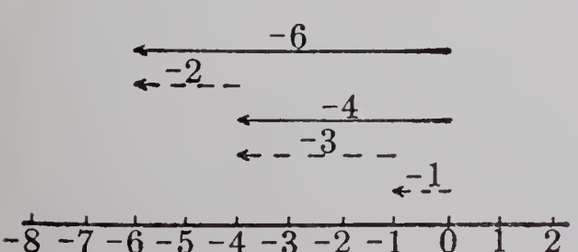


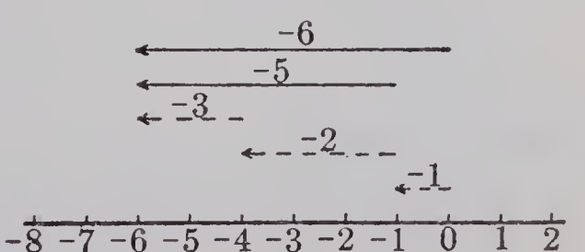
$(+2)+[(+3)+(+4)]$
$= (+2)+[+(3+4)]$
$= (+2)+(+7)$
$= +(2+7)$
$= +9$



(ii) $[(-1)+(-3)]+(-2)$
$= [-(1+3)]+(-2)$
$= (-4)+(-2)$
$= -(4+2)$
$= -6$

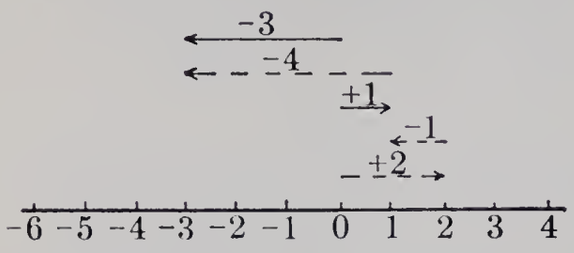
$(-1)+[(-3)+(-2)]$
$= (-1)+[-(3+2)]$
$= (-1)+(-5)$
$= -(1+5)$
$= -6$

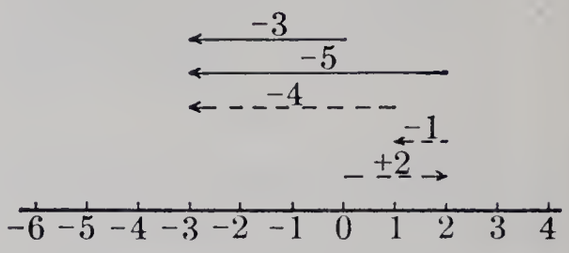




(iii) $[(+2) + (-1)] + (-4)$
$= [+(2 - 1)] + (-4)$
$= (+1) + (-4)$
$= -(4 - 1)$
$= -3$

$(+2) + [(-1) + (-4)]$
$= (+2) + [-(1 + 4)]$
$= (+2) + (-5)$
$= -(5 - 2)$
$= -3$





Section **B.12** (page 27)

1.
$$\begin{array}{r} 7x^2 - x + 2 \\ -10x^2 + 3x - 5 \\ x^2 - 2x + 7 \\ \hline -2x^2 \qquad + 4 \end{array}$$

or

$(7x^2 - x + 2) + (3x - 10x^2 - 5) + (x^2 - 2x + 7)$
$= 7x^2 - x + 2 + 3x - 10x^2 - 5 + x^2 - 2x + 7$
$= -2x^2 + 4$

2.
$$\begin{array}{r} 5x + 3y - 2 \\ 3x - 4y + 3 \\ \hline 8x - y + 1 \\ 6x + 3y - 6 \\ \hline 2x - 4y + 7 \end{array}$$

or

$(5x + 3y - 2) + (3x - 4y + 3) - (6x + 3y - 6)$
$= 5x + 3y - 2 + 3x - 4y + 3 - 6x - 3y + 6$
$= 2x - 4y + 7$

Chapter I

Section **1.8** (page 42)

1. (i) $\dfrac{2}{3} + \dfrac{-5}{4} - \dfrac{7}{-2}$

$= \left(\dfrac{2}{3} + \dfrac{-5}{4}\right) + \dfrac{-7}{-2}$

$= \dfrac{2(4) + (-5)(3)}{12} + \dfrac{7}{2}$

$= \dfrac{8 + (-15)}{12} + \dfrac{7}{2}$

$= \dfrac{-7}{12} + \dfrac{7}{2}$

$= \dfrac{(-7)1 + 7(6)}{12}$

$= \dfrac{-7 + 42}{12}$

$= \dfrac{35}{12}$

(ii) $\dfrac{5}{xy} - \dfrac{3}{xy} + \dfrac{-2}{xy}$

$= \left(\dfrac{5}{xy} - \dfrac{3}{xy}\right) + \dfrac{-2}{xy}$

$= \dfrac{5 - 3}{xy} + \dfrac{-2}{xy}$

$= \dfrac{2}{xy} + \dfrac{-2}{xy}$

$= \dfrac{2 + (-2)}{xy}$

$= \dfrac{0}{xy}$

$= 0$

Section **1.9** (page 44)

1. $\dfrac{-2}{-3} = \dfrac{2}{3}$; $\dfrac{-4}{3} = -1\frac{1}{3}$; $\dfrac{-5}{-5} = 1$; $\dfrac{4}{-1} = -4$

Then with the elements arranged in increasing order, the set is:

$$\left(\frac{4}{-1},\ -2,\ -1\tfrac{2}{3},\ \frac{-4}{3},\ \frac{-2}{-3},\ \frac{-5}{-5}\right)$$

NOTE: The sum $-1 + \left(-\frac{1}{3}\right) = \frac{-3 + (-1)}{3} = \frac{-4}{3} = -\frac{4}{3}$

It is sometimes convenient to write rational numbers in *mixed number form* when the numerator is greater than the denominator.

Thus, $-\frac{4}{3} = -1\frac{1}{3}$ where $-1\frac{1}{3}$ means $-\left(1 + \frac{1}{3}\right)$ or $-1 + \left(-\frac{1}{3}\right)$,

and $+\frac{19}{7} = +2\frac{5}{7}$ where $+2\frac{5}{7}$ means $+\left(2 + \frac{5}{7}\right)$ or $+2 + \left(+\frac{5}{7}\right)$.

Section 1.10 (page 47)

1. (i) Let $x = .374999\ldots$

$10{,}000x = 3749.99\ldots$

$1{,}000x = 374.99\ldots$

$\therefore 9{,}000x = 3{,}375$

$\therefore x = \frac{3{,}375}{9{,}000}$

$\therefore x = \frac{375}{1{,}000}$

$\therefore .374\dot{9} = \frac{375}{1{,}000} = \frac{3}{8}$

(ii) $.375 = \frac{375}{1{,}000} = \frac{3}{8}$

$\therefore$ both are equivalent to $\frac{3}{8}$.

Section 1.17 (page 56)

1. $x^4 = x \cdot x \cdot x \cdot x$

2. $(a \cdot b)^5 = a^5b^5$
or $(a \cdot b)(a \cdot b)(a \cdot b)(a \cdot b)(a \cdot b)$

3. $(a^2b^3)(a^4b^5) = a^6b^8$

4. $(d^3)^5 = d^{3\times5}$
$= d^{15}$

5. $(a^2b^3)^4 = a^{2\cdot4}b^{3\cdot4}$
$= a^8b^{12}$

6. $(3abcd)^2 = 3^2a^2b^2c^2d^2$
$= 9a^2b^2c^2d^2$

7. $32a^5b^{10}c^{15} = 2^5a^5b^{10}c^{15}$
$= (2ab^2c^3)^5$

8. $(-a^2)^3 = (-a^2)(-a^2)(-a^2)$
$= -a^6$

Section 1.19 (page 58)

1. $a^7 \div a^4 = a^{7-4}$
$= a^3$

2. $b^7 \times b^0 = b^{7+0}$
$= b^7$

3. $b^7 \div b^0 = b^{7-0}$
$= b^7$

4. $b^0 \times b^0 = b^{0+0}$
$= b^0$
$= 1$

5. $b^0 \div b^0 = b^{0-0}$
$= b^0$
$= 1$

6. $c^5 \div c^7 = \frac{c^5}{c^7}$
$= \frac{c^5 \div c^5}{c^7 \div c^5}$
$= \frac{1}{c^{7-5}}$
$= \frac{1}{c^2}$

7. $(x^2y^3) \div (xy^2) = x^{2-1}y^{3-2}$
$= xy$

8. $\dfrac{6p^4q^2}{3p^2q^4} = \dfrac{2p^{4-2}}{q^{4-2}}$
$= \dfrac{2p^2}{q^2}$

9. $\dfrac{6a^8b^6c^7}{2a^2b^4c^9} = \dfrac{3a^{8-2}b^{6-4}}{c^{9-7}}$
$= \dfrac{3a^6b^2}{c^2}$

10. $\left(\dfrac{3}{2}\right)^5 = \dfrac{3^5}{2^5}$

11. $\dfrac{4^3}{7^3} = \left(\dfrac{4}{7}\right)^3$

12. $\dfrac{a^2b^2}{c^2} = \left(\dfrac{ab}{c}\right)^2$

13. $\dfrac{x^3y^3}{a^3b^3} = \left(\dfrac{xy}{ab}\right)^3$

14. $\left(\dfrac{abc}{mnp}\right)^4 = \dfrac{a^4b^4c^4}{m^4n^4p^4}$

15. $\dfrac{8a^3}{125b^3} = \dfrac{2^3a^3}{5^3b^3} = \left(\dfrac{2a}{5b}\right)^3$

Section 1.21 (page 60)

1. $a^{-4} = \dfrac{1}{a^4}$

2. $\dfrac{1}{b^{-6}} = b^6$

3. $\dfrac{a^{-2}}{b^{-3}} = \dfrac{b^3}{a^2}$

4. $a^{-2}b^{-3} = \dfrac{1}{a^2b^3}$

5. $\dfrac{a^2b^3}{a^{-2}b^{-4}} = a^2 \cdot a^2 \cdot b^3 \cdot b^4$ or $= a^{2-(-2)}b^{3-(-4)}$
$= a^4b^7$ $= a^4b^7$

6. $a^2 \times a^{-3} = a^{2+(-3)}$
$= a^{-1}$

7. $(6x^4y^{-5}) \div (2x^2y^{-3}) = 3x^{4-2}y^{-5-(-3)}$
$= 3x^2y^{-2}$

8. $4m^5n^{-3} \times 3m^{-2}n^3 = 12m^{5-2}n^{-3+3}$
$= 12m^3n^0$
$= 12m^3$

9. $\dfrac{12a^{-4}b^{-6}}{(3a^{-1}b^{-3})(4a^{-3}b^{-3})} = \dfrac{12a^{-4}b^{-6}}{12a^{-4}b^{-6}}$
$= a^0b^0$
$= 1$

Chapter II

Section 2.1 (page 70)

1. $(10x + 6) - (11 - 15x) = 20x$
$\therefore 10x + 6 - 11 + 15x = 20x$
$\therefore 10x + 15x - 20x = 11 - 6$
$\therefore 5x = 5$
$\therefore x = 1$

By reversing the steps:
$x = 1$
$\therefore 5x = 5$
$\therefore 10x + 15x - 20x = 11 - 6$
$\therefore 10x + 6 - 11 + 15x = 20x$
$\therefore (10x + 6) - (11 - 15x) = 20x$
$\therefore$ the root is 1 .

2. $\{x \mid 8x + (x - 7) + 3 = 2x - (3 + x), x \in R\}$
$8x + (x - 7) + 3 = 2x - (3 + x)$
$\therefore 8x + x - 7 + 3 = 2x - 3 - x$
$\therefore 9x - 4 = x - 3$
$\therefore 9x - x = 4 - 3$
$\therefore 8x = 1$
$\therefore x = \dfrac{1}{8}$.

$\{x \mid 8x + (x - 7) + 3 = 2x - (3 + x), x \in R\} = \left\{\dfrac{1}{8}\right\}$

Verification. L.S. $= 8x + (x - 7) + 3$ R.S. $= 2x - (3 + x)$
$= 1 + (-6\frac{7}{8}) + 3$ $= \frac{1}{4} - (3\frac{1}{8})$
$= -2\frac{7}{8}$ $= -2\frac{7}{8}$

3. $3(x-5)-2(3+4x)=8-6(2x-1)$

$\therefore 3x-15-6-8x=8-12x+6$

$\therefore -5x-21=14-12x$

$\therefore 7x=35$

$\therefore x=5$

The solution set is $\{5\}$.

Verification. L.S. $=3(x-5)-2(3+4x)=3(0)-2(23)=-46$ R.S. $=8-6(2x-1)=8-6(9)=-46$

Section 2.2 (page 72)

1. $\frac{1}{3}(2x-1)-\frac{1}{5}(4x-6)=\dfrac{7}{15}$

Multiply each term by 15. (the L.C.D.)

$\therefore 15\times\frac{1}{3}(2x-1)-15\times\frac{1}{5}(4x-6)=15\times\dfrac{7}{15}$

$\therefore 5(2x-1)-3(4x-6)=7$

$\therefore 10x-5-12x+18=7$

$\therefore -2x=7+5-18$

$\therefore -2x=-6$

$\therefore x=3$

2. $\left\{x \mid \dfrac{x-2}{2}+\dfrac{x+3}{3}=\dfrac{3x+4}{4}-2\right\}$

$$\frac{x-2}{2}+\frac{x+3}{3}=\frac{3x+4}{4}-2$$

Multiply each term by 12. (the L.C.D.)

$\therefore 6(x-2)+4(x+3)=3(3x+4)-24$

$\therefore 6x-12+4x+12=9x+12-24$

$\therefore 6x+4x-9x=12-24$

$\therefore x=-12$

$$\left\{x \mid \frac{x-2}{2}+\frac{x+3}{3}=\frac{3x+4}{4}-2\right\}=\{-12\}$$

Verification. L.S. $=\dfrac{x-2}{2}+\dfrac{x+3}{3}=\dfrac{-14}{2}+\dfrac{-9}{3}=-10$ R.S. $=\dfrac{3x+4}{4}-2=\dfrac{-32}{4}-2=-10$

Section 2.3 (page 74)

1. $\{x \mid 3(x+2)(x-1)-2(x-4)^2=x^2+19,\ x\in I\}$

$3(x+2)(x-1)-2(x-4)^2=x^2+19$

$\therefore 3(x^2+x-2)-2(x^2-8x+16)=x^2+19$

$\therefore 3x^2+3x-6-2x^2+16x-32=x^2+19$

$\therefore 19x=57$

$\therefore x=3$

$\{x \mid 3(x+2)(x-1)-2(x-4)^2=x^2+19,\ x\in I\}=\{3\}$

Verification. L.S. $=3(x+2)(x-1)-2(x-4)^2=3(5)(2)-2(-1)^2=28$ R.S. $=x^2+19=9+19=28$

Section 2.7 (page 83)

1. $\{x \mid 21 - 8x \leqq 9 - 5x\}$

$$21 - 8x \leqq 9 - 5x$$
$$\therefore -8x + 5x \leqq 9 - 21$$
$$\therefore -3x \leqq -12$$
$$\therefore x \geqq 4$$
$$\{x \mid 21 - 8x \leqq 9 - 5x\} = \{x \mid x \geqq 4\}$$

Graph of $\{x \mid x \geqq 4\}$

2. $3(-4 - y) \geqq 2(2y + 3) - 11$

$$\therefore -12 - 3y \geqq 4y + 6 - 11$$
$$\therefore -4y - 3y \geqq 12 + 6 - 11$$
$$\therefore -7y \geqq 7$$
$$\therefore y \leqq -1$$

The solution set is $\{y \mid y \leqq -1, y \in I\}$.

Graph of $\{y \mid y \leqq -1, y \in I\}$

Section 2.8 (page 85)

1.

$$2x^2 \geqq 2(2x - 3)^2 - 3(2x - 5)(x + 1)$$
$$\therefore 2x^2 \geqq 2(4x^2 - 12x + 9) - 3(2x^2 - 3x - 5)$$
$$\therefore 2x^2 \geqq 8x^2 - 24x + 18 - 6x^2 + 9x + 15$$
$$\therefore 2x^2 - 8x^2 + 24x + 6x^2 - 9x \geqq 18 + 15$$
$$\therefore 15x \geqq 33$$
$$\therefore x \geqq 2\tfrac{1}{5}$$

2. $\{x \mid 1 - \dfrac{1 - 2x}{4} > \dfrac{x - 2}{6}, x \in I\}$

$$1 - \frac{1 - 2x}{4} > \frac{x - 2}{6}$$

Multiply each side by 12.

$$\therefore 12 - 3(1 - 2x) > 2(x - 2)$$
$$\therefore 12 - 3 + 6x > 2x - 4$$
$$\therefore 6x - 2x > -4 - 12 + 3$$
$$\therefore 4x > -13$$
$$\therefore x > -3\tfrac{1}{4}$$
$$\{x \mid 1 - \frac{1 - 2x}{4} > \frac{x - 2}{6}, x \in I\} = \{x \mid x > -3\tfrac{1}{4}, x \in I\}$$

Graph of $\{x \mid x > -3\tfrac{1}{4}, x \in I\}$

Section 2.11 (page 91)

1. $|2x + 1| = +1$ is equivalent to

(i) $2x + 1 = +1$ or (ii) $2x + 1 = -1$

$x = 0$ | $x = -1$

Verification.

If $x = 0$, L.S. $= |0 + 1|$ R.S. $= +1$
$= |+ 1|$
$= +1$

If $x = -1$, L.S. $= |-2 + 1|$ R.S. $= +1$
$= |-1|$
$= +1$

The roots of the equation are 0 and 1.

2. Since the absolute value of a real number cannot be negative, the solution set is $\varnothing$.

Chapter III

Section 3.3 (page 98)

Represent any integer by x.
Then 4 times the integer decreased by 5 is $4x - 5$.
But this is less than 7.
$\therefore 4x - 5 < 7$

Section 3.5 (page 101)

1. Represent the first number by $x - 1$.
Then the next two consecutive numbers are x and $x + 1$.
The sum of the three numbers $= (x - 1) + x + (x + 1)$
$= 3x$.
But this sum is 27.
$\therefore 3x = 27$
$\therefore x = 9$
The three numbers are 8, 9, and 10.
Verification. The numbers are consecutive,
and their sum $= 8 + 9 + 10 = 27$.

2. Represent any real number by x.
The number decreased by 2, then divided by 3 is $\frac{x - 2}{3}$.
But this number is greater than 5.
$\therefore \frac{x - 2}{3} > 5$
$\therefore x - 2 > 15$
$\therefore x > 17$
The required numbers are any real numbers greater than 17.
Verification. Any real number greater than 17 may be represented by $17 + d, d \in {}^+R$.

Thus, $\frac{17 + d - 2}{3} = \frac{15 + d}{3}$
$= 5 + \frac{d}{3}$
> 5

Section 3.6 (page 103)

1. Depends on the variable(s):
(i) distance and rate (ii) distance and time
(iii) length and width (iv) radius

(v) length, width, depth
(vi) radius
(vii) cost per article, number of articles
(viii) selling price, cost price
(ix) resistance, voltage
(x) area of base, height
(xi) principal, time, rate
(xii) supply and demand

2. (i) $A = lw, w = \frac{A}{l}, l = \frac{A}{w}$
(ii) $V = lwh$
(iii) $C = 2\pi r$
(iv) $A = \pi r^2$
(v) $V = \frac{4}{3}\pi r^3$
(vi) $d = rt, r = \frac{d}{t}, t = \frac{d}{r}$
(vii) $C = cn, c = \frac{C}{n}, n = \frac{C}{c}$
(viii) $S.P. - C.P. = G(\text{or } L)$
(ix) $I = Prt$

3.

ASSOCIATED VARIABLE(S) NOT STATED	LAW	EXPRESSION FOR ASSOCIATED VARIABLE
(i) width of rectangle	$w = \frac{A}{l}$	$w = 15$ (inches)
(ii) length of rectangle	$l = \frac{A}{w}$	$l = \frac{A}{8}$ (inches)
(iii) circumference and area	(a) $C = 2\pi r$ (b) $A = \pi r^2$	$C = 20\pi$(inches) $A = 100\pi$(sq. in.)
(iv) number of shirts	$n = \frac{C(\text{total})}{c(\text{per shirt})}$	$n = \frac{30}{x}$
(v) number of afternoons	$n = \frac{W(\text{total wages})}{w(\text{wages/afternoon})}$	$n = \frac{y}{1\frac{1}{2}}$
(vi) rate	$r = \frac{d}{t}$	$r = \frac{x-4}{y}$ (m.p.h.)
(vii) interest	$I = Prt$	$I = (3x+4)\left(\frac{5}{100}\right)(3)$

Section 3.8 (page 109)

1. Represent the number of dollars invested by x.
Then the interest is $.05x$ dollars.
But the interest is 48 dollars.

$\therefore .05x = 48$
$\therefore 5x = 4{,}800$
$\therefore x = 960$

The amount of the investment is \$960.

Verification. The interest on \$960 at 5% is $\$(.05 \times 960) = \$48$.

2. Represent the number of c.c. of water added by x.
Then the number of c.c. of acid in a 5% solution is $.05(x + 60)$.
But this number of c.c. of acid is $.12 \times 60 = 7.2$.

$\therefore .05(x + 60) = 7.2$
$\therefore 5(x + 60) = 720$
$\therefore 5x + 300 = 720$
$\therefore 5x = 420$
$\therefore x = 84$

Verification. There should be 84 c.c. of water added.
The number of c.c. of acid in 144 c.c. of a 5% solution is
$.05(144) = 7.2$, as required.

Chapter IV

Section 4.4 (page 122)

1. $2x^3 - 4x^2 - 2x = 2x(x^2 - 2x - 1)$
2. $a(x - y) + b(x - y) = (a + b)(x - y)$

Section 4.5 (page 124)

1. $ax + a + x + 1$
$= (ax + a) + (x + 1)$
$= (x + 1)a + (x + 1)$
$= (x + 1)(a + 1)$

2. $x^2 + y - xy - x$
$= (x^2 - xy) + (y - x)$
$= (x - y)x - (x - y)$
$= (x - y)(x - 1)$

3. $(a - b)^2 - ax + bx = (a - b)^2 - (ax - bx)$
$= (a - b)^2 - (a - b)x$
$= (a - b)(a - b - x)$

Section 4.6 (page 126)

1. METHOD 1
$a^2 - 5a - 36 = (a - 9)(a + 4)$

METHOD 2
$a^2 - 5a - 36$
$= a^2 - 9a + 4a - 36$
$= a(a - 9) + 4(a - 9)$
$= (a + 4)(a - 9)$

2. METHOD 1
$x^2 + 8xy - 20y^2$
$= (x + 10y)(x - 2y)$

METHOD 2
$x^2 + 8xy - 20y^2$
$= x^2 + 10xy - 2xy - 20y^2$
$= x(x + 10y) - 2y(x + 10y)$
$= (x - 2y)(x + 10y)$

3. METHOD 1
$2x^3 - 16x^2 - 18x$
$= 2x(x^2 - 8x - 9)$
$= 2x(x - 9)(x + 1)$

METHOD 2
$2x^3 - 16x^2 - 18x$
$= 2x(x^2 - 8x - 9)$
$= 2x(x^2 - 9x + x - 9)$
$= 2x[x(x - 9) + (x - 9)]$
$= 2x(x + 1)(x - 9)$

Section 4.7 (page 129)

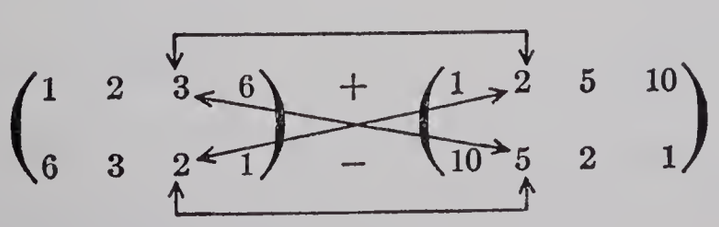
1.

$\because 6(-10) = -60$
$= -2 \times 2 \times 3 \times 5$
$= (+4)(-15)$

$-11 = (+4) + (-15)$

$\therefore 6a^2 - 11ab - 10b^2 = 6a^2 - 15ab + 4ab - 10b^2$
$= 3a(2a - 5b) + 2b(2a - 5b)$
$= (3a + 2b)(2a - 5b)$

By inspection:



$6a^2 - 11ab - 10b^2 = (3a + 2b)(2a - 5b)$

Section 4.8 (page 130)

1. In the expression $36x^2 - 30x + 25$:
(i) $36x^2$ and 25 are the squares of $6x$ and 5 respectively;
(ii) $-30x$ is not $-2(6x)(5)$;
$\therefore$ $36x^2 - 30x + 25$ is not a trinomial square.

2. $36x^2 - 60xy + 25y^2 = (6x - 5y)^2$
Since a positive real number has two square roots which are equal in magnitude but opposite in sign, $\therefore$ the square roots of $36x^2 - 60xy + 25y^2$ are $\pm(6x - 5y)$. These may be written $(6x - 5y)$ and $(5y - 6x)$.

3. The symbol $\sqrt{\ }$ indicates the principal square root which is the positive square root.
$\therefore$ $\sqrt{9x^2 + 60x + 100} = |3x + 10|$ (absolute value of $(3x + 10)$)

Section 4.9 (page 132)

1. (i) $x^4 - 16y^4$
$= (x^2 - 4y^2)(x^2 + 4y^2)$
$= (x - 2y)(x + 2y)(x^2 + 4y^2)$

(ii) $(a - b)^2 - c^2$
$= (a - b - c)(a - b + c)$

2. (i) $9 - (x - y)^2$
$= [3 + (x - y)][3 - (x - y)]$
$= (3 + x - y)(3 - x + y)$

(ii) $(a + b)^2 - (c - d)^2$
$= [(a + b) + (c - d)][(a + b) - (c - d)]$
$= (a + b + c - d)(a + b - c + d)$

Section 4.10 (page 134)

1. The terms $-2xy$ and $2zp$ suggest the following grouping:
$x^2 + y^2 - z^2 - p^2 - 2xy + 2zp$
$= (x^2 - 2xy + y^2) - (z^2 - 2zp + p^2)$
$= (x - y)^2 - (z - p)^2$
$= [(x - y) + (z - p)][(x - y) - (z - p)]$
$= (x - y + z - p)(x - y - z + p)$

2. $4y^2 + 6x - 9x^2 - 1$
$= 4y^2 - (9x^2 - 6x + 1)$
$= 4y^2 - (3x - 1)^2$
$= [2y + (3x - 1)][2y - (3x - 1)]$
$= (2y + 3x - 1)(2y - 3x + 1)$

Section 4.11 (page 136)

1. $2x^2 + 3x - 2 = 0$
$\therefore (2x - 1)(x + 2) = 0$

$\therefore 2x - 1 = 0$ or $x + 2 = 0$
$\therefore x = \frac{1}{2}$ $\qquad$ $x = -2$

The roots are -2, and $\frac{1}{2}$.

Verification.

L.S. $= 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) - 2$
$= \frac{1}{2} + \frac{3}{2} - 2$
$= 0$
R.S. $= 0$

L.S. $= 2(-2)^2 + 3(-2) -$
$= 8 - 6 - 2$
$= 0$
R.S. $= 0$

2.
$$x^2 - 6 = x$$
$$\therefore x^2 - x - 6 = 0$$
$$\therefore (x - 3)(x + 2) = 0$$

$\therefore x - 3 = 0$ or $x + 2 = 0$

$\therefore x = 3$ | $x = -2$

The roots are -2 and 3.

Section 4.12 (page 138)

1.
$$\begin{aligned} & a^4 + 4b^4 \\ &= a^4 + 4a^2b^2 + 4b^4 - 4a^2b^2 \\ &= (a^2 + 2b^2)^2 - (2ab)^2 \\ &= (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab) \\ &= (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2) \end{aligned}$$

2.
$$\begin{aligned} & 9y^4 - 51y^2 + 49 \\ &= 9y^4 - 42y^2 + 49 - 9y^2 \\ &= (3y^2 - 7)^2 - (3y)^2 \\ &= (3y^2 - 7 + 3y)(3y^2 - 7 - 3y) \\ &= (3y^2 + 3y - 7)(3y^2 - 3y - 7) \end{aligned}$$

Section 4.13 (page 140)

1. $2x^3 - 11x^2 + 5x + 4$

The integral factors of 4 are ± 1, ± 2, ± 4.

If $x = 1$, the expression $= 2 - 11 + 5 + 4 = 0$.

$\therefore x - 1$ is a factor of the expression.

$$\begin{array}{r|l} & 2x^2 - 9x - 4 \\ \hline x - 1 & 2x^3 - 11x^2 + 5x + 4 \\ & \underline{2x^3 - 2x^2} \\ & \quad -9x^2 + 5x \\ & \quad \underline{-9x^2 + 9x} \\ & \qquad -4x + 4 \\ & \qquad \underline{-4x + 4} \\ & \qquad\quad 0 \end{array}$$

$\therefore 2x^3 - 11x^2 + 5x + 4 = (x - 1)(2x^2 - 9x - 4)$

2. (i) $x^3 - y^3$

The factors of y^3 are $\pm y$, $\pm y^2$, $\pm y^3$.

If $x = y$, the expression $= y^3 - y^3$
$= 0$.

$\therefore x - y$ is a factor of the expression.

$$\begin{array}{r|l} & x^2 + xy + y^2 \\ \hline x - y & x^3 \qquad\quad - y^3 \\ & \underline{x^3 - x^2y} \\ & \quad x^2y \\ & \quad \underline{x^2y - xy^2} \\ & \qquad xy^2 - y^3 \\ & \qquad \underline{xy^2 - y^3} \\ & \qquad\quad 0 \end{array}$$

$\therefore x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

(ii) $x^3 + y^3$

If $x = -y$, the expression $= x^3 + (-y)^3$
$= 0$.

$\therefore x + y$ is a factor of the expression.

$$\begin{array}{r|l} & x^2 - xy + y^2 \\ \hline x + y & x^3 \qquad\quad + y^3 \\ & \underline{x^3 + x^2y} \\ & \quad -x^2y \\ & \quad \underline{-x^2y - xy^2} \\ & \qquad xy^2 + y^3 \\ & \qquad \underline{xy^2 + y^3} \\ & \qquad\quad 0 \end{array}$$

$\therefore x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

3. If $y = -3$, then $y^3 + 2y^2 - 7y - 12 = -27 + 18 + 21 - 12 = 0$.

$\therefore y + 3$ is a factor of the expression.

Chapter V

Section 5.3 (page 147)

1. $\dfrac{2x^4 - 2y^4}{6x - 6y}$

$= \dfrac{2(x^2 + y^2)(x + y)(x - y)}{3 \times 2(x - y)}$

$= \dfrac{(x^2 + y^2)(x + y)}{3}, \quad x \neq y$

2. $\dfrac{ax - bx + ay - by}{ax - bx - ay + by}$

$= \dfrac{x(a - b) + y(a - b)}{x(a - b) - y(a - b)}$

$= \dfrac{(x + y)(a - b)}{(x - y)(a - b)}$

$= \dfrac{x + y}{x - y}, \quad a \neq b$

3. $\dfrac{x^4 - 15x^2 + 9}{x^2 - 3x - 3} = \dfrac{(x^2 - 3)^2 - 9x^2}{x^2 - 3x - 3} = \dfrac{(x^2 - 3x - 3)(x^2 + 3x - 3)}{x^2 - 3x - 3}$

$= x^2 + 3x - 3, \quad (x^2 - 3x - 3) \neq 0$

Section 5.4 (page 149)

1. $\dfrac{2a^2 - ab - b^2}{(a + b)(b - a)} = \dfrac{(2a + b)(a - b)}{(a + b)(b - a)} = \dfrac{(2a + b)(a - b)(-1)}{(a + b)(b - a)(-1)}$

$= \dfrac{(2a + b)(a - b)(-1)}{(a + b)(a - b)}$

$= -\dfrac{(2a + b)}{a + b}, \quad a \neq b$

Section 5.5 (page 151)

1. $\dfrac{ax + ay}{a - b} \times \dfrac{a^2 - b^2}{a^2 + ab} \div (x + y)$

$= \dfrac{a(x + y)}{a - b} \times \dfrac{(a - b)(a + b)}{a(a + b)} \times \dfrac{1}{(x + y)}$

$= 1, \quad a \neq 0$ or b or $-b$; $x \neq -y$

2. $\dfrac{2x^2 - 3x - 2}{x^2 + x - 6} \times \dfrac{x^3 - 1}{4x^2 - 1} \div \dfrac{x^3 + x^2 + x}{2x^2 + 5x - 3}$

$= \dfrac{(2x + 1)(x - 2)}{(x + 3)(x - 2)} \times \dfrac{(x - 1)(x^2 + x + 1)}{(2x - 1)(2x + 1)} \times \dfrac{(2x - 1)(x + 3)}{x(x^2 + x + 1)}$

$= \dfrac{x - 1}{x}, \quad x \neq \pm\dfrac{1}{2}$ or 2 or -3; $x^2 + x + 1 \neq 0$

Section 5.6 (page 153)

1. $\dfrac{3x}{x - 2} - \dfrac{2x}{x + 2} + \dfrac{5x^2 - 1}{x^2 - 4} = \dfrac{3x}{x - 2} - \dfrac{2x}{x + 2} + \dfrac{5x^2 - 1}{(x + 2)(x - 2)}$

$= \dfrac{3x(x + 2) - 2x(x - 2) + 5x^2 - 1}{(x + 2)(x - 2)}$

$= \dfrac{3x^2 + 6x - 2x^2 + 4x + 5x^2 - 1}{(x + 2)(x - 2)}$

$= \dfrac{6x^2 + 10x - 1}{x^2 - 4}$

Section 5.7 (page 155)

1. $\dfrac{x^2 - 9x + 20}{x^2 - 7x + 12} - \dfrac{x^2 - 9x + 18}{x^2 - 9} = \dfrac{(x - 5)(x - 4)}{(x - 4)(x - 3)} - \dfrac{(x - 3)(x - 6)}{(x - 3)(x + 3)}$

$= \dfrac{x - 5}{x - 3} - \dfrac{x - 6}{x + 3}, \quad x \neq 3 \text{ or } 4$

$= \dfrac{(x - 5)(x + 3) - (x - 3)(x - 6)}{(x - 3)(x + 3)}$

$= \dfrac{(x^2 - 2x - 15) - (x^2 - 9x + 18)}{(x - 3)(x + 3)}$

$= \dfrac{x^2 - 2x - 15 - x^2 + 9x - 18}{(x - 3)(x + 3)}$

$= \dfrac{7x - 33}{x^2 - 9}$

Chapter VI

Section 6.2 (page 163)

1. $A \times A = \{(1, 1), (3, 1), (5, 1), (1, 3), (3, 3), (5, 3), (1, 5), (3, 5), (5, 5)\}$

$\therefore B = \{(3, 1), (5, 1), (5, 3)\}$

2. $N = \{1, 2, 3, 4, \ldots\}$

$N \times N$ cannot be listed, nor can the required subset. The required subset can be indicated as follows:

$\{(1, 3), (2, 4), (3, 5), (4, 6), (5, 7), \ldots\}$

3. $A \times A = \{(8, 8), (27, 8), (64, 8), (8, 27), (27, 27), (64, 27), (8, 64), (27, 64), (64, 64)\}$

$C = \phi$

Section 6.5 (page 171)

1. $A = \{0, 1, 2, 3, 4, 5\}$

if $x = 0$, $y = 0^2 + 1 = 1$
if $x = 1$, $y = 1^2 + 1 = 2$
if $x = 2$, $y = 2^2 + 1 = 5$
if $x = 3$, $y = 3^2 + 1 = 10$
if $x = 4$, $y = 4^2 + 1 = 17$
if $x = 5$, $y = 5^2 + 1 = 26$

Since both x, y must be elements of A,
$\therefore B = \{(0, 1), (1, 2), (2, 5)\}$.
Domain of B is $\{0, 1, 2\}$.
Range of B is $\{1, 2, 5\}$.

2. x, y are elements of $\{0, 1, 2, 3, \ldots\}$.

if $x = 0$, $y = 2 \times 0 - 1 = -1$
if $x = 1$, $y = 2 \times 1 - 1 = 1$
if $x = 2$, $y = 2 \times 2 - 1 = 3$
if $x = 3$, $y = 2 \times 3 - 1 = 5$
etc.

Since -1 is not in the set N_0,
$\therefore C = \{(1, 1), (2, 3), (3, 5), \ldots\}$.
The pattern indicates that:
the domain of C is $\{1, 2, 3, 4, \ldots\}$ or the domain of C is N;
the range of C is $\{1, 3, 5, 7, \ldots\}$ or the set of odd natural numbers.

3. Since $x, y \in R$ and $y = x$, for all $x \in R$,
then $D = \{(x, x) \mid x \in R\}$.
Since both x and y may assume any real value,
therefore the domain of D is R or $\{x \mid x \in R\}$;
the range of D is R or $\{y \mid y \in R\}$.

4. E is the set of ordered pairs represented by (x, y) which satisfy the defining sentence $x^2 + y^2 = 25$, $x, y \in I$.
To discover whether an ordered pair does or does not belong to the relation, it must be shown that the ordered pair does or does not satisfy the equation.

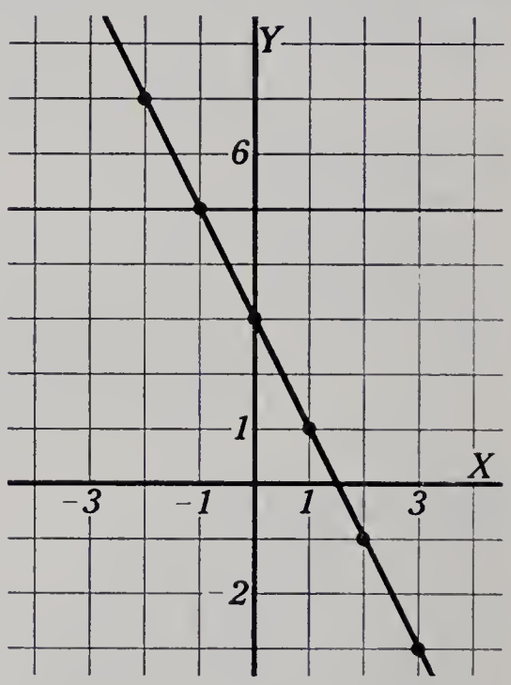
For the ordered pair $(3, 4)$: $x^2 + y^2 = 9 + 16 = 25$, $\therefore (3, 4) \in E$.
For the ordered pair $(0, 0)$: $x^2 + y^2 = 0 + 0 = 0$, $\therefore (0, 0) \notin E$.
For the ordered pair $(0, -5)$: $x^2 + y^2 = 0 + 25 = 25$, $\therefore (0, -5) \in E$.
For the ordered pair $(4, -3)$: $x^2 + y^2 = 16 + 9 = 25$, $\therefore (4, -3) \in E$.
For the ordered pair $(5, 0)$: $x^2 + y^2 = 25 + 0 = 25$, $\therefore (5, 0) \in E$.
For the ordered pair $(0, 25)$: $x^2 + y^2 = 0 + 625 = 625$, $\therefore (0, 25) \notin E$.
For the ordered pair $(4, 3)$: $x^2 + y^2 = 16 + 9 = 25$, $\therefore (4, 3) \in E$.
For the ordered pair $(6, -1)$: $x^2 + y^2 = 36 + 1 = 37$, $\therefore (6, -1) \notin E$.
The ordered pairs $(0, 0)$, $(0, 25)$, $(6, -1)$ do not satisfy E.

Section 6.6 (page 177)

1. $M = \{(x, y) \mid 2x + y = 3, x, y \in R\}$
For ease in constructing a table of values the defining equation may be written in the form $y = 3 - 2x$.
$\therefore M = \{(x, y) \mid y = 3 - 2x, x, y \in R\}$
The relation may also be written
$M = \{(x, 3 - 2x) \mid x \in R\}$,
because y (or $3 - 2x$) $\in R$
for all $x \in R$.

For M:

x	-2	-1	0	1	2	3
y	7	5	3	1	-1	-3



The partial graph is drawn.
From the graph:
the domain is $\{x \mid x \in R\}$;
the range is $\{y \mid y \in R\}$.

NOTE: The equation $y = 3 - 2x$ expresses *explicitly* what y is equivalent to in terms of x. Similarly, the equation $x = \dfrac{3 - y}{2}$ expresses explicitly what x is equivalent to in terms of y. Equations written in this manner are said to be written in *explicit form*. The equation $2x + y = 3$ or $2x + y - 3 = 0$ *implies* a relationship between x and y and some given numbers. Equations written in this manner are said to be written in *implicit form*.

The following are different forms of the same equation:

(i) $2x + y = 3$ (ii) $2x + y - 3 = 0$
(iii) $y = 3 - 2x$ (iv) $x = \dfrac{3 - y}{2}$

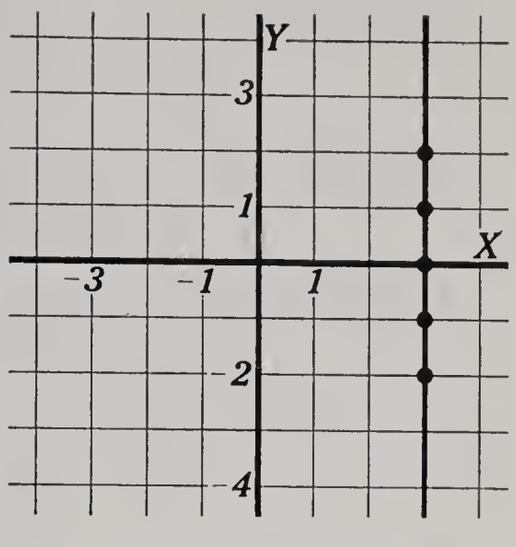
The explicit form of the equation is particularly useful if a table of values is to be constructed.

2. $N = \{(x, y) \mid x = 3, x, y \in R\}$

Since y does not appear in the equation $x = 3$, then since $y \in R$, y may have any real value, but x must always be 3.

For N:

x	3	3	3	3	3
y	-2	-1	0	1	2



The partial graph is drawn.
From the graph or otherwise:
the domain of N is $\{3\}$;
the range of N is $\{y \mid y \in R\}$.

NOTE: A table of values is really not necessary for the drawing of this graph. If x must always be 3 and y may be any real number, then all the points on the graph must be 3 units to the right of the y-axis, and the graph must be the line parallel to the y-axis and three units to the right of it.

In a similar manner, it may be argued that the graphs of the relations with the following defining sentences are as indicated:

(i) $x = -3, x, y \in R$: line parallel to y-axis, 3 units to the left of it;
(ii) $x = 0, x, y \in R$: the y-axis;
(iii) $y = 4, x, y \in R$: line parallel to x-axis, 4 units above it;
(iv) $y = -7, x, y \in R$: line parallel to x-axis, 7 units below it;
(v) $y = 0, x, y \in R$: the x-axis.

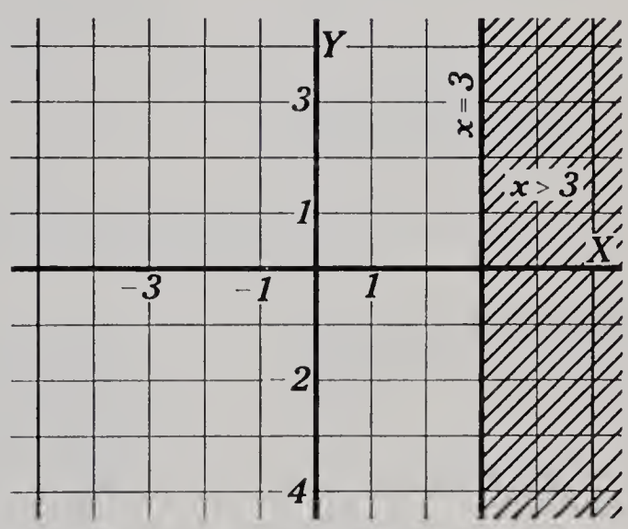
Section 6.9 (page 185)

1. $F = \{(x, y) \mid x \geqq 3, x, y \in R\}$

The corresponding equality is $x = 3$, $x, y \in R$.

The graph of the relation defined by $x = 3, x, y \in R$ is the straight line parallel to the y-axis and three units to the right of it.

The graph of the relation defined by $x \geqq 3, x, y \in R$ is, then, all the points on this line and (together with) all the points on the plane to the right of it (all the points with abscissa greater than 3).



The partial graph of F is shown.
The domain of F is $\{x \mid x \geqq 3, x \in R\}$;
the range of F is $\{y \mid y \in R\}$.

Chapter VII

Section 7.1 (page 203)

1.

$$\begin{cases} x + y = 3 & (1) \\ 2x + 3y = 8 & (2) \end{cases}$$

From (1) $\quad y = 3 - x \qquad (3)$

From (2) $\quad y = \dfrac{8 - 2x}{3} \qquad (4)$

By comparison $3 - x = \dfrac{8 - 2x}{3}$ (5)

$\therefore 9 - 3x = 8 - 2x$

$\therefore -x = -1$

$\therefore x = 1$ (6)

Substitute in (3) $y = 2$ (7)

The solution set of the system is $\{(1, 2)\}$.

Verification. L.S. (1) $= 1 + 2 = 3$ R.S. (1) $= 3$

L.S. (2) $= 2 + 6 = 8$ L.S. (2) $= 8$

2.

$$\begin{cases} \frac{1}{4}x + \frac{1}{2}y = 7 & (1) \\ \frac{1}{3}x + \frac{1}{4}y = 6 & (2) \end{cases}$$

Clear of fractions

$4 \times (1)$ $x + 2y = 28$ (3)

$12 \times (2)$ $4x + 3y = 72$ (4)

From (3) $y = \dfrac{28 - x}{2}$ (5)

From (4) $y = \dfrac{72 - 4x}{3}$ (6)

By comparison $\dfrac{28 - x}{2} = \dfrac{72 - 4x}{3}$

$\therefore 84 - 3x = 144 - 8x$

$\therefore 5x = 60$

$\therefore x = 12$ (7)

Substitute in (3) $12 + 2y = 28$

$\therefore 2y = 16$

$\therefore y = 8$

The solution set is $\{(12, 8)\}$.

Section 7.3 (page 208)

1.

$$\begin{cases} 3x + 2y = 13 & (1) \\ 5x - 2y = 3 & (2) \end{cases}$$

(1) + (2) $8x = 16$

$\therefore x = 2$ (3)

Substitute in (1) $6 + 2y = 13$

$\therefore 2y = 7$

$\therefore y = 3\frac{1}{2}$ (4)

The solution set is $\{(2, 3\frac{1}{2})\}$.

Verification. L.S. (1) $= 3x + 2y = 6 + 7 = 13$ L.S. (2) $= 5x - 2y = 10 - 7 = 3$

R.S. (1) $= 13$ R.S. (2) $= 3$

2.

$$\begin{cases} 5x + 2y = 9 & (1) \\ 2x + 3y = 8 & (2) \end{cases}$$

To eliminate x

$2 \times (1)$ $10x + 4y = 18$ (3)

$5 \times (2)$ $10x + 15y = 40$ (4)

(3) − (4) $-11y = -22$

$\therefore y = 2$ (5)

Substitute in (2)

$2x = 2$

$\therefore x = 1$ (6)

or To eliminate y

$3 \times (1)$ $15x + 6y = 27$ (3)

$2 \times (2)$ $4x + 6y = 16$ (4)

(3) − (4) $11x = 11$

$\therefore x = 1$ (5)

Substitute in (2)

$2 + 3y = 8$

$\therefore 3y = 6$

$\therefore y = 2$ (6)

The solution set is $\{(1, 2)\}$.

Verification. L.S. (1) $= 5 + 4 = 9$ R.S. (1) $= 9$

L.S. (2) $= 2 + 6 = 8$ R.S. (2) $= 8$

3.

$$\begin{cases} \frac{1}{2}x - \frac{1}{3}y = 0 & (1) \\ \frac{1}{4}x - \frac{3}{2}y = -4 & (2) \end{cases}$$

Clear the equations of fractions

$6 \times (1)$ $3x - 2y = 0$ (3)

$4 \times (2)$ $x - 6y = -16$ (4)

$3 \times (3)$ $9x - 6y = 0$ (5)

(4) − (5) $-8x = -16$

$\therefore x = 2$ (6)

Substitute in (3) $6 - 2y = 0$

$\therefore y = 3$ (7)

$$\{(x, y) \mid \tfrac{1}{2}x - \tfrac{1}{3}y = 0 \text{ and } \tfrac{1}{4}x - \tfrac{3}{2}y = -4,\ x, y \in R\} = \{(2, 3)\}$$

Verification. L.S. (1) $= 1 - 1 = 0$ R.S. (1) $= 0$

L.S. (2) $= \frac{1}{2} - \frac{9}{2} = -4$ R.S. (2) $= -4$

Section **7.4** (page 212)

1.

$$\begin{cases} x + 2y + 3z = 26 & (1) \\ 4x - 5y + 6z = 29 & (2) \\ 7x + 8y - 9z = -16 & (3) \end{cases}$$

Eliminate z from (1) and (2)

$2 \times (1)$ $2x + 4y + 6z = 52$ (4)

$4x - 5y + 6z = 29$ (2)

(4) − (2) $-2x + 9y = 23$ (5)

Eliminate z from (1) and (3)

$3 \times (1)$ $3x + 6y + 9z = 78$ (6)

$7x + 8y - 9z = -16$ (3)

(6) + (3) $10x + 14y = 62$ (7)

Eliminate x from (5) and (7)

$5 \times (5)$ $-10x + 45y = 115$ (8)

$10x + 14y = 62$ (7)

(8) + (7) $59y = 177$

$\therefore y = 3$ (9)

To obtain the corresponding value of x substitute in (5) or (7).

Substitute in (5)
$$-2x + 27 = 23$$
$$\therefore -2x = -4$$
$$\therefore x = 2 \qquad (10)$$

To obtain the corresponding value of z, substitute in (1) or (2) or (3).

Substitute in (1)
$$2 + 6 + 3z = 26$$
$$\therefore 3z = 18$$
$$\therefore z = 6 \qquad (11)$$

The solution is (2, 3, 6).

Verification. L.S. (1) $= 2 + 6 + 18 = 26$ R.S. (1) $= 26$
L.S. (2) $= 8 - 15 + 36 = 29$ R.S. (2) $= 29$
L.S. (3) $= 14 + 24 - 54 = -16$ R.S. (3) $= -16$

2.
$$\begin{cases} 5 + 4z = y + 3 & (1) \\ x + y = 3z & (2) \\ 7z = x + 2 & (3) \end{cases}$$

Arrange the equations with the variables on the left side
$$y - 4z = 2 \qquad (4)$$
$$x + y - 3z = 0 \qquad (5)$$
$$x - 7z = -2 \qquad (6)$$

Eliminate x from (5) and (6)

(5) − (6)
$$y + 4z = 2 \qquad (7)$$
$$y - 4z = 2 \qquad (4)$$

(7) + (4)
$$2y = 4$$
$$\therefore y = 2 \qquad (8)$$

(7) − (4)
$$8z = 0$$
$$\therefore z = 0 \qquad (9)$$

Substitute in (6)
$$x - 0 = -2$$
$$\therefore x = -2 \qquad (10)$$

The solution is $(-2, 2, 0)$.

3.
$$\begin{cases} a + b = 3 & (1) \\ b + c = 5 & (2) \\ c + d = 7 & (3) \\ a + 3d = 17 & (4) \end{cases}$$

Add
$$2a + 2b + 2c + 4d = 32$$
$$\therefore a + b + c + 2d = 16 \qquad (5)$$
$$a + b = 3 \qquad (1)$$

(5) − (1)
$$c + 2d = 13 \qquad (6)$$
$$c + d = 7 \qquad (3)$$

(6) − (3)
$$d = 6 \qquad (7)$$

Substitute in (3)
$$c + 6 = 7$$
$$\therefore c = 1 \qquad (8)$$

Substitute in (2)
$$b + 1 = 5$$
$$\therefore b = 4 \qquad (9)$$

Substitute in (1)
$$a + 4 = 3$$
$$\therefore a = -1 \qquad (10)$$

The solution is $(-1, 4, 1, 6)$.

Review D

Section D.2 (page 249)

Construction 2.

Construction: With centre B and any radius draw an arc cutting BA and BC on D and E, respectively.
With centres D and E and equal radii, draw arcs intersecting on F.
Draw ray BG on F.
Then ray BG bisects $\angle ABC$.

Construction 3.

Construction: With centre C and any radius draw an arc cutting AB on D and E, respectively.
With centres D and E and equal radii, draw arcs intersecting on F.
Draw ray CG on F.
Then ray CG is perpendicular to AB at C.

Section D.3 (page 252)

Construction 4.

Construction: With centre C and any suitable radius, draw an arc to cut AB on D and E.
With centres D and E and equal radii, draw arcs to intersect on F.
Join CF, cutting AB on G.
Then $CG \perp AB$.

Construction 5.

Construction: With centres A and B and equal radii, construct arcs intersecting on C and D.
Draw line FG on C and D.
Then line FG is the right bisector of AB.

Construction 6.

Construction: Mark two points C and D on AB.
With centre P and radius CD, construct an arc.
With centre D and radius CP, draw an arc cutting the former arc on Q.
Draw line PQ.
Then line PQ is parallel to line AB.

Chapter VIII

Section 8.3 (page 267)

Example 1.

Proof:

STATEMENTS	AUTHORITIES
1. $A \rightarrow B$	1. Hypothesis
2. A	2. Hypothesis
3. B	3. Law of Detachment 1, 2
4. $B \rightarrow C$	4. Hypothesis
5. C	5. Law of Detachment 3, 4
6. $C \rightarrow D$	6. Hypothesis
7. D	7. Law of Detachment 5, 6

Section 8.4 (page 269)
Example **1.**
Proof:

STATEMENTS	AUTHORITIES
1. $(A \text{ and } B) \rightarrow C$	1. Hypothesis
2. A and B	2. Hypothesis
3. C	3. Law of Detachment 1, 2
4. $C \rightarrow (D \text{ or } E)$	4. Hypothesis
5. D or E	5. Law of Detachment 3, 4

Example **2.**
Analysis:

I CAN PROVE	IF I CAN PROVE
1. B	1. D, $\because D \rightarrow B$.
2. D	2. $\sim C$, $\because \sim C \rightarrow D$.
3. $\sim C$	3. $\sim E$, $\because \sim E \rightarrow \sim C$.
4. $\sim E$ by hypothesis.	

Proof:

STATEMENTS	AUTHORITIES
1. $\sim E \rightarrow \sim C$	1. Hypothesis
2. $\sim E$	2. Hypothesis
3. $\sim C$	3. Law of Detachment 1, 2
4. $\sim C \rightarrow D$	4. Hypothesis
5. D	5. Law of Detachment 3, 4
6. $D \rightarrow B$	6. Hypothesis
7. B	7. Law of Detachment 5, 6

Section 8.5 (page 270)
Example.
(i) Polly flies.
(ii) A and B are equal angles.
(iii) No logical consequent.
(iv) $\angle P$ is less than 90°.
(v) $DE \parallel BC$ and $DE = \frac{1}{2} BC$
(vi) $\angle ABD$ and $\angle DBC$ are complementary angles.
(vii) $\angle ABD$ and $\angle DBC$ are supplementary angles.

Section 8.6 (page 275)
Example **1.**
Proof:

STATEMENTS	AUTHORITIES
1. $AD = AE$	1. Hypothesis
2. $DB = EC$	2. Hypothesis
3. $AD + DB = AE + EC$	3. Addition
4. $AB = AC$	4. Completion

Example **2.**
Proof:

STATEMENTS	AUTHORITIES
1. $AD = BC$	1. Hypothesis
2. $BC = EF$	2. Hypothesis
3. $AD = EF$	3. Transitive

Example 3.

Proof:

STATEMENTS	AUTHORITIES
1. $AB = AC$	1. Hypothesis
2. $DB = EC$	2. Hypothesis
3. $AB - DB = AC - EC$	3. Subtraction
4. $AD = AE$	4. Completion

Example 4.

Proof:

STATEMENTS	AUTHORITIES
1. $\angle ABC = \angle 1 + \angle 2 + \angle 3$	1. Completion
2. $\angle 1 = \angle 2 = \angle 3 = 30°$	2. Hypothesis
3. $\angle ABC = (30 + 30 + 30)°$	3. Replacement
4. $\angle ABC = 90°$	4. Completion

Example 5.

Proof:

STATEMENTS	AUTHORITIES
1. $\angle 1 + \angle 3 = 180°$	1. Hypothesis
2. $\angle 1 = \angle 2$	2. Hypothesis
3. $\angle 2 + \angle 3 = 180°$	3. Replacement

Section 8.7 (page 280)

Example 1.

Proof:

STATEMENTS	AUTHORITIES
1. $\angle ABC = 180°$	1. Hypothesis, definition
2. $2x + 2y = 180$	2. Completion
3. $x + y = 90$	3. Division
4. $\angle EBF = 90°$	4. Completion

Example 2.

Proof:

STATEMENTS	AUTHORITIES
1. ~~$\angle 4 = \angle 3$~~ $\angle ABD = \angle ACE$	1. Hypothesis
2. $\angle 1 + \angle 4 = \angle 2 + \angle 3$	2. Completion
3. $\angle 1 = \angle 2$	3. Hypothesis
4. $\angle 4 = \angle 3$	4. Subtraction

Example 3.

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $AB = CD$	1. $OB - OA = OD - OC$.
2. $OB - OA = OD - OC$	2. $OB = OD$ and $OA = OC$.
3. $OB = OD$ and $OA = OC$ by hypothesis and definition of circle.	

Proof:

STATEMENTS	AUTHORITIES
1. $OB = OD$	1. Hypothesis and definition of circle
2. $OA = OC$	2. Hypothesis and definition of circle
3. $OB - OA = OD - OC$	3. Subtraction
4. $AB = CD$	4. Replacement

Chapter IX

Section 9.2 (page 285)

Complementary Angle Theorem

Corollary:

Proof:

STATEMENTS	AUTHORITIES
1. $\angle 2$ is the complement $\angle 1$.	1. Hypothesis
2. $\angle 3$ is the complement $\angle 1$.	2. Hypothesis
3. $\angle 2 = \angle 3$	3. Complementary Angle Theorem

Section 9.4 (page 287)

Supplementary Angle Theorem

Corollary:

Proof:

STATEMENTS	AUTHORITIES
1. $\angle 2$ is the supplement $\angle 1$.	1. Hypothesis
2. $\angle 3$ is the supplement $\angle 1$.	2. Hypothesis
3. $\angle 2 = \angle 3$	3. Supplementary Angle Theorem

Example.

Proof:

STATEMENTS	AUTHORITIES
1. $\angle ABD = \angle DBC = 90°$	1. Hypothesis
2. $\angle ABE = \angle CBF$	2. Hypothesis
3. $\angle EBD = \angle DBF$	3. Complementary Angle Theorem
4. $2x - 60 = x + 10$	4. Replacement
5. $x = 70$	5. Subtraction, addition
6. $x + 10 = 80$	6. Addition
7. $y = 10$	7. Definition
8. $\angle ABE = 10°$	8. Replacement

Chapter X

Section 10.1 (page 299)

1. $P \leftrightarrow L, Q \leftrightarrow N, R \leftrightarrow M$; $PQ = LN, PR = LM, QR = NM$; $\angle QPR = \angle NLM, \angle PQR = \angle LNM, \angle QRP = \angle NML$

2. $X \leftrightarrow X, Y \leftrightarrow Z, K \leftrightarrow K$; $XY = XZ, XK = XK, YK = ZK$; $\angle XYK = \angle XZK, \angle XKY = \angle XKZ, \angle YXK = \angle ZXK$

3. $K \leftrightarrow E, G \leftrightarrow G, R \leftrightarrow S$; $KG = EG, KR = GS, GR = GS$; $\angle KGR = \angle EGS, \angle GRK = \angle GSE, \angle RKG = \angle SEG$

4. Each of the triangles is congruent to each of the others.

5. $\angle BAC = \angle EDF, \angle BCA = \angle EFD, AC = DF$

6. $\angle QRP = \angle TMP$, $\angle QPR = \angle TPM$, $\angle PQR = \angle PTM$
7. $\angle GHL = \angle GKL$, $HL = KL$, $GL = GL$

Section 10.3 (page 309)

Example 1.

Proof:

STATEMENTS	AUTHORITIES
1. $P \rightarrow \sim Q$	1. Hypothesis
2. P	2. Hypothesis
3. $\sim Q$	3. Law of Detachment 1, 2
4. $\sim Q \rightarrow R$	4. Hypothesis
5. R	5. Law of Detachment 3, 4
6. $R \rightarrow \sim S$	6. Hypothesis
7. $\sim S$	7. Law of Detachment 5, 6

Example 2.

Proof:

STATEMENTS	AUTHORITIES
1. $\angle BDE$ is the supplement $\angle ADE$.	1. Definition
2. $\angle CED$ is the supplement $\angle AED$.	2. Definition
3. $\angle ADE = \angle AED$	3. Hypothesis
4. $\angle BDE = \angle CED$	4. Supplementary Angle Theorem
5. In $\triangle$'s BED and CDE $\begin{cases} BD = CE \\ DE = ED \\ \angle BDE = \angle CED \end{cases}$	5. Hypothesis
6.	6. Reflexive
7.	7. 4, above
8. $\triangle BED \cong \triangle CDE$	8. sas
9. $BE = CD$	9. Definition

Section 10.4 (page 315)

Example 2.

Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s APB and AQB $\begin{cases} PB = QB \\ AB = AB \\ PA = QA \end{cases}$	1. Definition
2.	2. Reflexive
3.	3. Definition
4. $\triangle APB \cong \triangle AQB$	4. sss
5. $\angle PAB = \angle QAB$	5. Definition

Section 10.5 (page 317)

Example.

Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s DBC and ACB $\begin{cases} DC = AB \\ CB = BC \\ \angle DCB = \angle ABC \end{cases}$	1. Hypothesis
2.	2. Reflexive
3.	3. Hypothesis
4. $\triangle DBC \cong \triangle ACB$	4. sas
5. $DB = AC$	5. Definition
6. In $\triangle$'s DAB and ADC $\begin{cases} AB = DC \\ DA = AD \\ DB = AC \end{cases}$	6. Hypothesis
7.	7. Reflexive
8.	8. 5
9. $\triangle DAB \cong \triangle ADC$	9. sss
10. $\angle DAB = \angle ADC$	10. Definition

Chapter XI

Section **11.4** (page 337)

Example.

Proof:

STATEMENTS	AUTHORITIES
1. $AD = BD$	1. Hypothesis
2. $\angle 1 = \angle 2$	2. Isosceles Triangle Theorem
3. $AD = CD$	3. Hypothesis
4. $\angle 4 = \angle 3$	4. Isosceles Triangle Theorem
5. $\angle 1 + \angle 4 = \angle 2 + \angle 3$	5. Addition
6. $\angle BAC = \angle 1 + \angle 4$	6. Completion

Section **11.6** (page 340)

Right Bisector Theorem PART I

Proof:

STATEMENTS	AUTHORITIES
1. Join PA and PB.	1. Existence
2. In $\triangle$'s PAE and PBE $\left\{\begin{matrix} AE = BE \\ PE = PE \\ \angle PEA = \angle PEB \end{matrix}\right.$	2. Hypothesis
3.	3. Reflexive
4.	4. Definition
5. $\triangle PAE \cong \triangle PBE$	5. sas
6. $PA = PB$	6. Definition

Right Bisector Theorem PART II

Proof:

STATEMENTS	AUTHORITIES
1. Q is the midpoint of AB.	1. Existence
2. In $\triangle$'s PAQ and PBQ $\left\{\begin{matrix} PA = PB \\ AQ = BQ \\ PQ = PQ \end{matrix}\right.$	2. Hypothesis
3.	3. Definition
4.	4. Reflexive
5. $\triangle PAQ \cong \triangle PBQ$	5. sss
6. $\angle PQA = \angle PQB$	6. Definition
7. $\angle PQA + \angle PQB = 180°$	7. Definition, completion
8. $2\angle PQA = 180°$	8. Replacement, completion
9. $\angle PQA = 90°$	9. Division
10. $PQ \perp AB$	10. Definition
11. P is on the right bisector of AB.	11. Definition

Chapter XII

Section **12.1** (page 346)

1. $\triangle ABC$ is not scalene.
2. John is not studying.
3. AB is not perpendicular to CD.
4. $\angle ABC \neq 60°$.
5. It is raining.
6. $\angle ABC = \angle DEF$.
7. $\angle ABC \not> \angle DEF$.
8. AB is parallel to CD.
9. 5 is not greater than 10.
10. Jean is not the sister of Betty.

Example 3.

Proof:

STATEMENTS	AUTHORITIES
1. $AB = AC$	1. Assumption of negation of conclusion $AB \neq AC$
2. In $\triangle$'s ABD and ACD $\begin{cases} BD = CD \\ AD = AD \\ AB = AC \end{cases}$	2. Hypothesis 3. Hypothesis 4. 1, above
5. $\triangle ABD \cong \triangle ACD$	5. sss
6. $\angle ADB = \angle ADC = 90°$	6. Definition, completion
7. $AB = AC$ is false.	7. Since 6 contradicts the hypothesis
8. $AB \neq AC$	8. Law of Contradiction

Example 4.

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle B \neq \angle C$	1. $\angle B = \angle C$ is false.
2. $\angle B = \angle C$ is false	2. The assumption $\angle B = \angle C$ leads to a conclusion which contradicts a known fact.
3. The assumption $\angle B = \angle C$ leads to the conclusion that $AC = AB$ by the Isosceles Triangle Theorem. This contradicts the hypothesis.	

Proof:

STATEMENTS	AUTHORITIES
1. $\angle B = \angle C$	1. Assumption of negation of conclusion $\angle B \neq \angle C$
2. $AC = AB$	2. Isosceles Triangle Theorem
3. $\angle B = \angle C$ is false.	3. Since 2 contradicts the hypothesis
4. $\angle B \neq \angle C$	4. Law of Contradiction

Chapter XIII

Section 13.4 (page 359)

Parallel Line Theorem PART I (i)

Proof:

STATEMENTS	AUTHORITIES
1. $l \nparallel m$	1. Assumption of negation of conclusion
2. l and m have a point in common.	2. Definition
3. $\angle 1 \neq \angle 2$	3. Exterior Angle Theorem
4. $l \nparallel m$ is false.	4. 3 contradicts hypothesis
5. $l \parallel m$	5. Law of Contradiction

Parallel Line Theorem PART I (ii)

Proof:

STATEMENTS	AUTHORITIES
1. $\angle 3 = \angle 2$	1. Hypothesis
2. $\angle 3 = \angle 1$	2. Vertical Angle Theorem
3. $\angle 1 = \angle 2$	3. Replacement
4. $l \parallel m$	4. Parallel Line Theorem

Parallel Line Theorem PART I (iii)

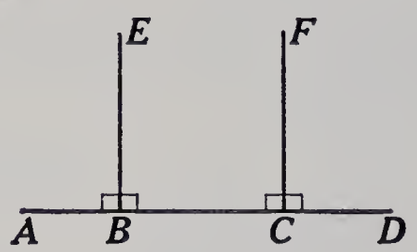
Proof:

STATEMENTS	AUTHORITIES
1. $\angle 4 + \angle 2 = 180°$	1. Hypothesis
2. $\angle 1 + \angle 4 = 180°$	2. Definition, completion
3. $\angle 1 + \angle 4 = \angle 4 + \angle 2$	3. Replacement
4. $\angle 1 = \angle 2$	4. Subtraction
5. $l \parallel m$	5. Parallel Line Theorem (i)

Parallel Line Theorem

Corollary:

Analysis:



I CAN PROVE	IF I CAN PROVE
1. $EB \parallel CF$	1. (i) A pair of alternate angles equal. (ii) A pair of corresponding angles equal. (iii) A pair of interior angles on the same side of the transversal supplementary.
2. (ii) or (iii) by hypothesis, using AD as a transversal.	

Proof:

STATEMENTS	AUTHORITIES
1. $\angle EBC = \angle FCD = 90°$	1. Hypothesis and definition
2. $BE \parallel CF$	2. Corresponding angles

Example 3.

Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s ADC and CBA $\begin{cases} CD = BA \\ AC = CA \\ AD = CB \end{cases}$	1. Hypothesis 2. Reflexive 3. Hypothesis
4. $\triangle ADC \cong \triangle CBA$	4. sss
5. $\angle BAC = \angle DCA$, $\angle BCA = \angle DAC$	5. Definition
6. $AB \parallel DC$, $BC \parallel AD$	6. Equal alternate angles
7. $ABCD$ is a parallelogram.	7. Definition

Section 13.7 (page 371)

Quadrilateral-Parallelogram Theorem

Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s ADB and CBD $\begin{cases} DB = BD \\ AD = CB \\ \angle ADB = \angle CBD \end{cases}$	1. Reflexive 2. Hypothesis 3. Equal alternate angles
4. $\triangle ADB \cong \triangle CBD$	4. sas
5. $\angle ABD = \angle CDB$	5. Definition
6. $AB \parallel DC$	6. Equal alternate angles
7. $ABCD$ is a parallelogram.	7. Definition

Parallelogram Theorem

Proof:

STATEMENTS	AUTHORITIES
PARTS (i) and (ii)	
1. Join BD.	1. Existence
2. $AD \parallel BC$	2. Definition
3. $\angle ADB = \angle CBD$	3. Equal alternate angles
4. $AB \parallel DC$	4. Definition
5. $\angle ABD = \angle BDC$	5. Equal alternate angles
6. In $\triangle$'s ABD and CDB $\begin{cases} BD = DB \\ \angle ABD = \angle CDB \\ \angle ADB = \angle CBD \end{cases}$	6. Reflexive 7. 5 8. 3
9. $\triangle ABD \cong \triangle CDB$	9. asa
10. $\angle A = \angle C$	10. Definition
11. $AD = CB$	11. Definition
12. $AB = CD$	12. Definition
13. $\angle ADB + \angle CDB = \angle ABD + \angle CBD$	13. 3, 5 Addition
14. $\angle ABC = \angle CDA$	14. Completion
PART (iii)	
1. Join AC.	1. Existence
2. $\angle CAD = \angle ACB$	2. 2, Equal alternate angles
3. In $\triangle$'s AED and CEB $\begin{cases} AD = CB \\ \angle EAD = \angle ECB \\ \angle EDA = \angle EBC \end{cases}$	3. Part (ii) 4. 2 5. Equal alternate angles
6. $\triangle AED \cong \triangle CEB$	6. asa
7. $AE = CE$, $BE = DE$	7. Definition
8. AC and BD bisect each other.	8. Definition

Section 13.8 (page 375)

Triangle Angle Sum Theorem

Proof:

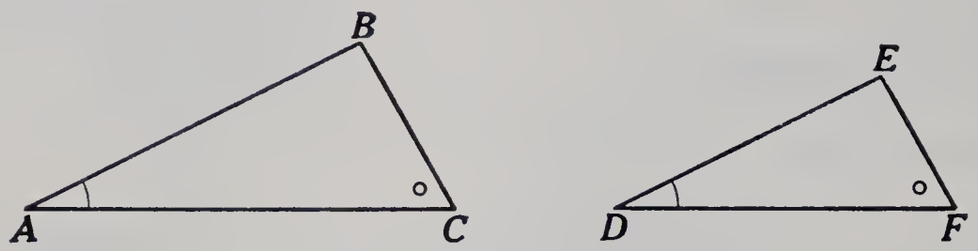
STATEMENTS	AUTHORITIES
1. Ray $AD \parallel BC$	1. Existence
2. $\angle 2 = \angle 4$	2. Equal alternate angles
3. $\angle DAC + \angle 3 = 180°$	3. Supplementary angles
4. $\angle 1 + \angle 4 + \angle 3 = 180°$	4. Completion
5. $\angle 1 + \angle 2 + \angle 3 = 180°$	5. 2 and Replacement

Corollary 1:

Proof:

STATEMENTS	AUTHORITIES
1. $\angle 1 + \angle 2 + \angle 3 = 180°$	1. Triangle Angle Sum Theorem
2. $\angle 4 + \angle 3 = 180°$	2. Definition
3. $\angle 1 + \angle 2 + \angle 3 = \angle 4 + \angle 3$	3. Replacement
4. $\angle 1 + \angle 2 = \angle 4$	4. Subtraction

Corollary 2:



Hypothesis: $\triangle ABC$ and $\triangle DEF$ with $\angle A = \angle D$ and $\angle C = \angle F$

Conclusion: $\angle B = \angle E$

Proof:

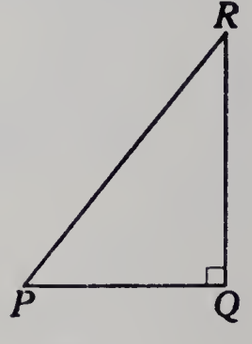
STATEMENTS	AUTHORITIES
1. $\angle A + \angle B + \angle C = 180°$	1. Triangle Angle Sum Theorem
2. $\angle D + \angle E + \angle F = 180°$	2. Triangle Angle Sum Theorem
3. $\angle A + \angle B + \angle C = \angle D + \angle E + \angle F$	3. Replacement
4. $\angle A = \angle D$, $\angle C = \angle F$	4. Hypothesis
5. $\angle B = \angle E$	5. Subtraction

Corollary 3:

Hypothesis: $\triangle PQR$ is a right triangle with $\angle Q = 90°$.

Conclusion: $\angle P$ and $\angle R$ are complementary.

Proof:



STATEMENTS	AUTHORITIES
1. $\angle P + \angle R + \angle Q = 180°$	1. Triangle Angle Sum Theorem
2. $\angle Q = 90°$	2. Hypothesis
3. $\angle P + \angle R + 90° = 180°$	3. Replacement
4. $\angle P + \angle R = 90°$	4. Subtraction
5. $\angle P$ and $\angle R$ are complementary.	5. Definition

Corollary 4:

Proof:

STATEMENTS	AUTHORITIES
1. $\angle C = \angle A + \angle B$	1. Hypothesis
2. $2\angle C = \angle A + \angle B + \angle C$	2. Addition
3. $\angle A + \angle B + \angle C = 180°$	3. Triangle Angle Sum Theorem
4. $2\angle C = 180°$	4. Replacement
5. $\angle C = 90°$	5. Division

Example 1.

Proof:

STATEMENTS	AUTHORITIES
1. $OA = OC$	1. Definition
2. $\angle 1 = \angle 3$	2. Isosceles Triangle Theorem
3. $OC = OB$	3. Definition
4. $\angle 2 = \angle 4$	4. Isosceles Triangle Theorem
5. $\angle 1 + \angle 2 = \angle 3 + \angle 4$	5. 2, 4 Addition
6. $\angle 1 + \angle 2 = \angle ACB$	6. Completion
7. $\angle ACB = 90°$	7. Triangle Angle Sum Theorem, Corollary

Example 2.

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\angle ADC = 3\angle x$	1. $\angle w + \angle z = 3\angle x$ ($\because \angle ADC = \angle z + \angle w$).
2. $\angle z + \angle w = 3\angle x$	2. $\angle y + \angle z + \angle z = 3\angle x$ ($\because \angle w = \angle y + \angle z$).
3. $\angle y + \angle z + \angle z = 3\angle x$	3. $2\angle x + \angle z = 3\angle x$ ($\because \angle y + \angle z = 2\angle x$).
4. $2\angle x + \angle z = 3\angle x$	4. $\angle z = \angle x$.
5. $\angle z = \angle x$ since $\angle z = \angle y$ and $\angle y + \angle z = 2\angle x$.	

Proof:

STATEMENTS	AUTHORITIES
1. $\angle y + \angle z = 2\angle x$	1. Hypothesis
2. $\angle y = \angle z$	2. Hypothesis
3. $2\angle z = 2\angle x$	3. Replacement
4. $\angle z = \angle x$	4. Division
5. $\angle w = \angle y + \angle z$	5. Hypothesis
6. $\angle w = 2\angle z$	6. 2 and Replacement
7. $\angle w = 2\angle x$	7. 4 and Replacement
8. $\angle ADC = \angle w + \angle z$	8. Triangle Angle Sum Theorem, Corollary
9. $\angle ADC = 3\angle x$	9. 4, 7, Replacement

Section 13.9 (page 381)

The aas *Congruence Theorem*

Proof:

STATEMENTS	AUTHORITIES
1. $\angle A = \angle D$, $\angle B = \angle E$	1. Hypothesis
2. $\angle C = \angle F$	2. Triangle Angle Sum Theorem, Corollary
3. In $\triangle$'s ABC and DEF $BC = EF$	3. Hypothesis
4. $\angle B = \angle E$	4. Hypothesis
5. $\angle C = \angle F$	5. 2
6. $\triangle ABC \cong \triangle DEF$	6. asa

Section 13.10 (page 383)

Right Triangle Congruence Theorem

Proof:

STATEMENTS	AUTHORITIES
1. On the ray opposite FE, G is the point such that $FG = CB$.	1. Existence
2. In $\triangle$'s ABC, DGF $\left\{ \begin{array}{r} AC = DE \\ BC = GF \\ \angle ACB = \angle DFG \\ = 90° \end{array} \right.$	2. Hypothesis 3. 1 4. Definition
5. $\triangle ABC \cong \triangle DGF$	5. sas
6. $DG = AB = DE$	6. Definition, Replacement
7. $\angle DFG = \angle ABC = \angle DEF$	7. Definition, Isosceles Triangle Th.
8. $\triangle ABC \cong \triangle DEF$	8. aas

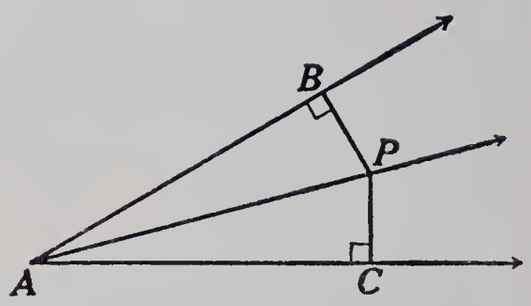
Section 13.11 (page 384)

Angle Bisector Theorem PART I

Hypothesis: PA is the bisector of $\angle BAC$, $BP \perp AB$, $PC \perp AC$.

Conclusion: $PB = PC$

Proof:



STATEMENTS	AUTHORITIES
1. In $\triangle$'s APB and APC $\left\{ \begin{array}{r} AP = AP \\ \angle BAP = \angle CAP \\ \angle ABP = \angle ACP \end{array} \right.$	1. Reflexive 2. Definition 3. Definition
4. $\triangle APB \cong \triangle APC$	4. aas
5. $PB = PC$	5. Definition

Angle Bisector Theorem PART II

Hypothesis: $PB \perp AB$, $PC \perp AC$, $PB = PC$

Conclusion: AP is the bisector of $\angle BAC$.

Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s APB and APC $\left\{ \begin{array}{r} AP = AP \\ PB = PC \\ \angle ABP = \angle ACP \end{array} \right.$	1. Reflexive 2. Hypothesis 3. Definition
4. $\triangle APB \cong \triangle APC$	4. Right Triangle Congruence Theorem
5. $\angle BAP = \angle CAP$	5. Definition
6. AP is the bisector of $\angle BAC$.	6. Definition

Chapter XIV

Section 14.4 (page 398)

Parallelogram Area Theorem

Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s ADE and BCF $\left\{ \begin{array}{r} AD = BC \\ \angle DAE = \angle CBF \\ \angle AED = \angle BFC \end{array} \right.$	1. Parallelogram Theorem 2. Parallel Line Theorem 3. Parallel Line Theorem
4. $\triangle ADE \cong \triangle BCF$	4. aas

5. $\triangle ADE = \triangle BCF$	5. Area postulate
6. Fig. $ADCF - \triangle BCF$ $= \text{Fig. } ADCF - \triangle ADE$	6. Area postulate
7. $\|\|^{gm} ABCD = \|\|^{gm} EFCD$	7. Completion

Corollary 1:
Proof:

STATEMENTS	AUTHORITIES
1. $HG = EF$	1. Parallelogram Theorem
2. $HG = DC$	2. Hypothesis
3. $DC = EF$	3. Replacement
4. $DC \parallel EF$	4. Hypothesis
5. $EFCD$ is a parallelogram.	5. Quadrilateral-Parallelogram Theorem
6. $\|\|^{gm} ABCD = \|\|^{gm} EFCD$	6. Parallelogram Area Theorem
7. $\|\|^{gm} EFGH = \|\|^{gm} EFCD$	7. Parallelogram Area Theorem
8. $\|\|^{gm} ABCD = \|\|^{gm} EFGH$	8. Replacement

Example.
Proof:

STATEMENTS	AUTHORITIES
1. $\|\|^{gm} ABCD = \|\|^{gm} AHGD$	1. Parallelogram Area Theorem
2. $\|\|^{gm} BEFC = \|\|^{gm} HEFG$	2. Parallelogram Area Theorem
3. $\|\|^{gm} ABCD + \|\|^{gm} BEFC$ $= \|\|^{gm} AHGD + \|\|^{gm} HEFG$	3. Addition
4. $\|\|^{gm} AHGD + \|\|^{gm} HEFG$ $= \|\|^{gm} AEFD$	4. Completion
5. $\|\|^{gm} ABCD + \|\|^{gm} BEFC$ $= \|\|^{gm} AEFD$	5. Replacement

Section 14.5 (page 401)

Parallelogram-Diagonal Theorem

Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s ABC and CDA $\begin{cases} AB = CD \\ BC = DA \\ CA = AC \end{cases}$	1. Parallelogram Theorem 2. Parallelogram Theorem 3. Reflexive
4. $\triangle ABC \cong \triangle CDA$	4. sss
5. $\triangle ABC = \triangle CDA$	5. Definition
6. $\triangle ABD = \triangle CDB$	6. As in 1 to 5

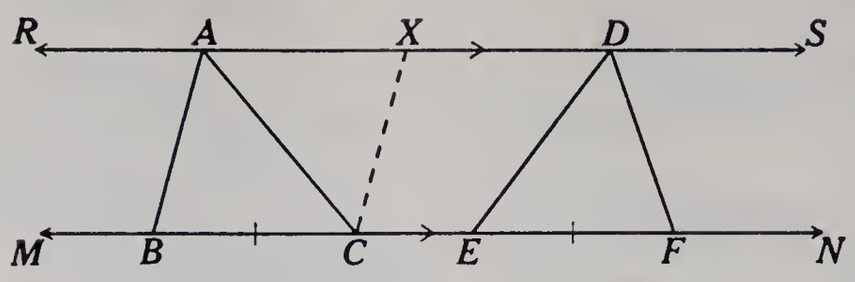
Section 14.6 (page 401)

Parallelogram-Triangle Area Theorem

Proof:

STATEMENTS	AUTHORITIES
1. CX is a line segment terminated on XY such that $CX \parallel AB$.	1. Existence
2. $2\ \triangle ABC = \|\|^{gm} ABCX$	2. Parallelogram-Diagonal Theorem
3. $\|\|^{gm} ABCX = \|\|^{gm} DGFE$	3. Parallelogram Area Theorem
4. $2\ \triangle ABC = \|\|^{gm} DGFE$	4. Replacement
5. $\triangle ABC = \frac{1}{2} \|\|^{gm} DGFE$	5. Division

Corollary 1:



Hypothesis: $\triangle$'s ABC and DEF on equal bases BC and EF, and between the same parallel lines MN and RS.

Conclusion: $\triangle ABC = \triangle DEF$

Proof:

STATEMENTS	AUTHORITIES
1. XC is a line segment with X on RS such that $XC \parallel AB$.	1. Existence
2. $ABCX$ is a parallelogram.	2. Definition
3. $\triangle ABC = \frac{1}{2} \parallel^{gm} ABCX$	3. Parallelogram-Diagonal Theorem
4. $\triangle DEF = \frac{1}{2} \parallel^{gm} ABCX$	4. Parallelogram-Triangle Area Theorem
5. $\triangle ABC = \triangle DEF$	5. Replacement

Example 1.

Proof:

STATEMENTS	AUTHORITIES
1. $\triangle ABD = \triangle ADC$	1. Parallelogram-Triangle Area Theorem
2. $\triangle ABE = \triangle EBD$	2. Parallelogram-Triangle Area Theorem
3. $2\triangle ABE = \triangle ADC$	3. 2, Completion
4. $2\triangle ADC = \triangle ABC$	4. 1, Completion
5. $4\triangle ABE = \triangle ABC$	5. Replacement
6. $\triangle ABE = \frac{1}{4}\triangle ABC$	6. Division

Example 2.

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $\triangle ABD = 2\triangle ECD$	1. $\triangle ECD = \triangle ACD$; $2\triangle ACD = \triangle ABD$.
2. $\triangle ECD = \triangle ACD$ by the Parallelogram-Triangle Area Theorem.	
3. $2\triangle ACD = \triangle ABD$ by the Parallelogram-Triangle Area Theorem.	

Proof:

STATEMENTS	AUTHORITIES
1. $\triangle ECD = \triangle ACD$	1. Parallelogram-Triangle Area Theorem
2. $\triangle ACD = \triangle ACB$	2. Parallelogram-Triangle Area Theorem
3. $2\ \triangle ACD = \triangle ABD$	3. Completion
4. $2\ \triangle ECD = \triangle ABD$	4. 1, 3, Replacement

Section 14.10 (page 407)

Equal Triangle-Parallel Line Theorem

Proof:

STATEMENTS	AUTHORITIES
1. $\triangle ABC = \triangle DBC$	1. Hypothesis
2. $\frac{1}{2}bh_1 = \frac{1}{2}bh_2$	2. Area of a triangle
3. $h_1 = h_2$	3. Division
4. $AX \parallel DY$	4. Parallel Line Theorem
5. $AD \parallel BC$	5. Parallelogram Theorem, Quadrilateral-Parallelogram Th.

Section 14.11 (page 414)

Example 2.

(ii) $x^2 = 1^2 + 1^2$
$\therefore x^2 = 1 + 1$
$\therefore x^2 = 2$
$\therefore x = \sqrt{2}$

(iii) $x^2 = 3^2 + 2^2$
$\therefore x^2 = 9 + 4$
$\therefore x^2 = 13$
$\therefore x = \sqrt{13}$

(iv) $5^2 = x^2 + 2^2$
$\therefore 25 = x^2 + 4$
$\therefore x^2 = 21$
$\therefore x = \sqrt{21}$

Example 5.

Proof:

STATEMENTS	AUTHORITIES
1. $a^2 = x^2 + y^2$	1. Pythagorean Theorem
2. $b^2 = w^2 + z^2$	2. Pythagorean Theorem
3. $c^2 = x^2 + w^2$	3. Pythagorean Theorem
4. $d^2 = y^2 + z^2$	4. Pythagorean Theorem
5. $a^2 + b^2 = x^2 + y^2 + w^2 + z^2$	5. Addition
6. $c^2 + d^2 = x^2 + w^2 + y^2 + z^2$	6. Addition
7. $a^2 + b^2 = c^2 + d^2$	7. Replacement

Section 14.12 (page 419)

Example 1.

(ii) $7^2 = 49, 9^2 = 81, 3^2 = 9$,
$\because 7^2 + 3^2 \neq 9^2$
$\therefore$ 7, 3, 9 units could not be the lengths of the sides of a right triangle.

(iii) $1.5^2 = 2.25, 2^2 = 4, 2.5^2 = 6.25$
$\because 1.5^2 + 2^2 = 2.5^2$
$\therefore$ 1.5, 2, 2.5 units could be the lengths of the sides of a right triangle.

(iv) $7^2 = 49,\ 24^2 = 576,\ 25^2 = 625$
$\because 7^2 + 24^2 = 25^2$
$\therefore$ 7, 24, 25 units could be the lengths of the sides of a right triangle.

(v) $42^2 = 1{,}764,\ 40^2 = 1{,}600,\ 58^2 = 3{,}364$
$\because 42^2 + 40^2 = 58^2$
$\therefore$ 42, 40, 58 units could be the lengths of the sides of a right triangle.

(vi) $2.5^2 = 6.25, 6^2 = 36, 6.5 = 42.25$
$\because 2.5^2 + 6^2 = 6.5^2$
$\therefore$ 2.5, 6, 6.5 units could be the lengths of the sides of a right triangle.

Section 14.13 (page 422)

Example 1.

$$\begin{aligned}\text{Trapezoid } ADEB &= \tfrac{1}{2}(5+3)\cdot 2 \text{ sq. units}\\ &= 8 \text{ sq. units}\\ \text{Trapezoid } BEFC &= \tfrac{1}{2}(3+7)\cdot 6 \text{ sq. units}\\ &= 30 \text{ sq. units}\\ \text{Trapezoid } ADFC &= \tfrac{1}{2}(5+7)\cdot 8 \text{ sq. units}\\ &= 48 \text{ sq. units}\end{aligned}$$

$\triangle ABC$ = trapezoid $ADFC$ − (trapezoid $ADEB$ + trapezoid $BEFC$)

$\therefore \triangle ABC = (48 - 38)$ sq. units

$\therefore \triangle ABC = 10$ sq. units

Example 2.

Proof:

STATEMENTS	AUTHORITIES
1. $\triangle ABC \cong \triangle DEB$	1. sas
2. $BE = a$ units	2. Definition
3. $\angle 1 = \angle 3$	3. Definition
4. $\angle 1 + \angle 2 = 90°$	4. Triangle Angle Sum Theorem
5. $\angle 3 + \angle 2 = 90°$	5. Replacement
6. $\angle CBE = 90°$	6. Completion, definition
7. Trapezoid $CADE = \triangle ABC + \triangle DEB + \triangle CBE$	7. Completion
8. $\frac{1}{2}(b+c)(c+b) = \frac{1}{2}cb + \frac{1}{2}cb + \frac{1}{2}a^2$	8. Trapezoid Area Theorem, Area of Triangle
9. $b^2 + 2bc + c^2 = 2bc + a^2$	9. Number axioms
10. $b^2 + c^2 = a^2$	10. Subtraction

Chapter XV

Section 15.3 (page 429)

Hypothesis: $a, b, c, d \neq 0$ and $\dfrac{a}{b} = \dfrac{c}{d}$

Conclusion: (ii) $\dfrac{a}{c} = \dfrac{b}{d}$

(iii) $\dfrac{b}{a} = \dfrac{d}{c}$

(iv) $\dfrac{a+b}{b} = \dfrac{c+d}{d}$

(v) $\dfrac{a-b}{b} = \dfrac{c-d}{d}$

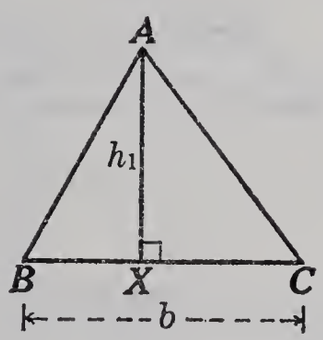
Proof:

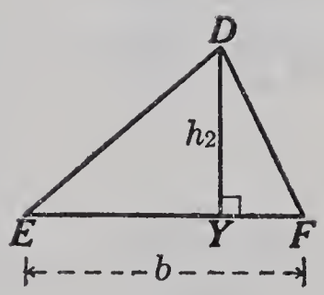
STATEMENTS	AUTHORITIES
(ii) 1. $\dfrac{a}{b} = \dfrac{c}{d}$	1. Hypothesis
2. $\dfrac{a}{c} = \dfrac{b}{d}$	2. Multiplication by $\dfrac{b}{c}$

(iii) 1.	$\frac{a}{b} = \frac{c}{d}$	1. Hypothesis
2.	$\frac{\frac{1}{a}}{\frac{1}{b}} = \frac{\frac{1}{c}}{\frac{1}{d}}$	2. Division
3.	$\frac{b}{a} = \frac{d}{c}$	3. Definition
(iv) 1.	$\frac{a}{b} = \frac{c}{d}$	1. Hypothesis
2.	$\frac{a}{b} + 1 = \frac{c}{d} + 1$	2. Addition
3.	$\frac{a+b}{b} = \frac{c+d}{d}$	3. Definition
(v) 1.	$\frac{a}{b} = \frac{c}{d}$	1. Hypothesis
2.	$\frac{a}{b} - 1 = \frac{c}{d} - 1$	2. Subtraction
3.	$\frac{a-b}{b} = \frac{c-d}{d}$	3. Definition

Section 15.4 (page 432)

Triangle Area Ratio Theorem





Corollary:

Hypothesis: $\triangle$'s ABC and DEF with altitudes of lengths h_1 and h_2 units, respectively, and bases of length b units

Conclusion: $\frac{\triangle ABC}{\triangle DEF} = \frac{h_1}{h_2}$

Proof:

STATEMENTS	AUTHORITIES
1. $\triangle ABC = \frac{1}{2} bh_1$ sq. units	1. Area of Triangle
2. $\triangle DEF = \frac{1}{2} bh_2$ sq. units	2. Area of Triangle
3. $\frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2} bh_1}{\frac{1}{2} bh_2}$	3. Division
4. $\frac{\triangle ABC}{\triangle DEF} = \frac{h_1}{h_2}$	4. Principle of Equivalent Fractions

Section 15.5 (page 435)

Triangle Proportionality Theorem

Proof:

STATEMENTS	AUTHORITIES
1. Join BE, DC.	1. Existence
2. $\frac{\triangle AED}{\triangle DEB} = \frac{AD}{DB}$	2. Triangle Area Ratio Th.
3. $\frac{\triangle AED}{\triangle DEC} = \frac{AE}{EC}$	3. Triangle Area Ratio Th.
4. $DE \parallel BC$	4. Hypothesis
5. $\triangle DEB = \triangle DEC$	5. Parallelogram-Triangle Area Th.
6. $\frac{\triangle AED}{\triangle DEB} = \frac{\triangle AED}{\triangle DEC}$	6. Replacement
7. $\frac{AD}{DB} = \frac{AE}{EC}$	7. Replacement

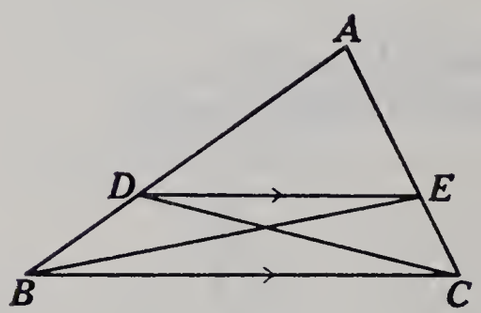
Triangle Proportionality Theorem

Corollary:

Hypothesis: $\triangle ABC$, $DE \parallel BC$

Conclusion: (i) $\frac{AB}{DB} = \frac{AC}{EC}$

(ii) $\frac{AB}{AD} = \frac{AC}{AE}$



Proof:

STATEMENTS	AUTHORITIES
(i) 1. $DE \parallel BC$	1. Hypothesis
2. $\frac{AD}{DB} = \frac{AE}{EC}$	2. Triangle Proportionality Th.
3. $\frac{AD + DB}{DB} = \frac{AE + EC}{EC}$	3. Adding unity
4. $\frac{AB}{DB} = \frac{AC}{EC}$	4. Completion
(ii) 1. $\frac{DB}{AD} = \frac{EC}{AE}$	1. Part (i) 2, inversion
2. $\frac{DB + AD}{AD} = \frac{EC + AE}{AE}$	2. Adding unity
3. $\frac{AB}{AD} = \frac{AC}{AE}$	3. Completion

Example.

Proof:

STATEMENTS	AUTHORITIES
1. $ST \parallel QR$	1. Hypothesis
2. $\frac{PQ}{QS} = \frac{PR}{RT}$	2. Triangle Proportionality Th.
3. $TV \parallel PQ$	3. Hypothesis
4. $\frac{RQ}{RV} = \frac{RP}{RT}$	4. Triangle Proportionality Th.
5. $\frac{PQ}{QS} = \frac{RQ}{RV}$	5. Replacement

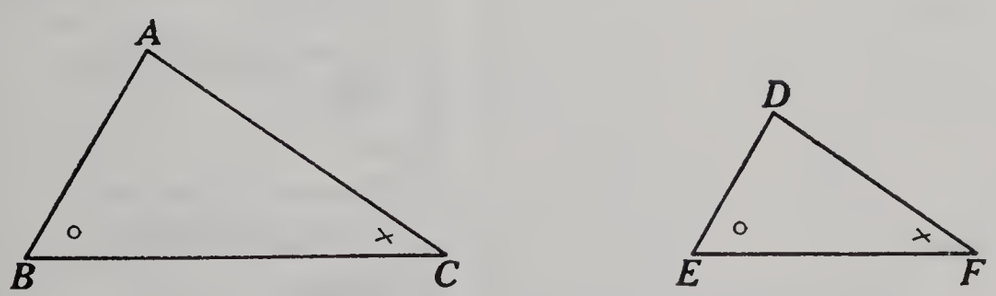
Section 15.7 (page 443)

The aaa *Similar Triangle Theorem*

Proof:

STATEMENTS	AUTHORITIES
1. X and Y are points on AC and AB such that $AX = DF$ and $AY = DE$.	1. Existence
2. In $\triangle$'s AXY and DFE $\begin{cases} AX = DF \\ AY = DE \\ \angle A = \angle D \end{cases}$	2. 1 3. 1 4. Hypothesis
5. $\triangle AXY \cong \triangle DFE$	5. sas
6. $\angle AXY = \angle DFE$	6. Definition
7. $\angle$ACB = $\angle$DFE	7. Hypothesis
8. $\angle$AXY = $\angle$ACB	8. Replacement
9. $XY \parallel CB$	9. Parallel Line Theorem
10. $\frac{AB}{AY} = \frac{AC}{AX}$	10. Triangle Proportionality Th.
11. $\frac{AB}{DE} = \frac{AC}{DF}$	11. 1, Replacement
12. $\frac{AB}{DE} = \frac{BC}{EF}$	12. Similar to 1 to 9 above
13. $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$	13. Replacement

The aaa *Similar Triangle Theorem*



Corollary: If two angles of one triangle are equal to two angles of another triangle, then the corresponding sides are proportional.

Hypothesis: $\triangle$'s ABC and DEF such that $\angle B = \angle E$ and $\angle C = \angle F$

Conclusion: $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s ABC and DEF $\begin{cases} \angle B = \angle E \\ \angle C = \angle F \\ \angle A = \angle D \end{cases}$	1. Hypothesis
2.	2. Hypothesis
3.	3. Triangle Angle Sum Th.
4. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$	4. aaa

Example 1.

Proof:

STATEMENTS	AUTHORITIES
(i) 1. In $\triangle$'s ABE and CDE $\begin{cases} \angle ABE = \angle CDE \\ \angle AEB = \angle CED \end{cases}$	1. Hypothesis
2.	2. Vertical Angle Theorem
3. $\triangle ABE \sim \triangle CDE$	3. aaa

Example 2.

Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s AXB and AYC $\begin{cases} \angle BAX = \angle CAY \\ \angle AXB = \angle AYC \end{cases}$	1. Reflexive
2.	2. Hypothesis
3. $\frac{AB}{AC} = \frac{AX}{AY}$	3. aaa
4. $AB \cdot AY = AC \cdot AX$	4. Multiplication

Section 15.8 (page 448)

Alternate proof for the Pythagorean Theorem

Proof:

STATEMENTS	AUTHORITIES
1.	1. Existence
2.	2. Existence
3.	3. Hypothesis, definition
4.	4. Reflexive
5.	5. aaa
6.	6. Cross-multiplication
7.	7. Hypothesis, definition
8.	8. Reflexive
9.	9. aaa
10.	10. Cross-multiplication
11.	11. Addition
12.	12. D
13.	13. Replacement

ANSWERS TO EXERCISES

Exercise A-1 (page 3)

17. (i) B, C, F; D, E, H; A, I (ii) A, I

18. $\{1, 2, 3, 5, 7, 11, 13, 17, 19, 23\}$

19. $\{\ \}$

20. $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30\}$

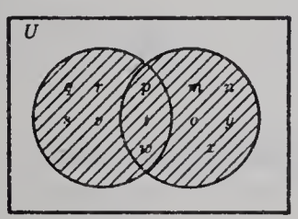
21. $\left\{\frac{3}{4}, \frac{3}{7}, \frac{3}{11}, \frac{4}{7}, \frac{4}{11}, \frac{7}{11}\right\}$

22. the set of points inside the circle
the set of points on the circle
the set of points outside the circle

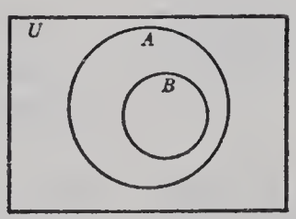
23. (i) the set of points inside both the circle and the rectangle
(ii) the set of points outside both the circle and the rectangle
(iii) the set of points inside the rectangle but outside the circle
(iv) the set of points outside the rectangle but inside the circle

Exercise A-2 (page 8)

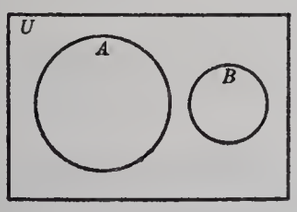
1. (i) and (ii)



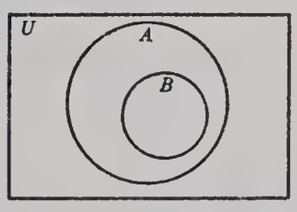
2.



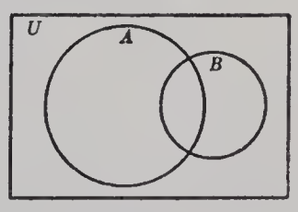
3. (i)



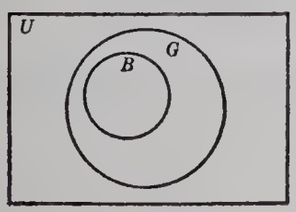
(ii)



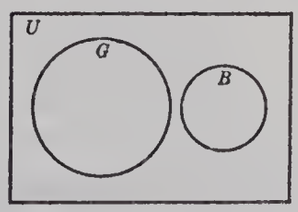
(iii)



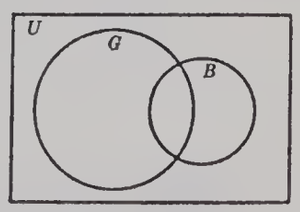
4. (i)



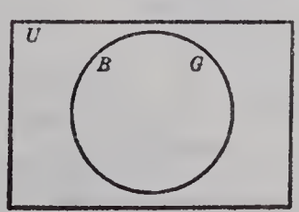
(ii)



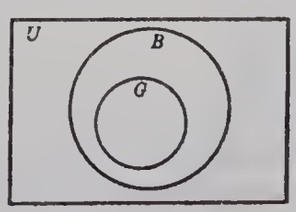
(iii)



(iv)

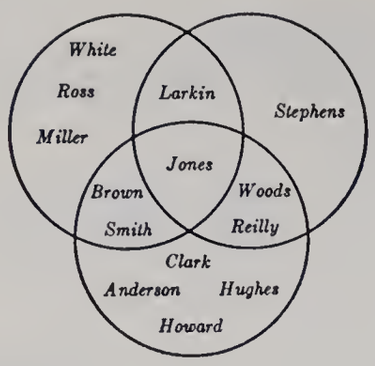


(v)



5. 21 **6.** 7 **7.** (i) 332 (ii) 300 **8.** 41

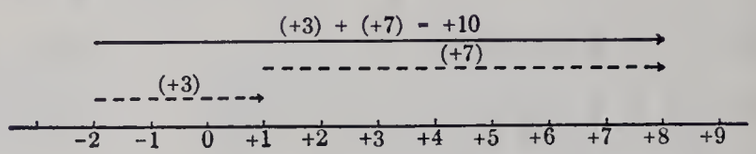
9.



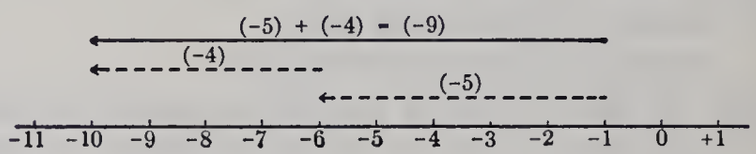
10. (i) 2 (ii) 2 (iii) 4 **11.** 9

Exercise B-1 (page 16)

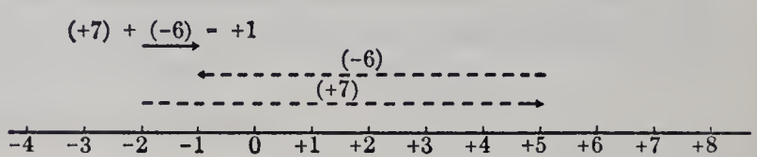
3. (i) $(+3) + (+7)$
$= +(3 + 7)$
$= +10$



(ii) $(-5) + (-4)$
$= -(5 + 4)$
$= -9$

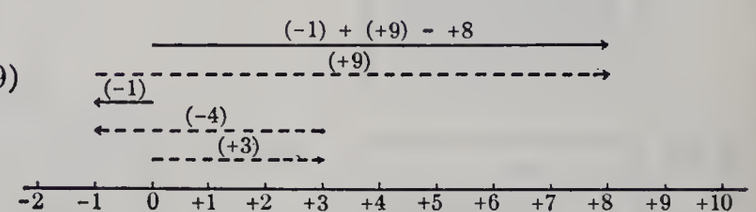


(iii) $(+7) + (-6)$
$= +(7 - 6)$
$= +1$

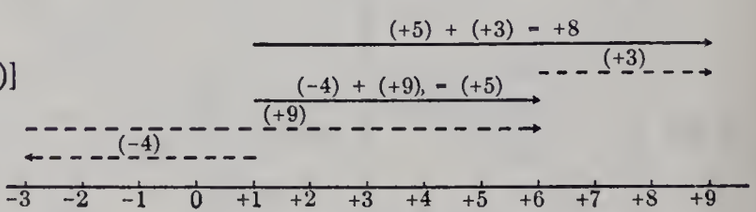


5. (i) +5 (ii) +7 (iii) −24 (iv) +14 (v) −9 (vi) 0
(vii) 0 (viii) −6 (ix) 0 (x) +12

6. (i) $[(+3) + (-4)] + (+9)$
$= (-1) + (+9)$
$= +8$

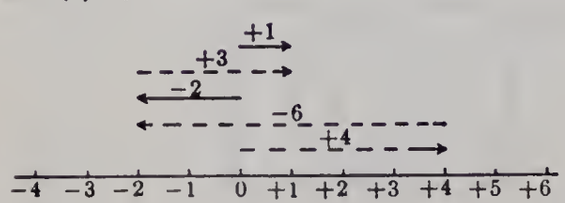


(ii) $(+3) + [(-4) + (+9)]$
$= (+3) + (+5)$
$= +8$

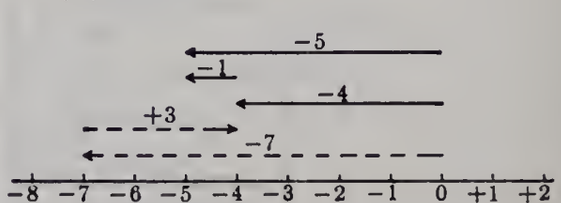


Exercise B-2 (page 19)

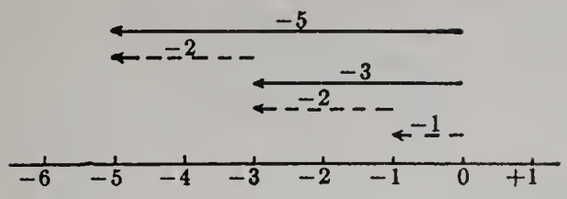
4. (i) +1

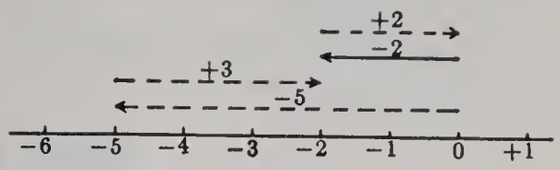


(ii) −5



4. (iii) -5 (iv) 0

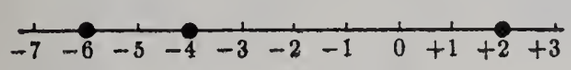




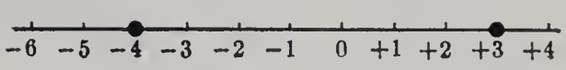
5. (i) -5 (ii) -1 (iii) -2 (iv) -4 (v) 0 (vi) 3

6. (i) 1 (ii) 7 (iii) -1 (iv) -1 **7.** (i) 4
(ii) 0 (iii) 6 (iv) 11 (v) 0 (vi) 2

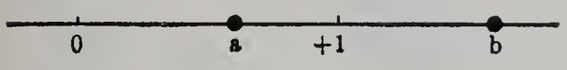
8. (i)



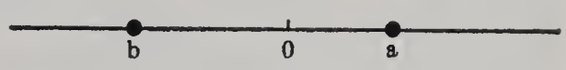
(ii)



(iii)



(iv)



Exercise B-3 (page 23)

5. (i) abc (ii) $-12x^3$ (iii) $-8a^3b^3$ (iv) 0 (v) $+x^3a^2$ (vi) $+a^7$

6. (i) $+1$ (ii) -1 (iii) $+1$ (iv) $+1$ (v) -1

7. (i) $+7x$ (ii) $-5b$ (iii) $+2x$ (iv) $+1$ (v) $-6ab$ (vi) $\dfrac{-b^2}{a}$
(vii) 0 (viii) $+4bc$

8. (i) $5x + 5y$ (ii) $-6m - 6n$ (iii) $-2a - 2b$
(iv) $3c - 3d$ (v) $(+3) + (-2)$ (vi) $a - b$

9. (i) $ab - ac$ (ii) $2m - mn$ (iii) $-2xa - 6xy$
(iv) $2x^3 - 3x^2y$ (v) $6y^2 + 9yz$ (vi) $-2mn - 10mp$

Exercise B-4 (page 28)

2. $a - b$ **3.** $2x - 6y$ **4.** $3x^2 - 3x - 1$

5. $a - 6b + 6c$ **6.** $4n + 2m + 3p$ **7.** $-6a + b - 2c$

8. 3 **9.** 2 **10.** 1

11. $-2a + 3b$ **12.** $3x - 9y$ **13.** 0

14. $-3a + 4b$ **15.** $x(a + 3b + 3c) + y(a + 2b - 2c)$

16. $x(3p + q - r) + y(r - p - 2q)$ **17.** $x(m - n - p) + y(n - m + p)$

18. $x^3(c - 3) + x^2(b - d) + x(a - 2) + 8$ **19.** $x^3(1 - k) + x^2(a + p) - x(h + b)$

20. $x^3(m - n + 2) + x^2(p - q) - x(p + 2) + n$

21. $b - d + 1$ **22.** $a - b - 4c$ **23.** $-3a - 2b - 1$

Exercise B-5 (page 29)

4. (i) $8a - b + 7c$ (ii) $7x - 15y$ (iii) $-14x + 28y$
(iv) $-4a - 4b + 12c$ (v) $11m^2 + 5n^2$ **5.** (i) equal
(ii) equal **6.** -2 **7.** (i) $a(a + b)$
(ii) $3(a + 2b - 4c)$ (iii) $-4p(m + 2 + 3q)$ (iv) $3(-4x + a - 5b)$

8. (i) $m - nx - ny + 5nz$ (ii) $3ax + 2bx + 3ay + 2by$
(iii) $4a^2 - 11ab - 3b^2$ (iv) $2x^2 - 5xy + 2y^2$

9. (i) $6x^2 + 13x + 6$ (ii) $25m^2 + 20m + 4$
(iii) $81a^2 - 25b^2$ (iv) $6a^2 + ab + 4ac - b^2 + 2bc$
(v) $6m^2 - 20mn + 7m - 21n + 6n^2$ (vi) $8x^3 + 36x^2 + 54x + 27$

10. (i) $16a^2 + 4ab$ (ii) $3m^3 - 2m^2 - m + 10$

Exercise 1-1 (page 34)

5. (i) $-\frac{1}{3}$ (ii) $-15pq$ (iii) $\frac{x+y}{a}$ (iv) 0
(v) $-\frac{1}{80}$ (vi) $\frac{1}{b+c}$ 6. $\frac{p}{-q}, -\frac{p}{q}, -\frac{-p}{-q}$

7. (i) $-\frac{2b}{a}$ (ii) undefined (iii) 1 (iv) $-\frac{1}{3}$

8. (i) $-\frac{3}{2}$ (ii) $\frac{4}{3}$ (iii) $\frac{b}{a}$ (iv) 1 (v) undefined
(vi) $-\frac{4}{3}$ 9. (i) $-\frac{8}{7}$ (ii) $\frac{2a+3b}{2}$ (iii) $\frac{xy}{b}$ (iv) $-\frac{1}{2c}$
(v) $\frac{4}{a}$ (vi) $-3b$ (vii) $-\frac{6y}{zx}$ (viii) $-\frac{a}{d}$ (ix) $-\frac{1}{4}$

10. (i) $\frac{4}{9}$ (ii) $-\frac{1}{2}$ (iii) $\frac{9}{16}$ 11. (i) -18 (ii) -18 12. 0

Exercise 1-2 (page 39)

11. $\frac{4}{5}$ 12. $\frac{-7}{5}$ 13. $\frac{65}{42}$ 14. 6 15. $-\frac{16}{5a^2}$
16. $\frac{5}{3p^2}$ 17. $-\frac{5}{3}$ 18. $\frac{-3r}{5s}$ 19. $\frac{1}{y}$ 20. $-\frac{ab}{4}$
21. $\frac{y}{x^2}$ 22. $\frac{x^3}{12y^2}$ 23. $\frac{6}{5}$ 24. $\frac{1}{2}$ 25. $-\frac{3}{2m^2}$

Exercise 1-3 (page 42)

19. 0 20. $\frac{1}{2}$ 21. $\frac{1}{8}$ 22. $\frac{13}{16}$ 23. -1
24. $\frac{1}{4}$ 25. $-\frac{1}{10}$ 26. $-\frac{1}{x}$ 27. $-\frac{9}{x}$ 28. $-\frac{8}{x}$
29. $\frac{1}{x}$ 30. $\frac{3}{x}$ 31. $\frac{1}{b}$ 32. $-\frac{8}{c}$ 33. $\frac{13}{5x}$
34. 0 35. $\frac{b-3a-3}{3ab}$ 36. $\frac{a^4 - 2b^4 + 3c^4}{a^2b^2c^2}$
37. $\frac{5a^2 - bc}{2abc}$ 38. $\frac{2b-a}{2b}$ 39. $\frac{2a^2 - 2b^2 + ab}{2ab}$

Exercise 1-4 (page 44)

1. $\frac{2}{3} > \frac{3}{5}$ 2. $-\frac{1}{2} > -\frac{3}{4}$ 3. $\frac{2}{3} > -1$
4. $-\frac{8}{9} > -\frac{9}{10}$ 5. $\frac{7}{17} > \frac{5}{13}$ 6. $-\frac{12}{5} > -\frac{8}{3}$

7. $\left\{\frac{3}{-5}, \frac{-2}{5}, \frac{-2}{-5}, -\frac{-3}{5}\right\}$

8. $\left\{\frac{-8}{3}, -\frac{7}{3}, -\frac{7}{-3}, \frac{-8}{-3}\right\}$

9. $\left\{\frac{-3}{4}, \frac{3}{-8}, -\frac{3}{-8}, -\frac{-3}{4}\right\}$

10. $\left\{-\frac{5}{6}, -\frac{1}{3}, \frac{1}{4}, \frac{-5}{-12}, -\frac{-2}{3}\right\}$

Exercise 1-5 (page 47)

5. $3.25\dot{0}$

6. $8.\dot{6}$

7. $.\dot{0}76923$

8. $.\dot{4}8\dot{1}$

9. $3.\dot{1}$

10. $18.\dot{5}71428$

11. $-33.2\dot{0}$

12. $-0.\dot{8}70967741935483$

13. $\frac{1}{3}$

14. $\frac{23}{99}$

15. $\frac{29}{9}$

16. $\frac{448}{333}$

17. $\frac{27,823}{9,999}$

18. $\frac{2,814}{275}$

19. $\frac{31,717}{90,000}$

20. $\frac{549}{370}$

21. $\frac{108,727}{3,333}$

24.

PROPERTY	N	N_0	I	Q
Addition: (1) Closed	√	√	√	√
(2) Commutative	√	√	√	√
(3) Associative	√	√	√	√
(4) Identity		√	√	√
(5) Inverse			√	√
Multiplication: (6) Closed	√	√	√	√
(7) Commutative	√	√	√	√
(8) Associative	√	√	√	√
(9) Identity	√	√	√	√
(10) Inverse				√*
(11) Distributive	√	√	√	√
*except 0				

Exercise 1-6 (page 54)

10. 3.14, 3.142, 3.1416

11. 1.41, 1.414, 1.4143

12. 15.35, 15.346, 15.3457

13. 17.43, 17.433, 17.4328

14. 2.16, 2.156, 2.1556

15. 18.23, 18.232, 18.2323

16. 15.24, 15.235, 15.2354

17. 3.67, 3.673, 3.6726

Exercise 1-7 (page 56)

4. (i) $12a^5b^3$ (ii) $3a^4b^6$ (iii) $3a^2b^9$
(iv) $30p^{10}q^{15}$ (v) $8x^{12}y^{18}$ (vi) $5m^5p^{20}$
5. (i) $32x^{10}y^{15}$ (ii) $1{,}187x^7y^{14}z^{21}$
6. (i) $4x^6y^2z^4$ (ii) $6b$ (iii) $9a^2b^4c^6$
(iv) $5abc^2$ (v) $-3x^2y^2z$ (vi) $6m^2q$

Exercise 1-8 (page 59)

3. (i) $7a^2b$ (ii) x^7 (iii) $5x^3y^2z^4$
(iv) $10d$ (v) $\dfrac{11a^2d^3}{c^2}$ (vi) $\dfrac{12n^2}{m^2y^2z^6}$

4. Since $5ab^2 \times 3abc = 15a^2b^3c$
5. (i) 5 (ii) $2xy^3$ (iii) $5y^2p^4$ (iv) $5a^2$ (v) $5a$
(vi) 0 (vii) $4x^3y^5z^2$ (viii) $5b^2c^3$ (ix) $6x^2y^3z$
6. (i) The length is $5xy$ inches. (ii) The time required is $6n^3$ hours.
(iii) The number of tons is $6s$. (iv) The price of oranges is \$1 per dozen.
(v) One side of the square is $5m$ inches.

Exercise 1-9 (page 61)

9. 3 **10.** $36a^{-1}b^2c^{-2}$ **11.** $5a^{-1}b^{+4}$ **12.** $7xy^{-1}$
13. $45x^2y^{-3}z^2$ **14.** 12 **15.** $9a^4c^{-4}$ **16.** $8x^{12}y^{-9}$
17. $-5a^6y^2$ **18.** $5x^8z^{-5}y^{-6}$ **19.** $12m^{-2}n^8y^4$ **20.** $-11a^2b^{-4}c^{-2}d^{-7}$
21. (i) $5ab^{-1}c^{-3}$ (ii) $-3x^2y^{-4}z^{-1}$ (iii) $-6m^2p^{-5}q^6$ (iv) $2xz^{10}$
22. 17 **23.** $\dfrac{17}{72}$

Exercise 1-10 (page 63)

1. 1.0×10^{-8} cm. **2.** 1.0×10^{-7} cm. **3.** 1.0×10^{-5} cm.
4. 3.0×10^{-4} cm. **5.** 4.0×10^{-4} cm. **6.** 8.0×10^{-4} cm.
7. 1.0×10^{-3} cm. **8.** 1.0×10^{-2} cm. **9.** 2.5×10^{-1} cm.
10. 1.8 cm. **11.** 1.75×10^2 cm. **12.** 3.3×10^4 cm.
13. 5.0688×10^8 cm. **14.** 5.538368×10^{11} cm. **15.** 1.3×10^{14} cm.
16. 1.0×10^{-16} sec. **17.** 1.0×10^{-7} sec. **18.** 1.0×10^{-5} sec.
19. 1.0×10^{-3} sec. **20.** 1.0 sec. **21.** 2.0×10^9 sec. **22.** 1.0×10^{17} sec.
23. 6,600,000,000,000,000,000,000 tons
24. 2,100,000,000,000,000,000,000,000,000 tons
25. 81,000,000,000,000,000,000 tons
26. 240,000 miles **27.** 0.000,017
28. 0.000,000,000,000,000,000,000,000,000,9 grams
29. 0.000,000,000,000,000,000,76 coulombs
30. 6,300,000,000,000,000,000
31. (i) 740,000 cycles/sec. (ii) 91,000,000 cycles/sec.
32. 300,000,000 metres/sec. **33.** 0.000,000,01 cm.

Practice Exercise 1-11 (page 64)

1. $-5a^3x^4$ **2.** $270m^5$ **3.** $6b^8$
4. $-2m^3$ **5.** 0 **6.** $1{,}000a$
7. $9x^5bc$ **8.** $\dfrac{a^3b^5}{2}$ **9.** $3x^2y^4$

10. $-5x^2y^3z^4$ **11.** $-6a^4b^6$
12. $7x^5$ **13.** 27
14. $-12ac - 8bc$ **15.** $12a^2 - 8ab + 12ac$
16. $-24bm + 12bp + 16bc$
17. (i) 5 (ii) -20
19. $\left\{\frac{3}{-4}, \frac{-3}{8}, -\frac{5}{16}, \frac{2}{3}, \frac{-7}{-8}\right\}$

Practice Exercise **1-12** (page 65)

1. $-\frac{1}{3}$ **2.** $-3ab$ **3.** $\frac{13}{12}$ **4.** -2
5. $-\frac{1}{5}$ **6.** $-\frac{bf}{c^2}$ **7.** $-\frac{b}{3}$ **8.** $\frac{1 - 5y - 4x}{xy}$
9. $-n$ **10.** $-\frac{1}{c}$ **11.** $\frac{1}{12}$ **12.** $\frac{52}{5}$ **13.** $\frac{4}{5}$
14. $-\frac{1}{xy^3}$ **15.** $-\frac{a}{4}$ **18.** $\left\{\frac{-3}{-2}, \frac{4}{3}, \frac{0}{2}, \frac{1}{-3}, \frac{-3}{8}, -\frac{5}{6}\right\}$

Practice Exercise **1-13** (page 66)

1. (i) $\frac{4}{-27}$ (ii) $\frac{1}{2}$ (iii) $\frac{9}{-8}$ (iv) $\frac{4}{-27}$
(v) $\frac{1}{8}$ (vi) 1 (vii) -5 (viii) 13 (ix) 58

2. (i) $\frac{1}{-2}$ (ii) 4 (iii) 4 (iv) 4 (v) 4 (vi) 256
(vii) $\frac{1}{4}$ (viii) 4 (ix) 1 (x) 1 (xi) 2 (xii) 2

3. (i) 554,000,000,000 (ii) 13,000 (iii) 0.072
(iv) 330 (v) 0.000,1 (vi) 0.82 (vii) 25
(viii) .000,000,065 (ix) 3 (x) 2,000,000 (xi) 0.008

4. (i) 6.6×10^9 (ii) 9.3×10^7 (iii) 1.0×10^9 (iv) 7.5×10^{-5}
(v) 2.3×10^{-10} (vi) 2.8×10^2 (vii) 2.8×10^{-1} (viii) 3.72×10^{-2}
(ix) 4.286×10^2 (x) 7.25×10^6 (xi) 1.8 (xii) 5×10^{-9}

Exercise **2-1** (page 70)

20. 9 **21.** 6 **22.** 3 **23.** -5 **24.** 3 **25.** -1
26. 2 **27.** 2 **28.** $\frac{1}{2}$ **29.** 0 **30.** $\{3\}$ **31.** $\{-3\}$
32. $\left\{\frac{5}{3}\right\}$ **33.** $\{33\}$ **34.** $\{13\}$ **35.** $\left\{-\frac{7}{3}\right\}$

Exercise **2-2** (page 73)

10. 2 **11.** 3 **12.** 1 **13.** 3 **14.** 7 **15.** 6
16. 1 **17.** 7 **18.** -7 **19.** -1 **20.** $\{-3\}$ **21.** $\left\{-\frac{8}{3}\right\}$
22. $\{-5\}$ **23.** $\{4\}$ **24.** $\{-3\frac{2}{5}\}$ **25.** $\left\{\frac{13}{14}\right\}$ **26.** $\{5\frac{1}{2}\}$ **27.** $\{7\}$
28. $\{6\}$ **29.** $\{-1\frac{1}{2}\}$ **30.** $\{9\}$

Exercise 2-3 (page 74)

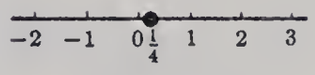
13. 4 14. 2 15. 5 16. 1 17. -5

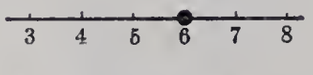
18. 4 19. $\frac{4}{3}$ 20. $\frac{4}{13}$ 21. $2\frac{1}{2}$ 22. $-24\frac{1}{2}$

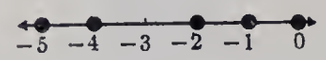
23. 3 24. $\{1\}$ 25. $\{4\frac{1}{2}\}$ 26. $\{2\}$ 27. $\left\{-\frac{2}{35}\right\}$

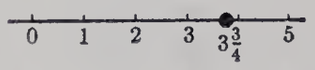
28. $\{8\}$ 29. $\{-2\}$ 30. $\{3\}$ 31. $\{1\}$ 32. $\{0\}$

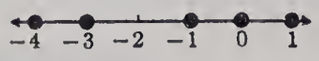
Exercise 2-4 (page 77)

9. $\left\{\frac{1}{4}\right\}$ 

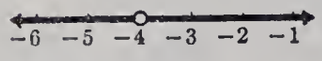
10. $\{6\}$ 

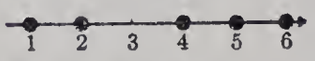
11. $\{. \ . \ . , -5, -4, -2, -1, 0, \ . \ . \ .\}$ 

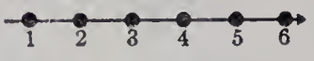
12. $\{3\frac{3}{4}\}$ 

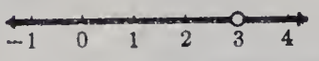
13. $\{. \ . \ . , -4, -3, -1, 0, 1, \ . \ . \ .\}$ 

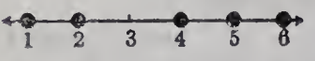
14. $\{. \ . \ . , 6, 7, 9, 10, 11, \ . \ . \ .\}$

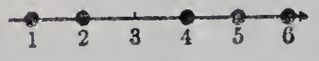
15. $\{x \mid x \neq -4, x \in R\}$ 

16. $\{1, 2, 4, 5, 6, \ . \ . \ .\}$ 

17. $\{1, 2, 3, 4, \ . \ . \ .\}$ 

18. (i) $\{x \mid x \neq 3, x \in R\}$ 

(ii) $\{. \ . \ ., 0, 1, 2, 4, 5, 6, \ . \ . \ .\}$ 

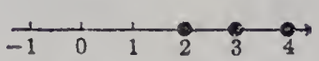
(iii) $\{1, 2, 4, 5, 6, \ . \ . \ .\}$ 

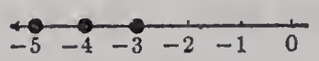
Exercise 2-5 (page 83)

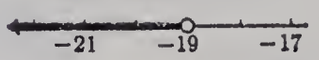
19. $x \leqq 13$ 20. $x = 1$ 21. $x < 1$ 22. $x > 2$

23. $y = 0.6$ 24. $x \leqq -2$ 25. $x \leqq 5$ 26. $x < 2$

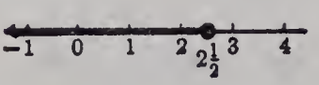
27. $x \geqq 0$ 28. $x < -6\frac{1}{2}$ 29. $x < -1\frac{3}{4}$

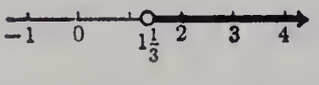
30. $\{x \mid x \geqq 2, x \in I\}$ 

31. $\{x \mid x \leqq -3, x \in I\}$ 

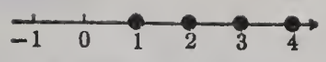
32. $\{x \mid x < -19, x \in R\}$ 

33. $\{x \mid x \geqq -2, x \in R\}$

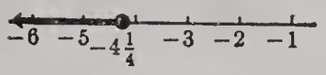
34. $\{x \mid x \leqq 2\frac{1}{2}, x \in R\}$ 

35. $\{x \mid x > 1\frac{1}{3}, x \in R\}$ 

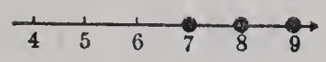
36. $\{x \mid x > 0, x \in I\}$



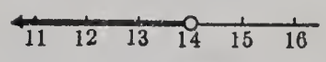
37. $\{x \mid x \leqq -4\frac{1}{4}, x \in R\}$



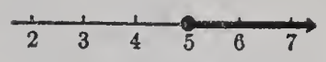
38. $\{x \mid x \geqq 7, x \in N\}$



39. $\{x \mid x < 14, x \in R\}$



40. $\{x \mid x \geqq 5, x \in R\}$

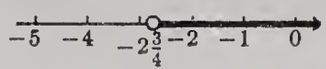


41. $\{x \mid x \leqq 7, x \in R\}$ 42. $\{y \mid y \geqq \frac{1}{2}, y \in R\}$ 43. $\{b \mid b \geqq -2, b \in R\}$

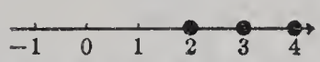
Exercise 2-6 (page 85)

1. $x < 2$ 2. $x < 3$ 3. $x > \frac{2}{3}$ 4. $y \geqq -1$

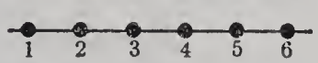
5. $\{x \mid x > -2\frac{3}{4}, x \in R\}$



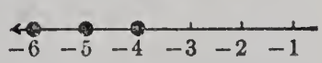
6. $\{x \mid x > 1, x \in I\}$



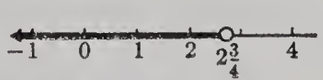
7. $\{x \mid x < 7, x \in N\}$



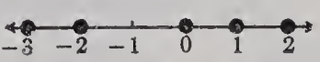
8. $\{x \mid x \leqq -4, x \in I\}$



9. $\{x \mid x < 2\frac{3}{4}, x \in R\}$



10. $\{x \mid x \neq -1, x \in I\}$



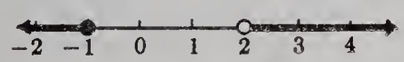
11. $\{x \mid x \leqq 1\frac{1}{2}, x \in R\}$ 12. $\{x \mid x > 4\frac{1}{3}, x \in R\}$

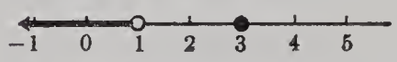
13. $\{x \mid x \leqq 12, x \in R\}$ 14. $\{x \mid x > 1\frac{2}{5}, x \in R\}$

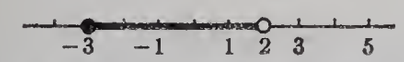
Exercise 2-7 (page 87)

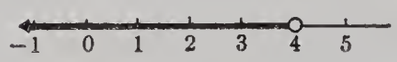
9.

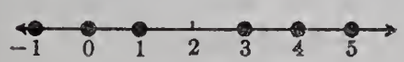
10.

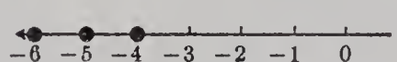
11. 

12. 

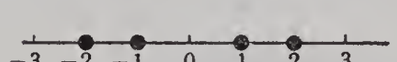
13. 

14. 

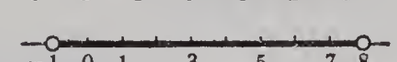
15. 

16. 

17.

18. 

19.

20. 

21.

22.

Exercise 2-8 (page 89)

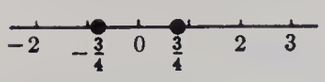
16. 13	17. 11	18. 0	19. 9	20. 3	21. 16
22. 7	23. 22	24. $4x + 3$	25. $(x + 1)^2$	26. $=$	27. $>$
28. $>$	29. $>$	30. $=$	31. $>$	32. 0	33. -3
34. 12	35. 14	36. -15			

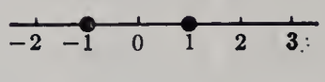
Exercise 2-9 (page 91)

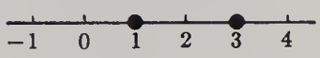
10. $\{4, -3\}$ 11. $\left\{\frac{2}{3}, -2\right\}$ 12. $\varnothing$ 13. $\left\{2, -\frac{2}{3}\right\}$

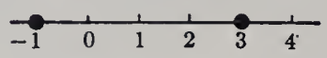
14. $\varnothing$ 15. $\left\{\frac{1}{2}, -2\right\}$ 16. $\left\{\frac{5}{2}, -\frac{7}{2}\right\}$ 17. $\left\{\frac{3}{2}, -\frac{5}{2}\right\}$

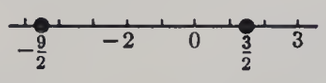
18. $\{-2, -6\}$ 19. $\left\{\frac{21}{4}, -\frac{21}{4}\right\}$ 20. $\left\{1, -\frac{5}{3}\right\}$ 21. $\varnothing$

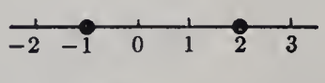
22. $\left\{\frac{3}{4}, -\frac{3}{4}\right\}$ 

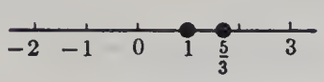
23. $\{-1, 1\}$ 

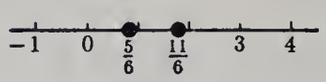
24. $\{3, 1\}$ 

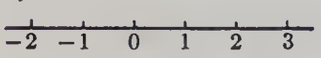
25. $\{-1, 3\}$ 

26. $\left\{\frac{3}{2}, -\frac{9}{2}\right\}$ 

27. $\{-1, 2\}$ 

28. $\left\{\frac{5}{3}, 1\right\}$ 

29. $\left\{\frac{11}{6}, \frac{5}{6}\right\}$ 

30. $\varnothing$ 

Practice Exercise 2-10 (page 92)

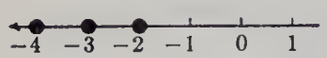
1. 7 2. -5 3. 5 4. 12 5. 1
6. $\{10\}$ 7. $\{5\}$ 8. $\{-5\}$ 9. $\{22\}$ 10. $\{3\frac{1}{2}\}$

Practice Exercise 2-11 (page 93)

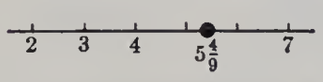
1. $3\frac{1}{4}$ 2. $3\frac{1}{2}$ 3. 3 4. 2 5. 3 6. 3
7. $\{3\}$ 8. $\{-1\}$ 9. $\left\{\frac{9}{5}\right\}$ 10. $\left\{\frac{1}{2}\right\}$ 11. $\{5\}$

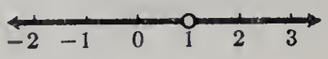
Practice Exercise 2-12 (page 93)

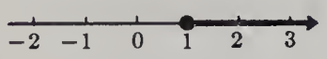
1. $\{x \mid x < 17, x \in R\}$ 2. $\{x \mid x \leqq \frac{1}{5}, x \in R\}$ 3. $\{x \mid x < 18, x \in R\}$
4. $\{x \mid x \geqq 12, x \in R\}$ 5. $\{a \mid a < -4, a \in R\}$ 6. $\{x \mid x \leqq 0, x \in R\}$

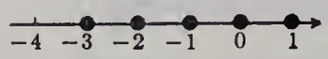
7. 

8.

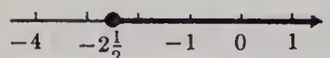
9. 

10. 

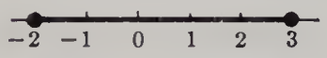
11. 

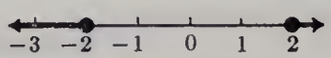
12. 

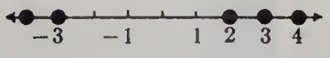
Practice Exercise 2-13 (page 94)

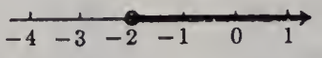
1. 

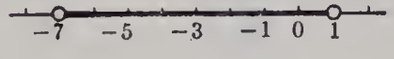
2.

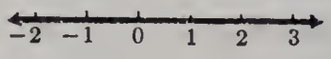
3. 

4. 

5. 

6. 

7. 

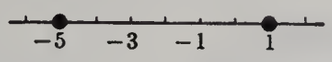
8. 

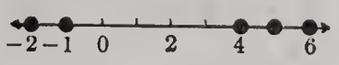
Practice Exercise 2-14 (page 94)

1. $\emptyset$ **2.** (i) a (ii) 0 (iii) $-a$ **3.** $\frac{14}{17}$ **4.** 6

5. 2 **6.** 7 **7.** $-x + 4$ **8.** -22

9. -5 **10.** $4a - 12b$ **11.** -51 **12.** $3b - 4c$

13. $\{1, -5\}$ 

14. $\{x \mid x > 3\frac{1}{2} \text{ or } x < -\frac{1}{2}, x \in I\}$ 

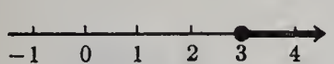
Review Exercise 2-15 (page 95)

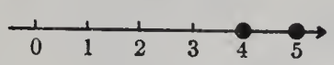
1. 39 **2.** 0 **3.** $-5a^2b + 4ac^2$

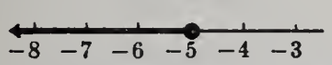
4. $y^3 + 3y^2 + 3y + 1$ **5.** $2a - 2b + 2c$ **6.** $8xy - 4y^2$

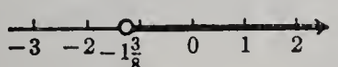
7. $-x^2 - 10x + 39$ **8.** $-4a + 3b - 6c$ **9.** -2

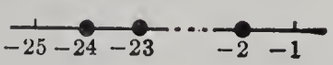
10. $2x^2 - x + 3$ **11.** $2x + 7$ **12.** 8 **13.** $\{-8\}$

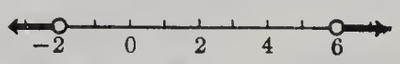
14. 

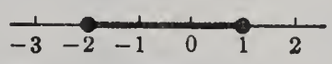
15. $\{x \mid x > 3, x \in I\}$ 

16. 

17. (i) 

(ii) 

18. (i) $\{y \mid y > 6 \text{ or } y < -2, y \in R\}$ 

(ii) $\{x \mid -2 \leqq x \leqq 1, x \in R\}$ 

Exercise 3-1 (page 99)

8. Represent any number by x.
The number with 21 added is $x + 21$.
Eight times the number diminished by 21 is $8x - 21$.
But these numbers are equal.
$\therefore\ x + 21 = 8x - 21$

9. Represent any number by x.
Seven added to eight times the number is $8x + 7$.
Five subtracted from 12 times the number is $12x - 5$.
But these numbers are equal.
$\therefore\ 8x + 7 = 12x - 5$

10. Represent any number by x.
A number which exceeds this number by 25 is $x + 25$.
The sum of the numbers is $x + (x + 25) = 2x + 25$.
But this sum is 83.
$\therefore\ 2x + 25 = 83$

11. Represent any number by $x - 1$.
Then the next two consecutive numbers are x and $x + 1$.
The sum of the numbers is $(x - 1) + x + (x + 1) = 3x$.
But the sum is 36.
$\therefore\ 3x = 36$

12. Represent any real number by x.
The number decreased by $\frac{1}{2}$ is $x - \frac{1}{2}$.
Three-quarters of the number increased by $1\frac{1}{2}$ is $\frac{3}{4}x + 1\frac{1}{2}$.
But the first of these two numbers is less than the second.

$$\therefore x - \frac{1}{2} < \frac{3}{4}x + 1\frac{1}{2}$$

Exercise 3-2 (page 102)

1. 23, 48	**2.** 2,584, 3,840	**3.** 11, 17, 44	**4.** 30
5. 60	**6.** 18, 22	**7.** 39, 28, 65	**8.** 48
9. 14, 16, 18	**10.** 19, 21, 23	**11.** 82, 84	**12.** $x > -4, x \in R$

Exercise 3-4 (page 107)

6. $6\frac{1}{2}$ ft., $13\frac{1}{2}$ ft.	**7.** 10 in.	**8.** 30°, 70°, 80°	**9.** 20 ft., 28 ft.
10. 10 ft. by 20 ft., 8 ft. by 25 ft.		**11.** $14\frac{1}{2}$ ft. by $17\frac{1}{2}$ ft.	

Exercise 3-5 (page 109)

1. \$3,000, \$1,000	**2.** \$5	**3.** \$900	**4.** \$2,400
5. \$20,000	**6.** \$2,325, \$3,875	**7.** 4.8 gal.	**8.** 50 gal

Exercise 3-6 (page 111)

1. 6 mi.	**2.** 3,107.5 ft.	**3.** 300 mi.	**4.** 250 mi.
5. 1,143 ft. per sec.	**6.** 120 m.p.h.	**7.** 2.4 mi.	**8.** No solution
9. 125 m.p.h.	**10.** 2.56 sec.	**11.** 25 mi.	
12. 35 m.p.h., 45 m.p.h.	**13.** 365 m.p.h., 438 m.p.h.		

Exercise 3-7 (page 115)

1. \$362, \$181, \$693	**2.** 42 yrs., 21 yrs.	**3.** (*d*)	**4.** \$200, \$350, \$100
5. 16 yrs., 8 yrs.	**6.** \$10	**7.** (*b*)	**8.** \$240
9. 29 yrs., 11 yrs.	**10.** \$3,200	**11.** (*e*)	**12.** 26 yrs.
13. \$22.95, \$7.65	**14.** 25, 15	**15.** \$15	**16.** 25
17. \$2.40	**18.** (*d*)	**19.** 41	**20.** \$60, \$70, \$45
21. (*a*)	**22.** \$80	**23.** \$20	**24.** (*c*)
25. $\frac{g^2}{100 - 2g}$	**26.** 6 mi.	**27.** 20, 15	**28.** \$4,800, \$5,300
29. \$76.77, \$68.61	**30.** 120 mi.	**31.** \$15, 60	**32.** 3 pints

Exercise 4-2 (page 123)

13. $3(x - 2)$	**14.** $2a(3a - 1)$	**15.** $x(x - y)$
16. $a(b - a)$	**17.** $a(a - b - 1)$	**18.** $x(1 - x - y)$
19. $3x(2x - y)$	**20.** $ax(x + a - 1)$	**21.** $\frac{1}{2}(x - y - z)$
22. $5x^2(2 - 3x)$	**23.** $2a(1 - 2a + 4a^2)$	**24.** $\frac{1}{2}r(a + b + c)$
25. $6r^3(1 + 2r + 4r^2)$	**26.** $ab(c - d + e)$	**27.** $a^2(a^2 - 3a - 6)$
28. $a(x + y + 1)$	**29.** $(x + y)(a - b)$	**30.** $(x + y)(a + 1)$
31. $(a - b)(x + 2)$	**32.** $(p - q)(2 - x)$	**33.** $(x - 2)(a - 1)$
34. $2x(4x - y)$	**35.** $2(x - y)(1 - 2z)$	**36.** $(x - y)(1 + x + y)$
37. $2(a + b)(a - b - 2)$	**38.** $(x + y)(a - b + 1)$	**39.** $a(a - 2)$
40. $2a(a - 2)$		

Exercise 4-3 (page 124)

1. $(x + y)(h + k)$ **2.** $(x + y)(h + m)$ **3.** $(x + y)(a - b)$
4. $(a - b)(x - y)$ **5.** $(a - x)(a - y)$ **6.** $(a + b)(a + b + 2)$
7. $(x - y)(x + z)$ **8.** $(x - y)(x - y + 1)$ **9.** $(x - y)(x - y - 1)$
10. $(z + y)(y - x)$ **11.** $(1 + x)(1 + b)$ **12.** $(x^2 + 1)(x - 1)$
13. $(x - a)(1 + x - a)$ **14.** $(x - 1)(a - 1)$ **15.** $(2a - d)(3b + d)$
16. $(2x - 3y)(a + b)$ **17.** $(ab - 1)(ab - 2)$ **18.** $(1 - x)(1 - x - y)$
19. $(x - y)(a - b)$ **20.** $(2 - x)(1 - y)$ **21.** $p(x - y)(1 - x)$
22. $(a + 1)(3x + y)$ **23.** $(x^2 + y^2)$ **24.** $c(1 - r)^2$

Exercise 4-4 (page 127)

7. $(x - 3)(x - 2)$ **8.** $(x - 3)(x + 2)$ **9.** $(x + 1)(x + 6)$
10. $(x - 6)(x + 1)$ **11.** $(x - 4y)(x - 2y)$ **12.** $(a - 4)(a + 2)$
13. $(b + 5)(b - 4)$ **14.** $(c - 2b)(c + b)$ **15.** $(x + 12y)(x + y)$
16. $(x + 5)(x - 3)$ **17.** $(x - 5)(x - 3)$ **18.** $(x - 15y)(x + y)$
19. $(a + 2)(a + 9)$ **20.** $(a + 8)(a - 2)$ **21.** $(a - 4)(a - 4)$
22. $(b + 12)(b - 2)$ **23.** $(x + 6y)(x + 4y)$ **24.** $(a - 12)(a + 2)$
25. $(y + 8)(y - 3)$ **26.** $(y - 6)(y + 3)$ **27.** $(a - 6)(a + 5)$
28. $(x + 4)(x + 8)$ **29.** $(x - 5y)(x - 5y)$ **30.** $(y - 7z)(y + z)$
31. $a(a - 17)(a - 1)$ **32.** $x(x - 5)(x + 2)$ **33.** $3(a + 8)(a - 1)$
34. $2x(x - 13)(x + 2)$ **35.** $2b(a - 6)(a + 2)$ **36.** $q^2(p - 9)(p + 2)$
37. $(x - 20)(x + 5)$ **38.** $(ab - 15)(ab + 4)$ **39.** $(x^2 - 32)(x^2 + 3)$
40. $(a + b + 4)(a + b + 3)$ **41.** $(x - y - 4)(x - y + 3)$
42. $(a - b - 6)(a - b + 2)$

Exercise 4-5 (page 129)

1. $(2x + 1)(x + 2)$ **2.** $(3a - 2)(a - 1)$ **3.** $(2y - 3)(y + 1)$
4. $(2x - 1)(x - 7)$ **5.** $(5m - 2)(m + 1)$ **6.** $(7p + 5)(p - 1)$
7. $(3a + 1)(a + 5)$ **8.** $(5x - 1)(x + 3)$ **9.** $(3x - y)(x - 7y)$
10. $(3a - 2)(2a - 1)$ **11.** $(4x + 3)(x - 1)$ **12.** $(3a - b)(3a + 5b)$
13. $(7x + 2y)(x - 2y)$ **14.** $(5a + 3)(a - 2)$ **15.** $(3x - 4y)(x - 2y)$
16. $(4a - 3)(2a + 1)$ **17.** $(3p + 2q)(2p + 3q)$ **18.** $(7 - 3x)(1 + 2x)$
19. $(5 + 3y)(1 - 3y)$ **20.** $(5a - 1)(5a - 1)$ **21.** $(2a + 3b)(2a + 3b)$
22. $(3x - 4y)(2x + y)$ **23.** $(9 - 2b)(1 + 3b)$ **24.** $(4y - 3)(3y + 1)$
25. $(5 - 2a)(2 - 3a)$ **26.** $(8x + y)(x - 4y)$ **27.** $(5 - 3m)(3 + 2m)$
28. $(5x^2 - 3)(3x^2 + 4)$ **29.** $(8y^2 + 5)(2y^2 + 3)$ **30.** $(4x^2 - 3y^2)(4x^2 - 3y^2)$
31. $(9 + 4b)(2 - b)$ **32.** $(3x + 2)(8x - 1)$ **33.** $(6x - 5y)(x + 6y)$
34. $(4 - 5x)(3 - 4x)$ **35.** $(9a - 4)(2a + 3)$ **36.** $(16h - 25k)(h + k)$
37. $\left(\frac{2x}{y} - 3\right)\left(\frac{2x}{y} - 1\right)$ **38.** $\left(6x + \frac{5}{x}\right)\left(x - \frac{3}{x}\right)$
39. $(3a - 3b + 2)(a - b - 3)$ **40.** $(x - m)(x - n)$
41. $(ax + 2)(bx + 3)$ **42.** $(ax + c)(bx + d)$ **43.** $(ax - m)(bx + n)$

Exercise 4-6 (page 131)

22. $72x$ **23.** $28y$ **24.** $120y$ **25.** 100
26. 49 **27.** 25 **28.** $\pm(12x - 5)$ **29.** $\pm(8y^2 + 9)$
30. $\pm(2ab - 11)$ **31.** $|3a - 20|$ **32.** $|7 - 10a|$ **33.** $|13x - 1|$
34. $|x + 1| - |x - 1|$ **35.** $2|a - 3| - 3|2 - a|$ **36.** $9a^2 - 4$ **37.** $(1 - 8ab)^2$
38. $\left(x + \frac{1}{2}\right)^2$ **39.** $\left(x - \frac{1}{x}\right)^2$ **40.** $\left(\frac{x}{3} + 1\right)^2$ **41.** $\left(\frac{x}{4} - \frac{y}{3}\right)^2$

Exercise 4-7 (page 132)

16. $(b + 1)(b - 1)$ **17.** $(1 + x)(1 - x)$ **18.** $(1 + 4y)(1 - 4y)$
19. $(6a + 5)(6a - 5)$ **20.** $(12 + 7y)(12 - 7y)$ **21.** $(mn + p)(mn - p)$
22. $a(a + 2)(a - 2)$ **23.** $xy(x + y)(x - y)$ **24.** $2(a + 4b)(a - 4b)$
25. $10x(x + 3)(x - 3)$ **26.** $3(1 + 4x)(1 - 4x)$ **27.** $2a(a + 5b)(a - 5b)$
28. $(x^2 + 1)(x + 1)(x - 1)$ **29.** $(1 + 4a^2)(1 + 2a)(1 - 2a)$
30. $2(9a^2 + b^2)(3a + b)(3a - b)$ **31.** $(x + y + z)(x + y - z)$
32. $(a - b + 1)(a - b - 1)$ **33.** $(2x - 3 + y)(2x - 3 - y)$
34. $(x^2 + x + 1)(x^2 - x + 1)$ **35.** $(x^2 + xy + y^2)(x^2 - xy + y^2)$
36. $(a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$ **37.** $\left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right)$
38. $(3a - 6b - 1)(3a - 6b + 1)$ **39.** $(a + b + c + d)(a - b - c - d)$
40. $(x - y + p - q)(x - y - p + q)$ **41.** $(a + b + c)(a - b - c)$
42. $(a + b - c)(a - b + c)$ **43.** $(x + 3)(x - 3)(x + 1)(x - 1)$
44. $a - 3$ **45.** (i) $(a + 4)(a + 1)(a - 4)(a - 1)$
(ii) $(a^4 + b^4)(a^2 + b^2)(a + b)(a - b)$

Exercise 4-8 (page 134)

7. $(x - 3 + y)(x - 3 - y)$ **8.** $(a - 5 + m)(a - 5 - m)$
9. $(a^2 - 1 + b)(a^2 - 1 - b)$ **10.** $(x + 2 + y)(x + 2 - y)$
11. $(3x + 1 + 2y)(3x + 1 - 2y)$ **12.** $(5 - a + b)(5 - a - b)$
13. $(a^2 + a + 1)(a^2 - a + 1)$ **14.** $(5x - 1 + y)(5x - 1 - y)$
15. $(4a^2 - 1 + b^2)(4a^2 - 1 - b^2)$ **16.** $(3a + 4c + 5b)(3a + 4c - 5b)$
17. $(a + 4b - c)(a + 4b + c)$ **18.** $(2m - p - 8q)(2m + p + 8q)$
19. $(a - c - b - d)(a - c + b + d)$

Exercise 4-9 (page 136)

7. 1, 7 **8.** 2, −3 **9.** 2, −1 **10.** 4, −3
11. 5, −4 **12.** 5, 3 **13.** 7, −9 **14.** −12, −1
15. $\frac{1}{2}$, −2 **16.** $-\frac{2}{5}$, $-\frac{5}{2}$ **17.** 1, 1 **18.** −3, −3
19. 7, −6 **20.** 5, −2 **21.** 1, −1 **22.** 10, −10
23. $\frac{1}{2}$, $-\frac{1}{2}$ **24.** .5, −.5 **25.** 5, −5 **26.** 7, −7
27. 6, 1 **28.** 10, −9 **29.** 12, −1 **30.** 10, 5
31. 13, −13 **32.** $\frac{2}{5}$, $-\frac{2}{5}$ **33.** 6, −6 **34.** $\frac{5}{7}$, $-\frac{5}{7}$
35. 5, −4 **36.** 5, −7 **37.** −1, −4 **38.** 3, 4
39. 17, −1 **40.** −17, −4 **41.** 8, 8 **42.** 11, −8
43. 0, 9 **44.** 8, −7 **45.** $-\frac{5}{2}$, 1 **46.** $2\frac{1}{3}$, −2
47. $-1\frac{1}{2}$, 2 **48.** $-\frac{1}{7}$, − 5 **49.** 8, −5 **50.** 8, 9
51. 8, 10; −8, −10 **52.** 56 m.p.h., 40 m.p.h.

Exercise 4-10 (page 138)

1. $(x^2 + x + 1)(x^2 - x + 1)$ **2.** $(x^2 + x - 1)(x^2 - x - 1)$
3. $(2x^2 + 2x + 1)(2x^2 - 2x + 1)$ **4.** $(a^2 + ab + 3b^2)(a^2 - ab + 3b^2)$
5. $(b^2 + bc - 4c^2)(b^2 - bc - 4c^2)$ **6.** $(2y^2 + 2y + 3)(2y^2 - 2y + 3)$
7. $(x^2 + 4x + 8)(x^2 - 4x + 8)$ **8.** $(8a^2 + 3a + 3)(8a^2 - 3a + 3)$

9. $(5a^2 + 7ab - 4b^2)(5a^2 - 7ab - 4b^2)$ **10.** $(9x^2 + 6x + 10)(9x^2 - 6x + 10)$
11. $(11 + 12y - y^2)(11 - 12y - y^2)$ **12.** $(2x^2 + 6x + 9)(2x^2 - 6x + 9)$
13. $(b^4 - b^2 + 1)(b^2 + b + 1)(b^2 - b + 1)$ **14.** $(4y + 7)(4y - 7)(y + 1)(y - 1)$
15. $(2 + 2x + x^2)^2(2 - 2x + x^2)^2$ **16.** $(x^2 + 4x + 1)(x^2 - 4x + 1)$

Exercise 4-11 (page 140)

15. $(x - 1)(x + 2)(x - 1)$ **16.** $(x + 1)(x - 2)(x + 1)$
17. $(x - 2)(x^2 + x + 1)$ **18.** $(x + 2)(3x - 1)(2x - 1)$
19. $(x - 1)(2x^2 + 17x + 1)$ **20.** $(a - 3)(a^2 + a + 3)$
21. $(y - 2)(y - 3)(y - 2)$ **22.** $(a - 3)(2a + 1)(a + 1)$
23. $(x + 3)(x^2 - x - 1)$ **24.** $(y - 1)(y - 7)(y + 2)$
25. $(x - 3)(3x - 1)(x - 4)$ **26.** $(y - 5)(y^2 + 5y + 25)$
27. $(x + 2)(3x - 1)(2x + 1)$ **28.** $(x + 3)(2x - 1)(2x - 1)$
30. $(x - y)(x^2 + xy - y^2)$ **31.** $(x + y)(2x + y)(x + y)$
32. $(x - 2a)(x^2 + 2ax + 4a^2)$ **33.** $(x + 3y)(x^2 - 3xy + 9y^2)$
34. $(2a - b)(5a - b)(a + b)$ **35.** $(a - x)(5a^2 - ax - x^2)$
37. $(x - 1)(x - 2)(x - 3)(x - 4)$ **38.** 6
39. -12 **40.** $-1, -6, (2x + 1)$ **41.** $a = -4, b = 5, x^2 - x + 1$

Practice Exercise 4-12 (page 142)

1. $(x - 8)(x + 7)$ **2.** $x(x + 1)(x - 1)$
3. $(a^2 + 1)(a + 1)(a - 1)$ **4.** $2(x - 3)^2$
5. $a(a - 5)(a + 4)$ **6.** $(4x + 1)(x - 1)$
7. $3x(3x + 1)(3x - 1)$ **8.** $(3x - 1)(9x^2 + 3x + 1)$
9. $(2a - b + 2)(2a - b - 2)$ **10.** $(x - 2y + xy)(x - 2y - xy)$
11. $(y - 1)(x + 2)$ **12.** $(a - b + c)(a - b - c)$
13. $(x^2 - 3y - x)(x^2 + 3y + x)$ **14.** $(10b^2 - 1)^2$
15. $(20b^2 + 1)(5b^2 - 1)$ **16.** $(10b^2 + 2b - 1)(10b^2 - 2b - 1)$
17. $(4y + 1)(16y^2 - 4y + 1)$ **18.** $(1 - xy)(1 + xy + x^2y^2)$
19. $10(3a^2 + 7ab - 3b^2)$ **20.** $xy(1 + xy)(1 - xy)$
21. $a^2(6a + 1)(4a - 1)$ **22.** $(x - 4)(x + 1)(x - 1)$
23. $(y + 1)(x - 1)$ **24.** $(3a - 10b)^2$
25. $xy(x - y + xy)$ **26.** $(x - m)(m - y)$
27. $(4x^2 + x + 1)(4x^2 - x + 1)$ **28.** $a(a - 7)(a + 6)$
29. $(2x + 1)(2x - 1)(x + 2)(x - 2)$ **30.** $(6b^2 + 2bc - c^2)(6b^2 - 2bc - c^2)$
31. $(3x + 1)(3x - 1)(x^2 + 4)$ **32.** $2(5x + 1)(2x - 3)$
33. $(a^2 + 1)(1 + 4b^2)$ **34.** $(a + 1)(a - 1)(1 + 2b)(1 - 2b)$
35. $(a - 2b + 1)(a - 2b - 1)$ **36.** $(a + 1)(a - 1)(1 + b)(1 - b)$
37. $3(3x - 2)(x + 2)$ **38.** $2y^2(4y + 3)(2y - 3)$
39. $(2x^2 + 3xy + 3y^2)(2x^2 - 3xy + 3y^2)$ **40.** $(a + b)(a - b)(2a + 3b)(2a - 3b)$
41. $y^2(5x - 4)(2x + 3)$ **42.** $2y^2(5x^4y^2 - 11x^2 + 6y^2)$
43. $(x + 1)(x - 1)(x + 2)(x - 2)$ **44.** $(x^2 + 1)(x - 1)^2$

Practice Exercise 4-13 (page 143)

1. $(2x + 1)(x - 1)$ **2.** $(x + 1)(x - 1)(1 - x)$
3. $(16x + 1)(x - 1)$ **4.** $5(1 + 4a)(1 - 4a)$
5. $(1 - 3y + 3y^2)(1 - 3y - 3y^2)$ **6.** $(y + 1)(2y - 1)^2$
7. $(a + 5)(a + 2)(a - 2)$ **8.** $(a - b)(a - 1)$
9. $4y(4y + 1)(4y - 1)$ **10.** $y(4y + 1)(16y^2 - 4y + 1)$
11. $ab(ab - 3)(ab + 1)$ **12.** $y^2(5x - 7)(x + 1)$
13. $(y - 2)(y + 5)(y - 4)$ **14.** $(4y + 5)(3y - 4)$
15. $5(1 - 2b)(1 + 2b + 4b^2)$ **16.** $(x + 1)(x - 1)(a - 1)$
17. $(a^2 + 3a + 8)(a^2 - 3a + 8)$ **18.** $(a + 3)^2(a - 3)^2$

Practice Exercise 4-14 (page 143)

1. $ax(x - a + 1)$
2. $a(x - 1)^2$
3. $(ax - 2)(ax + 1)$
4. $(ax - 1 + a)(ax - 1 - a)$
5. $a^2(x^2 + ax - a^2)(x^2 - ax - a^2)$
6. $x^2(a + 2x)(a - 2x)(a + x)(a - x)$
7. $2a(5x - 1)(x + 4)$
8. $(y - 2)(2y - 3)(3y + 1)$
9. $(x + 2)(ax - 1)$
10. $(ax + y)(a^2x^2 - axy + y^2)$
11. $(2a - 9)(a + 2)$
12. $(8a - 3)^2$
13. $(1 + x)(1 - x)(1 + 4x)(1 - 4x)$
14. $3x(1 + 3x)(1 - 3x)$
15. $(3a - b)(a - b)$
16. $(x + y)(x - y)^2$
17. $(5x^2 + 2ax + a^2)(5x^2 - 2ax + a^2)$
18. $(x^2 + a^2)(5x + 3a)(5x - 3a)$
19. $(a + 1)(2a - 1)(a + 3)$
20. $(x + 2)(3x - 2)(x - 3)$

Practice Exercise 4-15 (page 144)

16. $(2x - 1)(x + 2)$
17. $(3 + x)(1 - 2x)$
18. $y(4y - 3)(y - 1)$
19. $2y(y + 3)(y - 3)$
20. $3(5x + 4)(2x - 1)$
21. $(5x - 6y)^2$
22. $(x - y)(x - y - 2)$
23. $(x - 5y + 1)(x - 5y - 1)$
24. $(a - b)(a - b - 1)$
25. $(a - 1)(a - 3)$
26. $3a(3a + 1)(9a^2 - 3a + 1)$
27. $9a^2(3a + 1)(3a - 1)$
28. $(2x - y)(a - b)$
29. $(1 - d)(c + x)$
30. $(9x + 1)(8x - 1)$
31. $3(2y - 3)(y + 1)$
32. $y(6y - 1)(y + 4)$
33. $(ac - bd)(ab - cd)$
34. $(1 + x)(1 - x)^2$
35. $2(7x - 1)^2$
36. $a(a^2 + 9)(a + 3)(a - 3)$
37. $(xy - 1 + x)(xy - 1 - x)$
38. $(5x - 3)(3x + 1)$
39. $(11 + 5y)(1 - y)$
40. $5(8 - 3a)(1 + 2a)$
41. $(x - a)(1 + a)$
42. $(x - y)(a + b)$
43. $(a + 1)(a - 1)(1 + x)(1 - x)$
44. $(b^2 + bc + c^2)(b^2 - bc + c^2)$
45. $(b^2 + 2c^2)(b + c)(b - c)$
46. $(x^2 + x - 1)(x^2 - x - 1)$
47. $(x^2 + 1)(x + 2)(x - 2)$
48. $4x(1 + x)(1 - x + x^2)$
49. $(1 + 2x + 2x^2)(1 - 2x + 2x^2)$
50. $(8x^2 + 4xy + y^2)(8x^2 - 4xy + y^2)$
51. $(8x^2 + y^2)(8x^2 - y^2)$
52. $(y - 1)(y + 2)(y + 3)$
53. $(y - 1)(y + 1)^2$
54. $(y - 1)(4y^2 + y + 1)$
55. $y(4y + 1)(y - 1)$
56. $x(6x - 1)(2x + 3)$
57. $2x(2x - 9)^2$
58. $(y + 2)(3y - 5)(3y - 1)$
59. $(9y - 4)(y - 3)$
60. $x(4x + 1)(16x^2 - 4x + 1)$
61. $(x - 3)(2x - 1)^2$

Practice Exercise 4-16 (page 145)

1. $15x^8$
2. $-\frac{1}{2}x^3y^3$
3. $-2a^3 + 2a^2 + 6a$
4. $8x^2 - 10xy - 3y^2$
5. $6a^3 - 5a^2b - 3ab^2 + 2b^3$
6. $4a^2$, $a, b \neq 0$
7. $-\frac{2}{3}xy$, $x, y \neq 0$
8. $-a^2 + 2a + 1$, $a \neq 0$
9. $2a^2 + 3ab - b^2$, $2a - 3b \neq 0$
10. -12
11. 1
12. 5
13. -28
14. -32
15. 0
16. $2 - 4x$
17. $-a^3 + 2a^2 + 4a$
18. $-12x^2 + 15xy - 20y^2$
19. $2a$, $a \neq 0$
20. $2x$

Exercise 5-1 (page 147)

13. $3ab$, $c \neq 0$
14. $\frac{1}{3r}$, $p, q \neq 0$
15. $\frac{x + 4}{x + 1}$, $x \neq 2$

16. $\frac{x+y}{z}$ 17. $\frac{x-3}{x}$, $x \neq 4$ 18. $\frac{x^2+2x+4}{x+2}$, $x \neq 2$

19. $x^2 - y^2$, $x^2 + y^2 \neq 0$ 20. $\frac{x-4}{x+1}$, $x \neq 0$ or 1 21. $\frac{9a^2+3ab+b^2}{a(3a+b)}$, $b \neq 3a$

22. 1, $x \neq 1$ or 2 or 5 23. $\frac{x-5y}{2(x-y)}$, $x \neq 5y$ 24. $\frac{(x^2+y^2)(x+y)}{x^2+xy+y^2}$, $x \neq y$

25. 1, $m \neq -3$ or -1 or 2 or 4 26. $\frac{x^2-2x+8}{x}$, $x^2 + 2x + 8 \neq 0$

27. $\frac{a}{b(x^2-y^2)}$, $x^2 + y^2 \neq 0$ 28. $a - x + 2$, $a \neq 2 - x$

29. $\frac{x+y}{x-y}$, $a \neq -b$ 30. $\frac{1}{x+1}$, $x \neq 1$, $x^2 + 1 \neq 0$

31. $\frac{a-b-c}{a}$, $a \neq b - c$ 32. $\frac{x-7}{x^2+x-4}$, $x \neq 1$

33. $\frac{x-b}{x+b}$, $x \neq a$ 34. $\frac{x+z-y}{x-z+y}$, $z \neq x - y$

35. $m + 3$, $m \neq 3$, $m^2 + 2 \neq 0$ 36. $\frac{a-b}{a+b}$, $a^2 - ab + b^2 \neq 0$

37. $\frac{a-2b}{a-2b+4}$, $a \neq 2b + 4$ 38. $\frac{x^2+1}{x}$, $x \neq \pm 1$

39. $\frac{1}{a^2+b^2}$, $a^4 - a^2b^2 + b^4 \neq 0$ 40. $x^2 - 1$, $x^4 + x^2 + 1 \neq 0$

Exercise 5-3 (page 151)

7. $\frac{2x}{y}$, $y \neq 0$ 8. $\frac{4z}{5x}$, $x, y \neq 0$ 9. 1, $a, b, c \neq 0$ 10. $\frac{4}{9}$, $a, b, c \neq 0$

11. $\frac{1}{(a+2)(a-1)}$, $a \neq \pm 1$ or -2 12. $\frac{1}{x+y}$, $x \neq 0$ or $\pm y$

13. $\frac{y(a-b)}{ax}$, $x \neq y$, $a \neq -b$ 14. $\frac{3a}{2b}$, $x, y \neq 0$, $2a \neq 3b$

15. $\frac{(x+2)(x-4)}{x(x-2)}$, $x \neq -1$ or 3 16. $\frac{xy}{(y-x)(y+x)}$, $y \neq \pm x$

17. $\frac{1}{6(a-b)}$, $a \neq \pm b$ 18. $x^2 - x + 1$, $x \neq \pm 1$, $x^2 + x + 1 \neq 0$

19. 1, $a \neq b$, $c \neq d$ 20. 1, $x \neq -3$ or -1 or 4

21. 1, $x \neq 1$ or 2 or 3 22. -1, $a \neq \pm 5$ or -7 or -2 or 1 or 3

23. $\frac{b-c-a}{a-c+b}$, $a \neq b - c$, $a \neq -b + c$, $a \neq b + c$

24. $\frac{1}{x^2-xy+y^2}$, $x \neq \pm y$, $x^2 - xy + y^2 \neq 0$, $x^2 + xy + y^2 \neq 0$

25. $\frac{(a^2+b^2)(a+b)^2}{a^4+a^2b^2+b^4}$, $a \neq b$, $a^4 + a^2b^2 + b^4 \neq 0$

26. $\frac{a(a-1)}{a^2-a+1}$, $a \neq 0$ or ± 1, $a^2 + 1 \neq 0$, $a^4 + a^2 + 1 \neq 0$

Exercise 5-4 (page 153)

1. $\dfrac{3a + 4b - 6c}{12}$ **2.** $\dfrac{6x^2 + 3x - 2}{30}$ **3.** $\dfrac{3bc + 4ac - 2ab}{abc}$

4. $\dfrac{3 - 4x + 5x^2}{x^2}$ **5.** $\dfrac{pq^2 + pr^2}{rq}$ **6.** $\dfrac{a^2c + ab^2 + bc^2}{abc}$

7. $\dfrac{z + x + y}{xyz}$ **8.** $\dfrac{5ab - 3b + 2a}{a^2b^2}$ **9.** $\dfrac{x - 2}{30}$

10. $\dfrac{4a + b}{12}$ **11.** $\dfrac{6x - 5}{x^2 - 1}$ **12.** $\dfrac{16a + 5b}{a^2 - b^2}$

13. $\dfrac{6 - 8a + 6b}{4a^2 - 9b^2}$ **14.** $\dfrac{-2b^3}{a^4 + a^2b^2 + b^4}$ **15.** $\dfrac{9x^2 + 10xy - 7y^2}{(x - y)^2(x + y)^2}$

16. $\dfrac{(x^2 + 1)^2}{x(x^2 + x + 1)}$ **17.** $\dfrac{1}{(a + 1)(a + 2)(a + 3)}$ **18.** $\dfrac{1}{a - b}$

19. $\dfrac{5xy - 2x + 4y}{xy(x - y)}$ **20.** $\dfrac{2a^3}{1 - a^4}$, $a \neq 0$ **21.** $\dfrac{x^4 + y^4}{xy(x^4 - y^4)}$

22. $\dfrac{4a - 4}{(a - 2)^2(a + 2)}$ **23.** $\dfrac{4x^3}{81 - x^4}$ **24.** $\dfrac{m + 2}{(m + 1)(m + 3)}$, $m \neq -2$

25. $\dfrac{1}{1 - x^4}$ **26.** $\dfrac{5p}{(2p - 1)(p + 1)(3p + 1)}$ **27.** $\dfrac{1}{a + 2}$, $a \neq -3$ or -7

Exercise 5-5 (page 155)

1. $\dfrac{x^2 + y^2}{xy}$, $a, b \neq 0$ **2.** $\dfrac{1}{x}$, $x \neq 0$ **3.** $\dfrac{bc + ac + ab}{abc}$, $a, b, c \neq 0$

4. $\dfrac{3b - 2a}{6ab}$ **5.** $8x + 2$, $x \neq 0$ **6.** $2b$, $a, m \neq 0$

7. $\dfrac{40m + 9n}{15}$, $m \neq n$ **8.** $\dfrac{-3a - c}{a + c}$, $b \neq a$ or c

9. $\dfrac{4m - 10}{(m - 5)(m + 2)}$ **10.** $\dfrac{2x - 7}{(x + 5)(x - 5)}$, $x \neq 0$

11. $\dfrac{4a^2}{x(x + 2a)}$, $x \neq 2a$ **12.** $\dfrac{x^2 + 4xy + 8y^2}{x^2 - 16y^2}$, $x \neq 0$, $x \neq -4y$

13. $\dfrac{a + 5b}{6}$, $a^2 - ab + b^2 \neq 0$, $a^2 + ab + b^2 \neq 0$

14. $\dfrac{3a^2 + 5}{a^4 - 16}$, $a \neq 0$, $a \neq \pm 2$ **15.** $\dfrac{-x^2 + 2x + 25}{x^2 - 16}$, $x \neq 2$ or 3

Practice Exercise 5-6 (page 155)

1. $\dfrac{a - b}{b}$, $a \neq -b$ **2.** $\dfrac{a + b - c}{a - b - c}$, $a \neq -(b + c)$ **3.** $\dfrac{x(x + 3)}{x - 4}$, $x \neq 3$

4. $\dfrac{2(a - 2)}{a + 6}$, $a \neq 6$ **5.** $\dfrac{x^2 + 1}{x - 2}$, $x \neq -2$ **6.** $\dfrac{a^2 - 2}{2(a^2 + 4a + 16)}$, $a \neq 4$

7. $\dfrac{2(x - 3)}{x + 3}$, $x \neq 0$ or 3 **8.** $\dfrac{a - b}{ab}$, $a, b \neq 0$, $a \neq -b$

9. $\frac{a+b-c}{a-b-c}$, $a \neq -(b+c)$ **10.** $\frac{4x+7y}{2(x+4y)}$, $3x \neq 5y$

11. $\frac{(x+y)(x^2+y^2)}{x}$, $x \neq y$ **12.** $\frac{4x^2+2xy+y^2}{2x+y}$, $2x \neq y$

13. $x+1$, $x \neq 1$ **14.** x^2+xy+y^2, $x^2-xy+y^2 \neq 0$

Practice Exercise 5-7 (page 156)

1. $\frac{a}{d}$, $b, c \neq 0$ **2.** $\frac{ax}{by}$, $a, b, x, y \neq 0$ **3.** $\frac{1}{x}$, $x \neq \pm 2$, $x^2+2x+4 \neq 0$

4. $\frac{x}{x+4}$, $x \neq \pm 3$ or 4 **5.** $\frac{(x+y)^2}{x(x-y)}$, $x \neq \pm y$ **6.** 1, $m \neq 0$ or ± 3

7. $\frac{a-3}{a+4}$, $a \neq -2$ or 1 or 4 or ± 5 **8.** 1, $a \neq -2b$, $c \neq d$, $m \neq n$

9. $\frac{6b}{(a+b)(a^2+b^2)}$, $a \neq b$ **10.** $\frac{16x^2+4xy+y^2}{(x+2y)(x^2+4y^2)}$, $x \neq 2y$, $y \neq \pm 4x$

11. $\frac{(a+b)^2(a^2+b^2)}{a(a-b)}$, $a \neq b$, $a^2+b^2 \neq 0$

Practice Exercise 5-8 (page 157)

1. $\frac{pt+qr}{qt}$ **2.** $\frac{5bc-3ac+4ab}{abc}$ **3.** $\frac{10y-21x}{6xy}$

4. $\frac{3c-6a+b}{abc}$ **5.** $\frac{19x-21y}{15}$ **6.** $\frac{9y-x}{15}$

7. $\frac{13x-13}{6}$ **8.** $-\frac{-11a-b}{30}$ **9.** $\frac{5x+1}{x^2-1}$

10. $\frac{5a+9b}{a^2-b^2}$ **11.** $\frac{-4x}{x^2-1}$ **12.** $\frac{6a^2-4b^2}{a^2-b^2}$

13. $\frac{x^2+17x}{(x-3)(x+3)(x-4)}$ **14.** 1, $x \neq -1$ or 2

Practice Exercise 5-9 (page 157)

1. $\frac{2}{x^2+1}$ **2.** $\frac{5}{a-b}$, $a \neq -b$ **3.** $\frac{2x-11}{(x-3)(x-4)(x-5)}$

4. $\frac{5x-9y}{x(x^2-y^2)}$ **5.** $\frac{-a^2-9a+1}{a^2-9}$, $b \neq 0$ **6.** $\frac{2(x^2-3x+3)}{(x-1)(x-2)(x-3)}$

7. $\frac{1-2m}{(m-4)(m-5)}$, $m \neq 3$ or 4 **8.** $\frac{x^2-6x+16}{3(x-3)}$ **9.** $\frac{3a^2+4b^2}{a^4-b^4}$

Practice Exercise 5-10 (page 158)

1. 1, $a, b, c \neq 0$ **2.** $\frac{4c}{5a}$, $a, b \neq 0$ **3.** $\frac{1}{(m-1)(m+2)}$, $m \neq -2$ or ± 1

4. $\frac{x(a+2x)}{a^2}$, $a \neq -4x$ or $2x$ **5.** $a(a^2+b^2)$, $a \neq \pm b$ **6.** $\frac{(x-1)^3}{(x-3)^3}$, $x \neq 2$

7. $\frac{(a-b)^2}{b(a+b)}$, $b \neq 0$ or $-a$ **8.** $\frac{6y}{(x+y)(x^2+y^2)}$, $x \neq y$

9. $\dfrac{2(a-3)}{3a(a+6)}$, $a \neq 0$ or ± 4 or ± 5 **10.** $\dfrac{a^2-36}{a(a-4)}$, $a \neq 2$ or 6

11. $\dfrac{b^2+2bc+c^2-a^2}{a+c-b}$, $a \neq -b \pm c$ **12.** 1, $x \neq \pm y$, $x^2+y^2 \neq 0$

Review Exercise 5-11 (page 158)

1. $3ab^2$, $a, b \neq 0$ **2.** $\dfrac{4yz+3xz+12x}{24xyz}$ **3.** $x+3$, $x \neq 0, 3, \pm 4$

4. x^6-1 **6.** $\dfrac{6a+3b}{2(a^2-b^2)}$ **7.** $6x^4+7x^3-34x^2+34x-12$

8. $2\frac{5}{7}$ **9.** $\dfrac{-2}{(x-4)(x-6)}$, $x \neq 1$ or 2 **10.** 0

11. $9x^4-30x^3+37x^2-20x+4$ **12.** $(x-2)(x+2)(x^2+2x+4)(x^2-2x+4)$

13. $(x-2y)(x+2y+1)$ **14.** $(x-y)(2x-2y-1)(2x-2y+1)$

15. $x(3x+2y)(3x-2y)$ **16.** $(a+b+c)(a-b-c)$

17. $(a^2+4)(a-1)(a+1)$ **18.** $(3y+1)(2y-3)$

19. $(2m-1)(m^3+2)$ **20.** $1-5x+10x^2-20x^3$

22. 28, 17 **23.** $1\frac{7}{10}$ **24.** x^2+2x-3

25. $a+29b-11c$ **26.** 1 **28.** $\dfrac{a^3+a^2-2}{2a^2+2a+1}$, $a \neq 1$

29. $2x-9$ **30.** $\dfrac{1}{a^2-b^2}$, $a, b \neq 0$, $a \neq b$, $a^2+b^2 \neq 0$

31. $\dfrac{3a+2}{(a+1)(a-1)(a-2)}$ **32.** 1,600 **33.** $\dfrac{a-5}{a^2+4}$, $a \neq 0$ or ± 2

Exercise 6-1 (page 161)

1. (Ted, George), (Ted, Andy), (Ted, Michael), (Ted, Martin), (George, Andy), (George, Michael), (George, Martin), (Andy, Michael), (Andy, Martin), (Michael, Martin)

2. (Aluminum, Diamond), (Aluminum, Iron), (Aluminum, Copper), (Aluminum, Gold), (Diamond, Iron), (Diamond, Copper), (Diamond, Gold), (Iron, Copper), (Iron, Gold), (Copper, Gold)

3. $R_1 = \{(-1, -3), (1, 3), (0, 0)\}$
$R_2 = \{(-3, -3), (-1, -1), (0, 0), (1, 1), (3, 3), (5, 5)\}$
$R_3 = \{(0, 0), (-1, 1), (1, 1)\}$
$R_4 = \{(-3, -1), (-1, 1), (1, 3), (3, 5)\}$
$R_5 = \{(-3, 5), (-1, 5), (0, 5), (1, 5), (3, 5), (-3, 3), (-1, 3), (0, 3), (1, 3), (-3, 1), (-1, 1), (0, 1), (-3, 0), (-1, 0), (-3, -1)\}$

4. $P_1 = \left\{(3, 0), (3, \frac{1}{3}), (3, \frac{1}{2}), (3, 1), (3, 2), (2, 0), (2, \frac{1}{3}), (2, \frac{1}{2}), (2, 1), (1, 0), (1, \frac{1}{3}), (1, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{3}), (\frac{1}{3}, 0)\right\}$

$P_2 = \left\{(\frac{1}{3}, 3), (3, \frac{1}{3}), (\frac{1}{2}, 2), (2, \frac{1}{2}), (1, 1)\right\}$ $P_3 = \left\{(0, 1), (\frac{1}{2}, 2), (1, 3)\right\}$

Exercise 6-2 (page 163)

6. $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$

7. $\{(5, 4), (5, 6), (5, 8), (7, 4), (7, 6), (7, 8)\}$

8. $\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 9)\}$

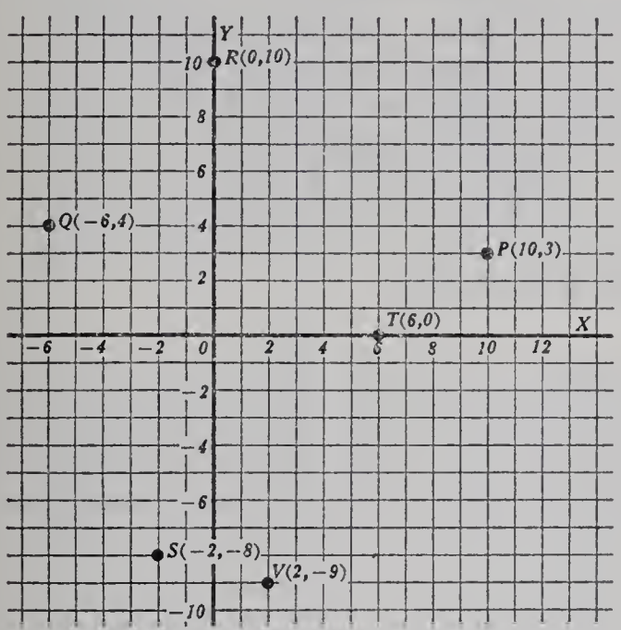
9. $\{(1, 2), (2, 4), (3, 6), (4, 8), \ldots\}$ **10.** $\{(1, 1)\}$

11. (Ham., Mont.), (Ham., Ott.), (Ham., Tor.), (Mont., Ham.), (Mont., Ott.), (Mont., Tor.), (Ott., Ham.), (Ott., Mont.), (Ott., Tor.), (Tor., Ham.), (Tor., Mont.), (Tor., Ott.)

Exercise 6-3 (page 168)

2. $A(2, 3)$, $B(-3, 3)$, $C(-2, -1)$, $D(4, -3)$, $E(3, 0)$, $F(0, 2)$, $G(-4, -3)$, $H(0, -3)$, $I(4, 2)$, $J(-5, 3)$, $K(-1, 0)$, $L(3, -2)$, $M(1\frac{1}{2}, 1\frac{1}{2})$, $N(-2\frac{1}{2}, 1)$, $Q(-4, -1\frac{1}{2})$

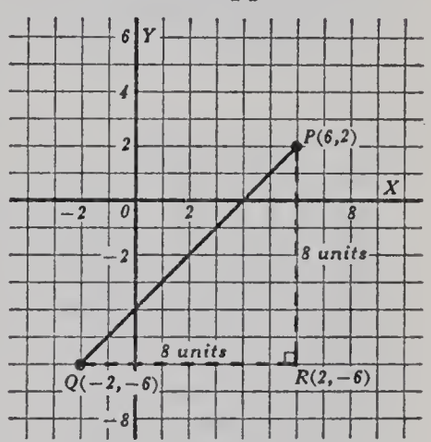
3.



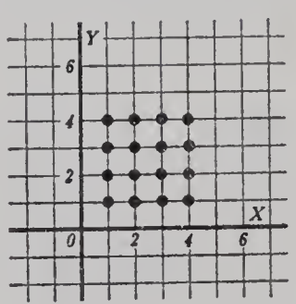
4. 10, 8 **5.** $(-3, -3)$, $(2, 5)$, $(7, 1)$

6. 5 **7.** 5

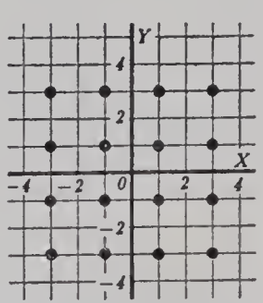
8. Length $= \sqrt{36 + 4}$ units
$= 6.3$ units approx.



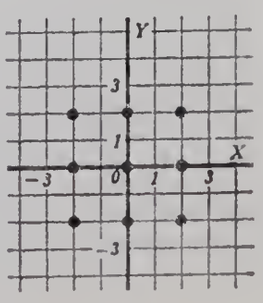
9. $A \times A = \{(1, 4), (2, 4), (3, 4), (4, 4),$
$(1, 3), (2, 3), (3, 3), (4, 3),$
$(1, 2), (2, 2), (3, 2), (4, 2),$
$(1, 1), (2, 1), (3, 1), (4, 1)\}$



10. $A \times A = \{(-3, 3), (-1, 3), (1, 3), (3, 3),$
$(-3, 1), (-1, 1), (1, 1), (3, 1),$
$(-3, -1), (-1, -1), (1, -1), (3, -1),$
$(-3, -3), (-1, -3), (1, -3), (3, -3)\}$



11. $A \times A = \{(-2, 2), (0, 2), (2, 2),$
$(-2, 0), (0, 0), (2, 0),$
$(-2, -2), (0, -2), (2, -2)\}$



Exercise 6-4 (page 171)

3. (2, 6), (18, 3)

4. (i) $\{(0, 1), (1, 2), (2, 3)\}$
Domain is $\{0, 1, 2\}$.
Range is $\{1, 2, 3\}$.

(ii) $\{(3, 0), (3, 1), (3, 2), (3, 3)\}$
Domain is $\{3\}$.
Range is $\{0, 1, 2, 3\}$.

(iii) $\{(0, 1), (1, 2), (0, 2), (1, 3), (0, 3), (2, 3)\}$
Domain is $\{0, 1, 2\}$.
Range is $\{1, 2, 3\}$.

(iv) $\{(0, 0)\}$
Domain is $\{0\}$.
Range is $\{0\}$.

(v) $\emptyset$
Domain is $\emptyset$.
Range is $\emptyset$.

(vi) $\{(0, 0), (1, 1)\}$
Domain is $\{0, 1\}$.
Range is $\{0, 1\}$.

5. $A \times B = \{(-1, 2), (0, 2), (1, 2), (-1, 0), (0, 0), (1, 0), (-1, -2), (0, -2), (1, 2)\}$
$\{(-1, -2), (0, 0), (1, 2)\}$
Domain is $\{-1, 0, 1\}$.
Range is $\{-2, 0, 2\}$.

6. (i) $\{x \mid x \in R\}$; $\{y \mid y \in R\}$
(ii) $\{-2\}$; $\{y \mid y \in R\}$
(iii) $\{x \mid x \in R\}$; $\{3\}$
(iv) $\{0\}$; $\{y \mid y \in R\}$

Exercise 6-5 (page 178)

4. (i) Typical tables

x	0	1	2
y	5	4	3

(ii)

x	0	1	2
y	-5	-4	-3

(iii)

x	0	1	2
y	-6	-4	-2

(iv)

x	0	3	6
y	0	-1	-2

(v)

x	0	1	2
y	1	-1	-3

(vi)

x	0	1	2
y	0	1	4

(vii)

x	0	2	4
y	4	3	2

(viii)

x	0	1	2
y	2	5	8

(ix)

x	1	3	5
y	0	4	12

(x)

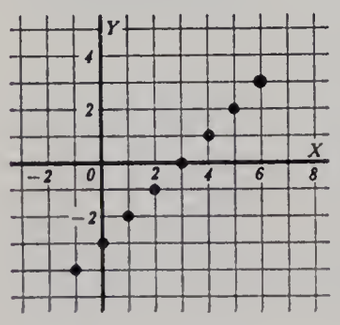
x	2	5	8
y	-1	1	3

(xi) No ordered pairs

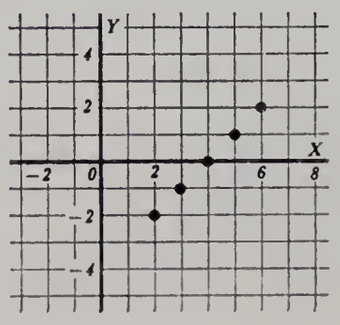
(xii)

x	0	1	4
y	0	1	2

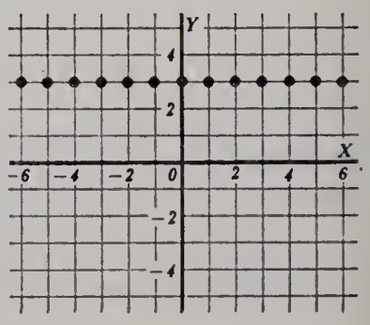
5. (i)



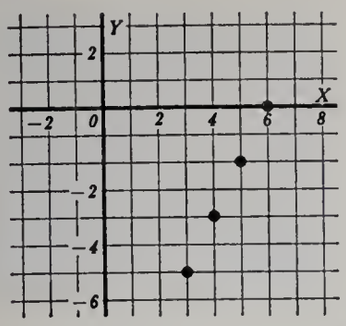
(ii)



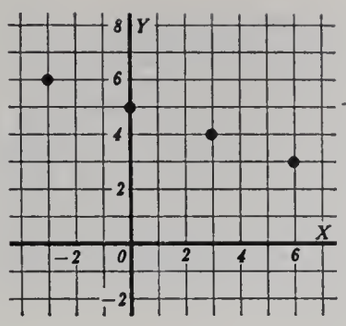
(iii)



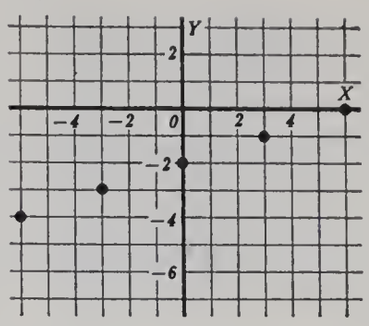
(iv)



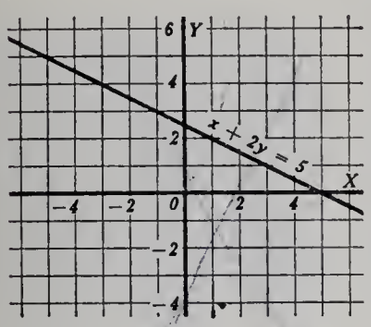
(v)



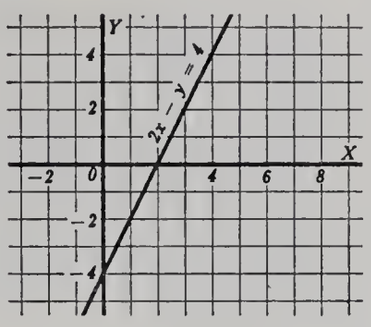
(vi)



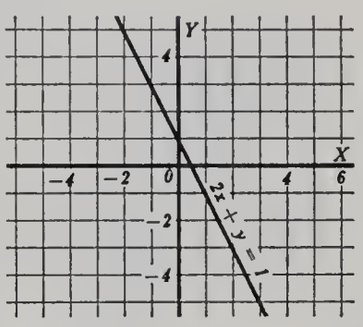
6. (i)



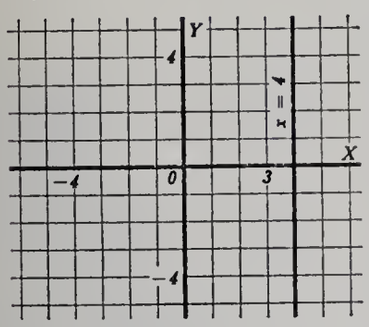
(ii)



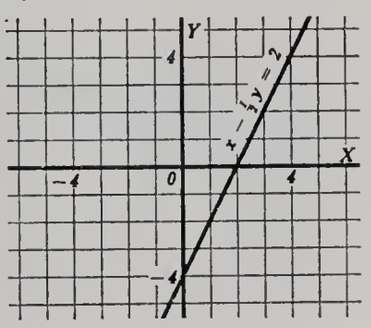
(iii)



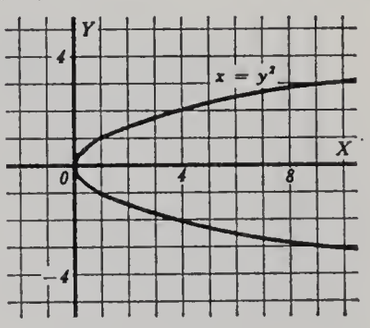
(iv)



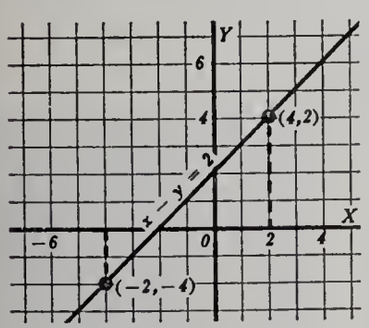
(v)



(vi)

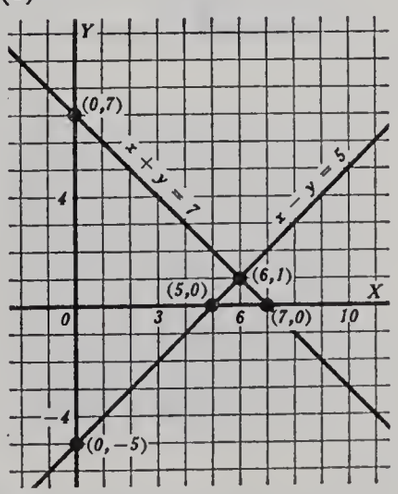


7.



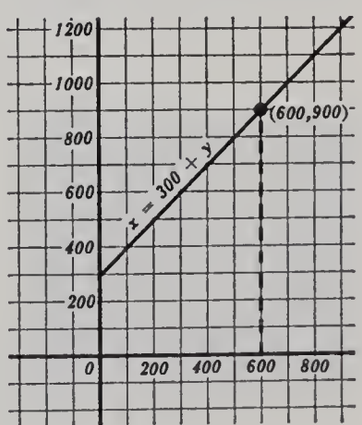
Ordinates are 2, −4.

8. (i)



(ii) (7, 0), (0, 7);
(5, 0), (0, −5);
(6, 1)

9.



Cost is $9.00.

Exercise 6-6 (page 182)

2. (i) 3, 2

(ii) $1\frac{1}{4}$, −5

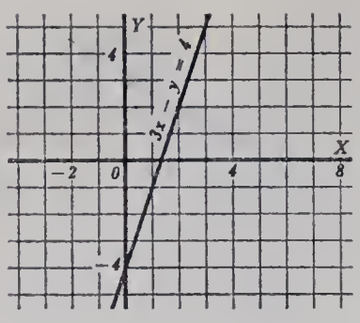
(iii) 0, 0

(iv) $-2\frac{1}{3}$, $-1\frac{2}{5}$

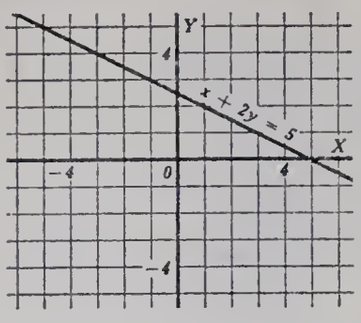
(v) $-\frac{3}{5}, \frac{3}{4}$

(vi) −3, 2

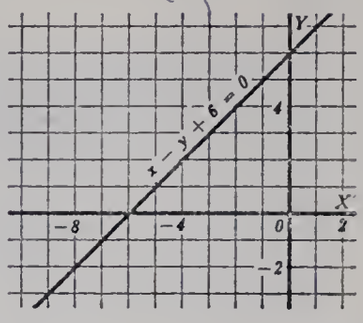
3.



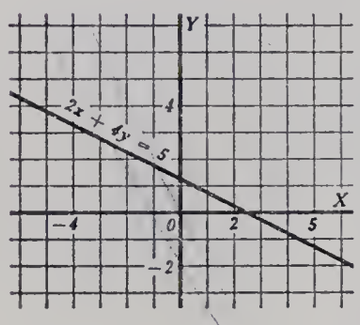
4.



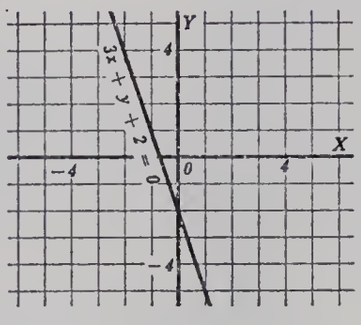
5.



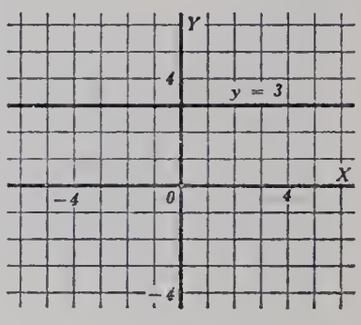
6.



7.



8.



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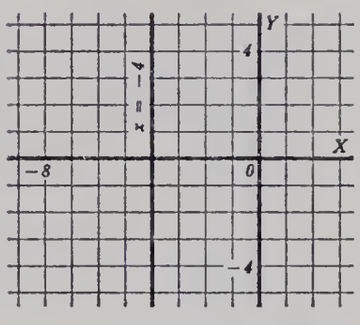

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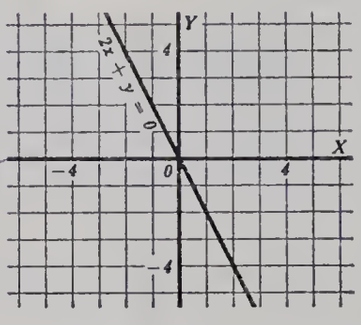
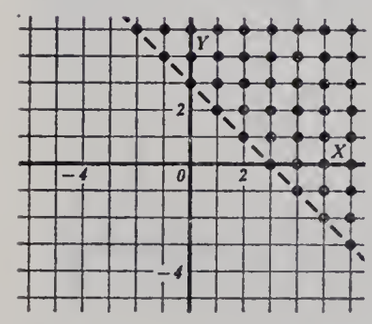
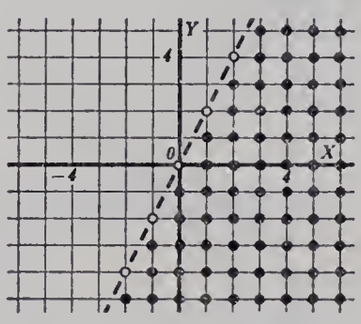


Exercise 6-7 (page 185)

2.



3.



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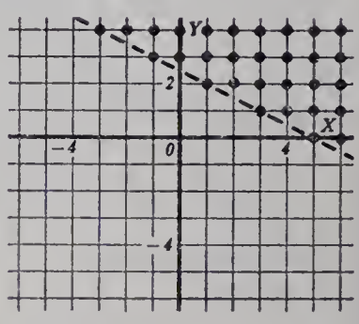

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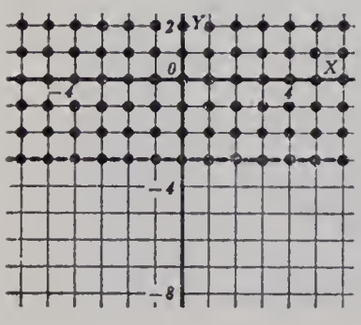
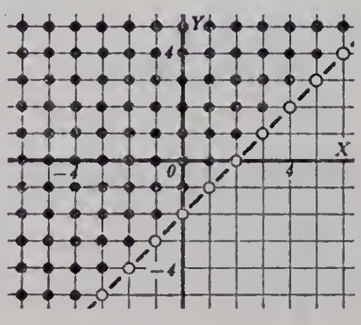


6.



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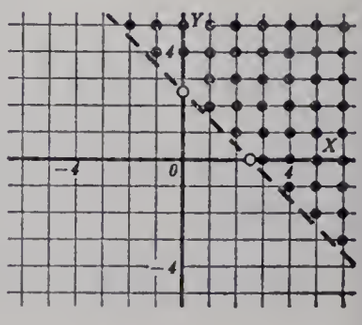

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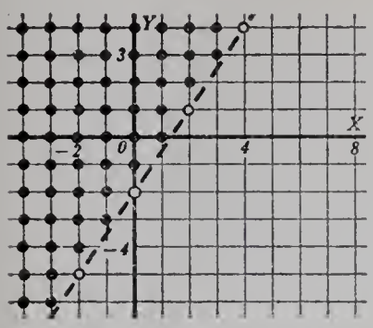

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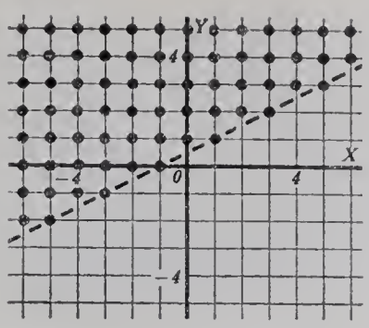

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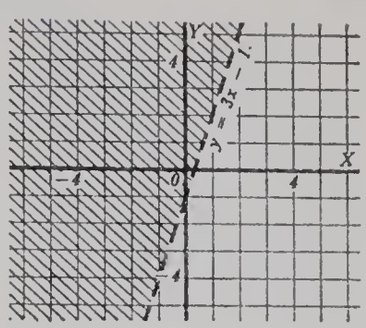
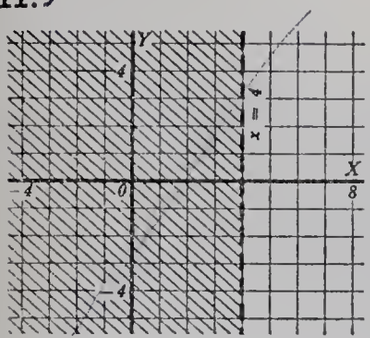


11.



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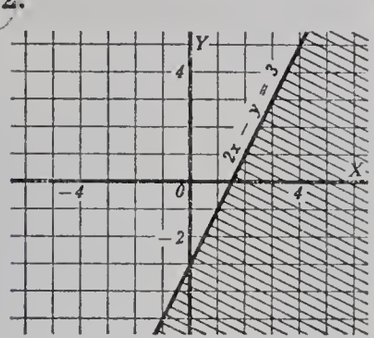

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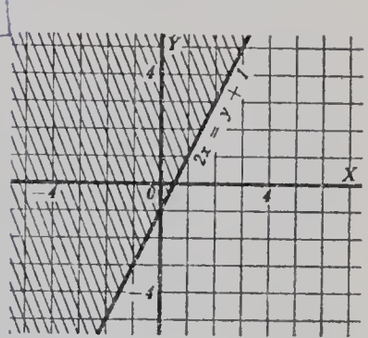

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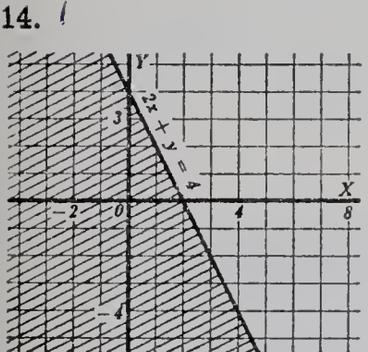

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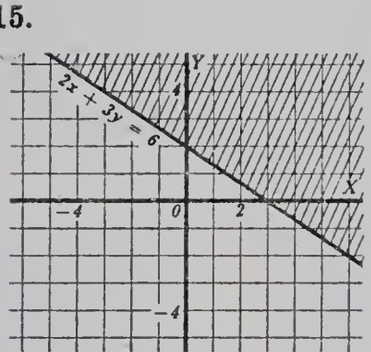

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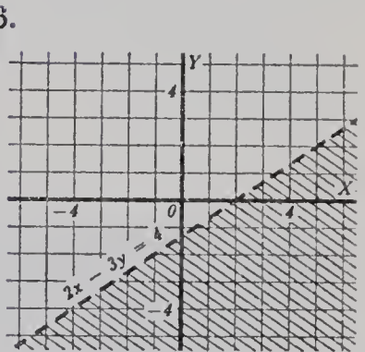

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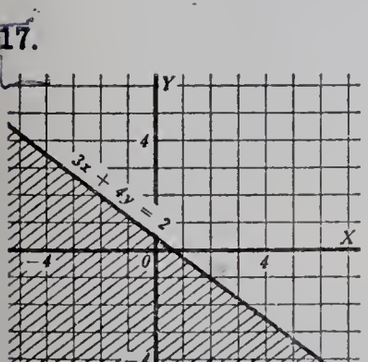

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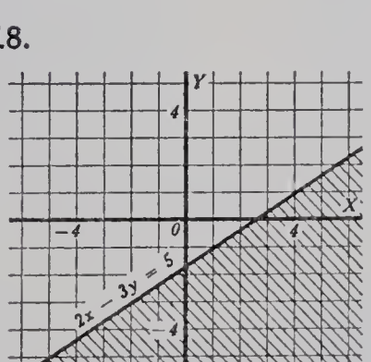

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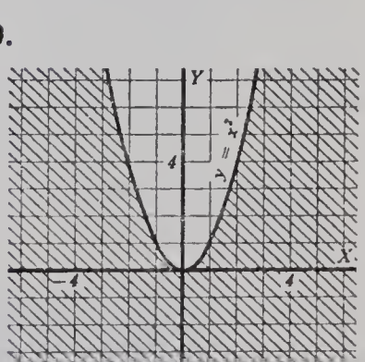

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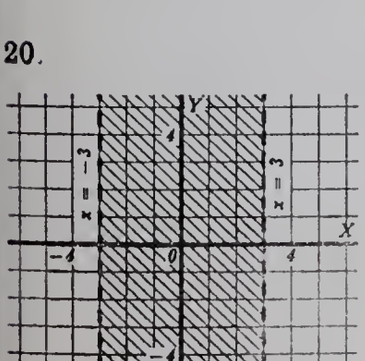
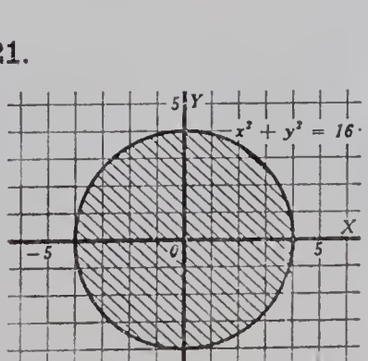
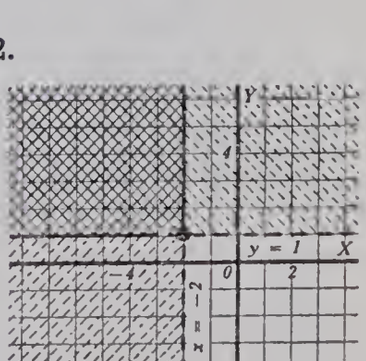


21.



22.



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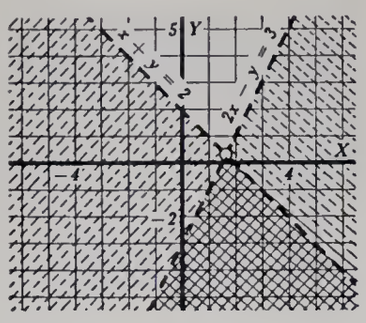
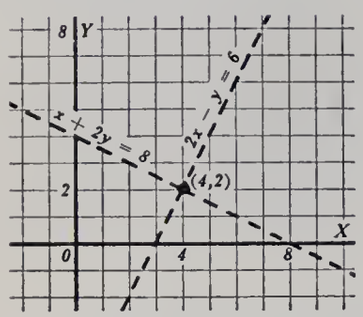


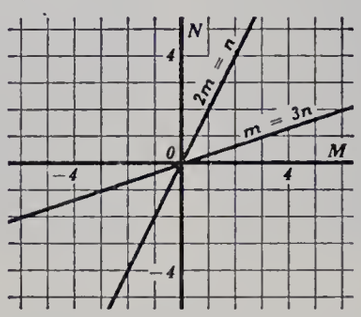
Exercise 6-8 (page 188)

5. (i)



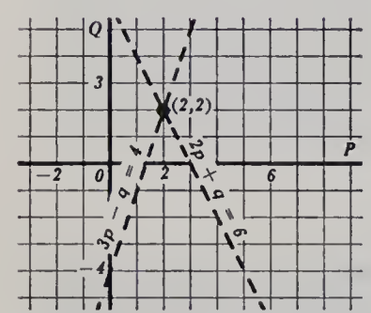
$A = \{(4, 2)\}$
Domain is $\{4\}$.
Range is $\{2\}$.

(ii)



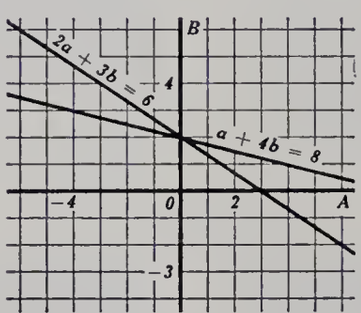
$B = \{(m, n) \mid m = 3n \text{ or } 2m = n, m, n \in R\}$
Domain is $\{m \mid m \in R\}$.
Range is $\{n \mid n \in R\}$.

(iii)



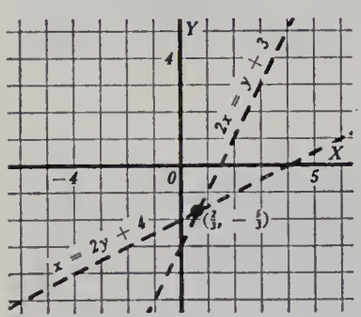
$C = \{(2, 2)\}$
Domain is $\{2\}$.
Range is $\{2\}$.

(iv)



$D = \{(a, b) \mid 2a + 3b = 6 \text{ or } a + 4b = 8, a, b \in R\}$
Domain is $\{a \mid a \in R\}$.
Range is $\{b \mid b \in R\}$.

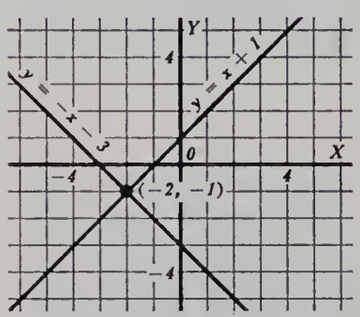
6.



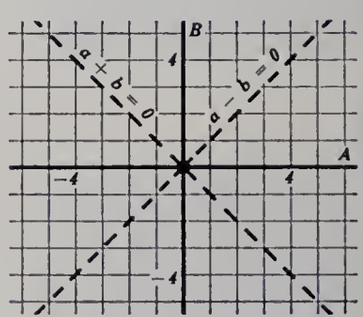
Intersection set is

$$\left\{\left(\frac{2}{3}, -\frac{5}{3}\right)\right\}.$$

7.



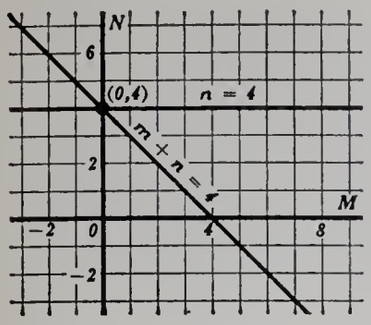
8.



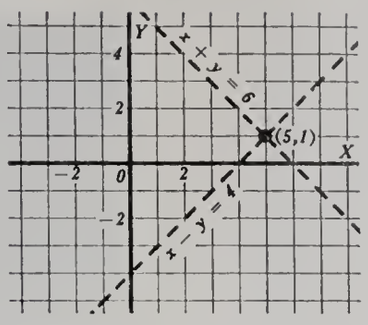
Intersection set is

$\{(0, 0)\}$.

9.

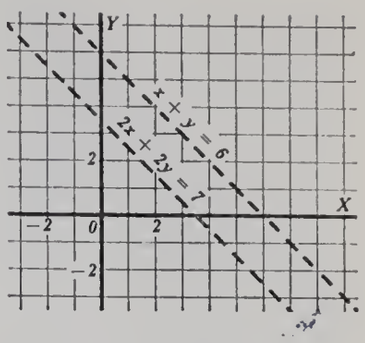


10. (i)



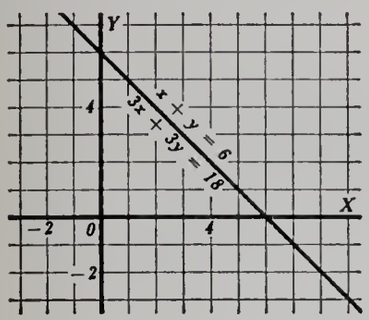
$A \cap B = \{(5, 1)\}$

(ii)



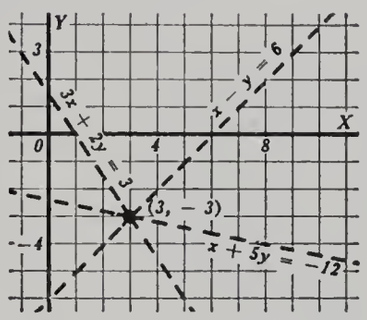
$A \cap C = \emptyset$

(iii)



$A \cap D$ is
$\{(x, y) \mid x + y = 6, x, y \in R\}$

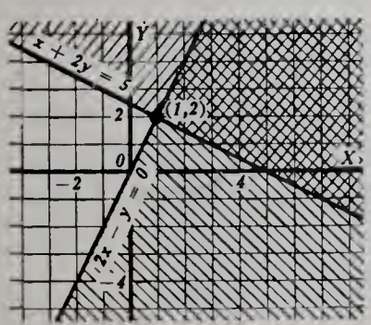
11.



Intersection set is $\{(3, -3)\}$.

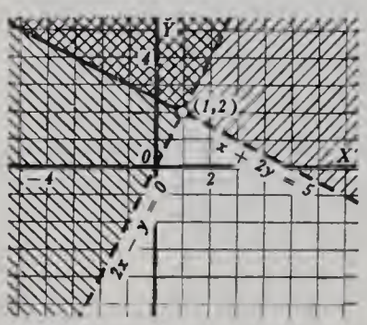
Exercise 6-9 (page 191)

1.



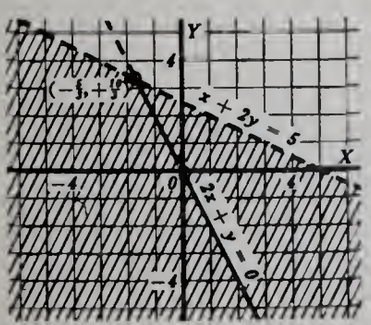
Domain is $\{x \mid x \in R\}$.
Range is $\{y \mid y \in R\}$.

2.



Domain is $\{x \mid x \in R\}$.
Range is $\{y \mid y > 2, y \in R\}$.

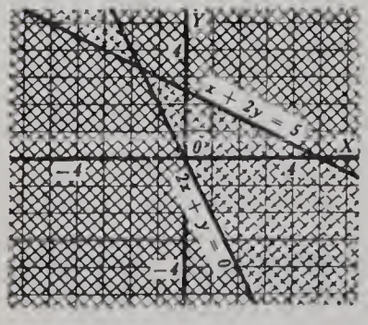
3.



Domain is $\{x \mid x \geqq -\frac{5}{3}, x \in R\}$.

Range is $\{y \mid y \leqq \frac{10}{3}, y \in R\}$.

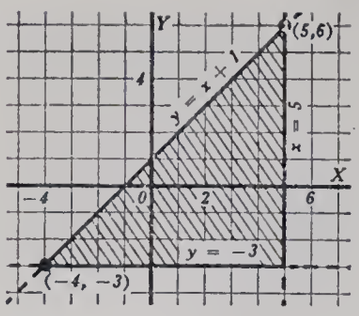
4.



Domain is $\{x \mid x \in R\}$.

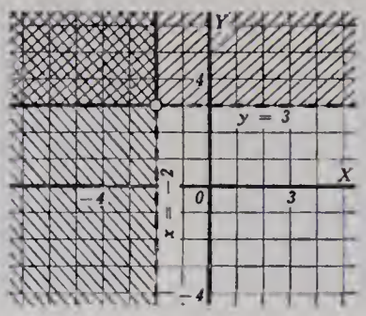
Range is $\{y \mid y \in R\}$.

5.



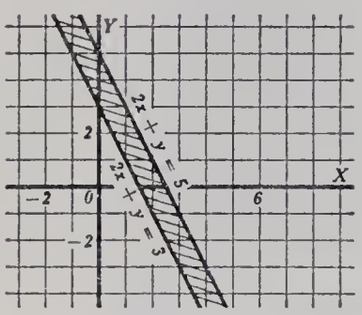
Domain is $\{x \mid -4 \leqq x \leqq 5, x \in R\}$.
Range is $\{y \mid -3 \leqq y \leqq 6, y \in R\}$.

6.



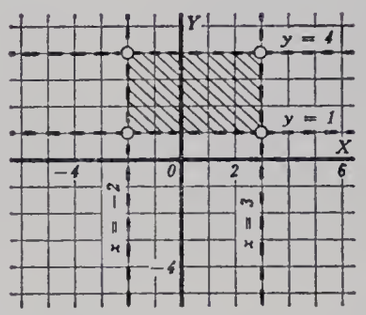
Domain is $\{x \mid x \in R\}$.
Range is $\{y \mid y \in R\}$.

7.



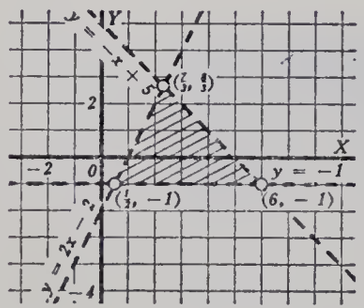
Domain is $\{x \mid x \in R\}$.
Range is $\{y \mid y \in R\}$.

8.



Domain is $\{x \mid -2 < x < 3, x \in R\}$.
Range is $\{y \mid 1 < y < 4, y \in R\}$.

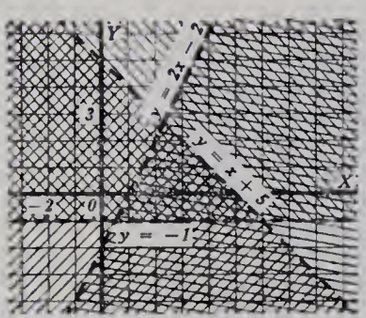
9. (i)



Domain is $\{x \mid \frac{1}{2} < x < 6, x \in R\}$.

Range is $\{y \mid -1 < y < \frac{8}{3}, y \in R\}$.

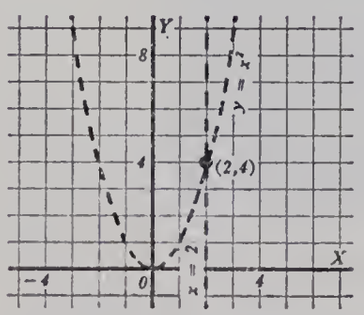
(ii)



Domain is $\{x \mid x \in R\}$.

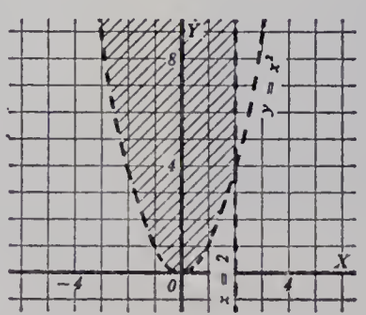
Range is $\{y \mid y \in R\}$.

10.



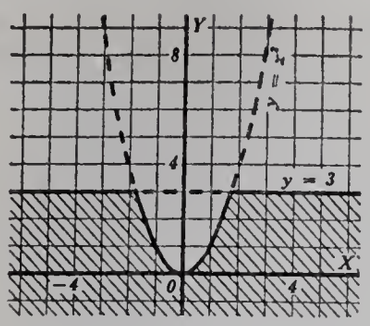
Domain is $\{2\}$.
Range is $\{4\}$.

11.



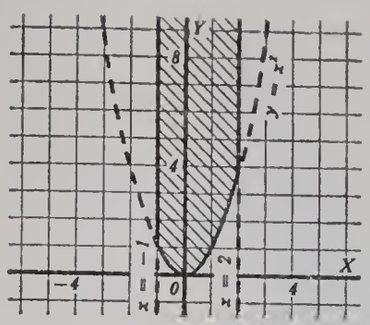
Domain is $\{x \mid x < 2, x \in R\}$.
Range is $\{y \mid y > 0, y \in R\}$.

12.



Domain is $\{x \mid x \in R\}$.
Range is $\{y \mid y \leqq 3, y \in R\}$.

13.



Domain is $\{x \mid -1 \leqq x \leqq 2, x \in R\}$.
Range is $\{y \mid y \geqq 0, y \in R\}$.

14. $\{(x, y) \mid y \leqq x + 2$ and $y \leqq 2$ and $y \geqq x - 1$ and $y \geqq -1, x, y \in R\}$

15. $\{(x, y) \mid y \leqq 3x$ and $y \geqq 0$ and $y \leqq -3x + 12, x, y \in R\}$

16. $\{(x, y) \mid x \geqq 0$ and $y \geqq 0$ and $y < \dfrac{12 - 3x}{4}, x, y \in R\}$ or alternatively

$\{(x, y) \mid x \geqq 0$ and $y \geqq 0$ and $4y + 3x - 12 < 0, x, y \in R\}$

17. $\{(x, y) \mid x^2 + y^2 \leqq 9$ and $y \geqq x + 2, x, y \in R\}$

18. $\{(x, y) \mid y \leqq \dfrac{3x + 6}{2}$ and $y \geqq \dfrac{3x}{2}, x, y \in R\}$ or alternatively

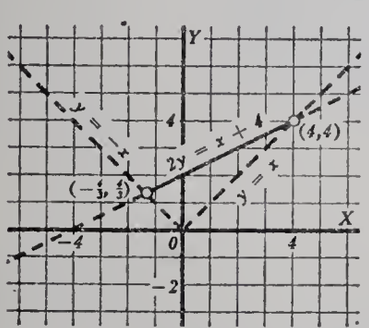
$\{(x, y) \mid 3x - 2y + 6 \geqq 0$ and $3x - 2y \leqq 0, x, y \in R\}$

19. $\{(x, y) \mid y \leqq \dfrac{6 - 3x}{2}$ and $y < 2x + 2$ and $y > 1 - x, x, y \in R\}$ or alternatively

$\{(x, y) \mid 3x + 2y - 6 \leqq 0$ and $2x - y + 2 > 0$ and $x + y - 1 > 0, x, y \in R\}$

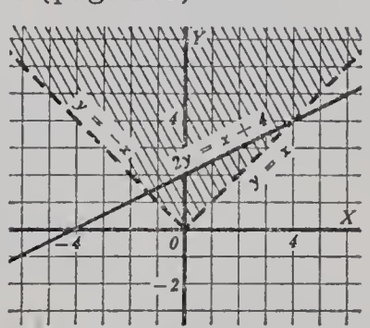
Practice Exercise 6-10 (page 193)

1.



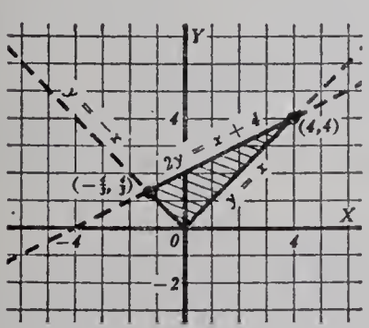
Domain is $\{x \mid -\dfrac{4}{3} < x < 4, x \in R\}$.
Range is $\{y \mid \dfrac{4}{3} < y < 4, y \in R\}$.

2.



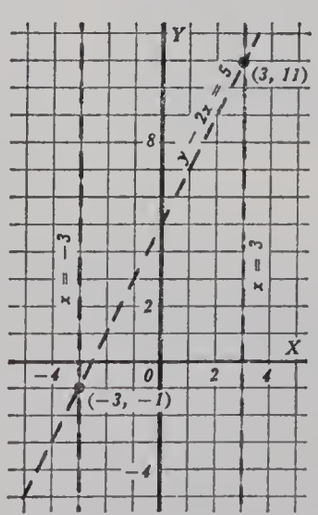
Domain is $\{x \mid x \in R\}$.
Range is $\{y \mid y \in R\}$.

3.



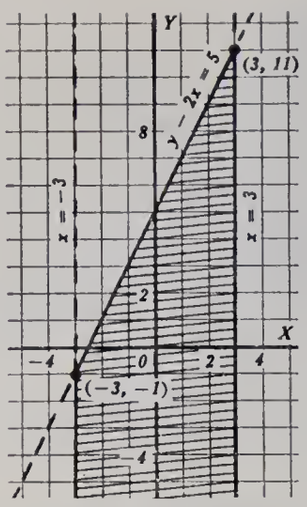
Domain is $\{x \mid -\dfrac{4}{3} \leqq x \leqq 4, x \in R\}$.
Range is $\{y \mid 0 \leqq y \leqq 4, y \in R\}$.

4.



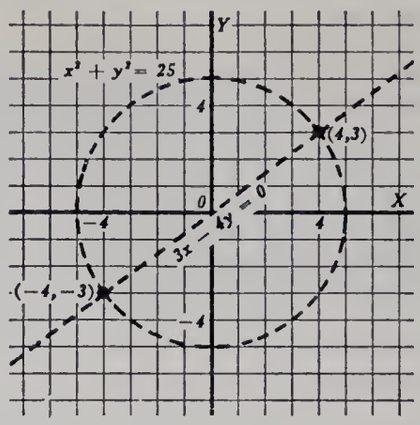
Domain is $\{-3, 3\}$. Range is $\{-1, 11\}$.

5.



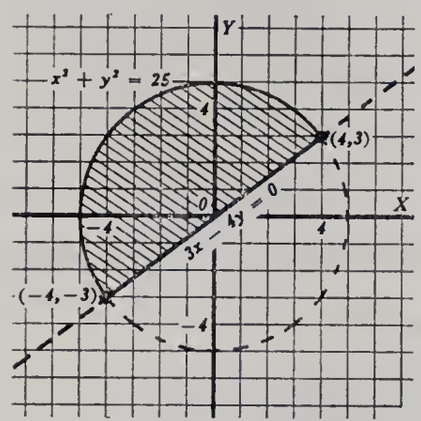
Domain is $\{x \mid -3 \leqq x \leqq 3, x \in R\}$.
Range is $\{y \mid y \leqq 11, y \in R\}$.

6.



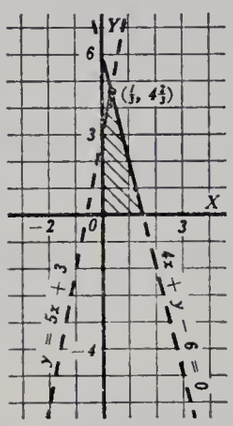
Domain is $\{-4, 4\}$.
Range is $\{-3, 3\}$.

7.



Domain is $\{x \mid -5 \leqq x \leqq 4, x \in R\}$.
Range is $\{y \mid -3 \leqq y \leqq 5, y \in R\}$.

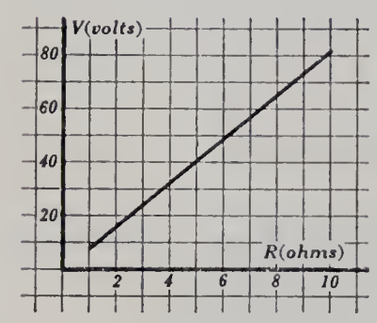
8.



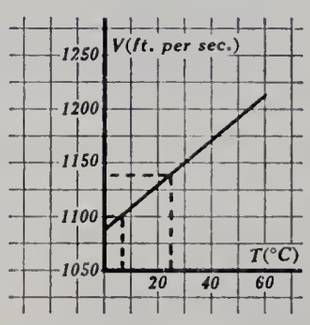
$(\frac{1}{3}, 4\frac{2}{3})$

Exercise 6-11 (page 197)

1.

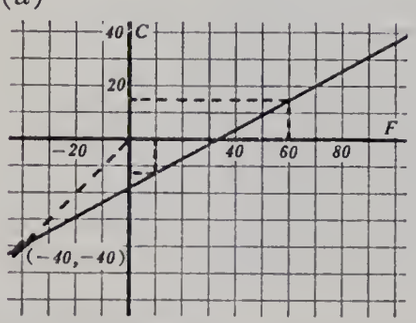


2. (a)



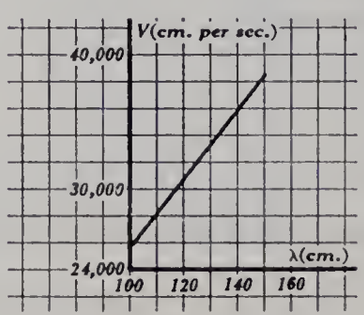
(b) (i) 1,140 ft. per sec. (ii) 6°C

3. (a)



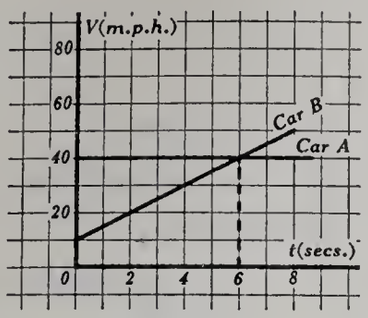
(b) (i) 60°F (ii) −12°C (iii) −40°C

4.



5. (a) $\{(t, v) \mid v = 32t, v, t \in {}^+R\}$ (b) $\{(t, v) \mid v = 10 + 20t, v, t \in {}^+R\}$

6.



(i) 6 secs.

(ii) For car A:
$\{(t, v) \mid v = 40, t \in {}^+R\}$
For car B:
$\{(t, v) \mid v = 10 + 5t, t, v \in {}^+R\}$

Exercise **7-1** (page 203)

13. (3, −2) **14.** (3, 1) **15.** (1, 5)
16. (−4, 6) **17.** (9, 5) **18.** (−2, 4)
19. (7, 9) **20.** $\left\{\left(\frac{3}{4}, \frac{2}{3}\right)\right\}$ **21.** $\{(3, 4)\}$
22. $\{(1, -1)\}$ **23.** $\left\{\left(-\frac{5}{6}, \frac{5}{6}\right)\right\}$ **24.** $\{(4, 6)\}$
25. (5, −2) **26.** $(7\frac{1}{7}, -1\frac{4}{7})$ **27.** (12, 8)

Exercise **7-2** (page 206)

7. (1, 2) **8.** (4, 1) **9.** (2, 2)
10. (2, 1) **11.** (6, 2) **12.** (9, 4)
13. (1, −2) **14.** $(1\frac{1}{2}, 3)$ **15.** (3, 5)
16. (6, 4) **17.** (20, 12) **18.** (2, 5)
19. $\{(2, 5)\}$ **20.** $\{(3, 4)\}$ **21.** $\{(-2, 4)\}$
22. $\{(4, 6)\}$ **23.** $\{(-1, 3)\}$

Exercise **7-3** (page 209)

7. (3, 2) **8.** (2, 3) **9.** (1, 1)
10. (−2, 1) **11.** (3, 0) **12.** (10, 3)
13. (8, 3) **14.** $\left(\frac{10}{3}, -2\right)$ **15.** (4, 6)
16. $\{(5, -1)\}$ **17.** $\{(5, 3)\}$ **18.** $\{(2, -1)\}$
19. $\{(3, 2)\}$ **20.** $\{(2\frac{1}{7}, 3\frac{3}{7})\}$ **21.** $\{(10, 6)\}$
22. $\{(3, 4)\}$ **23.** $\{(15, 3)\}$ **24.** $\{(20, 15)\}$
25. (7, 9) **26.** (4, 3) **27.** (−4, 8)
28. $\left\{\left(-\frac{5}{8}, -\frac{1}{4}\right)\right\}$ **29.** $\{(-3, -3)\}$

Exercise **7-4** (page 213)

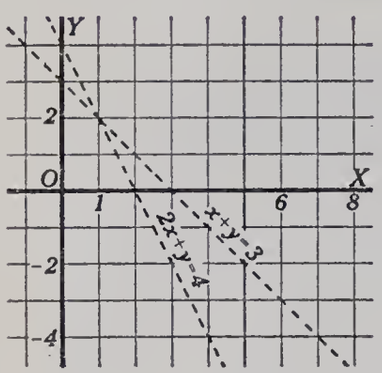
1. (1, 2, 3) **2.** (2, 3, 4) **3.** (1, 3, 4)
4. (8, 6, 3) **5.** (11, 9, 1) **6.** (12, 12, 12)
7. $\left(\frac{1}{3}, 4, 2\right)$ **8.** (4, 0, 5) **9.** (3, 5, −2)
10. (3, 1, 2) **11.** (9, 11, 13) **12.** (−2, 2, 0)
13. (2, −1, −3, 1) **14.** (−31, 112, 13, −92)

Exercise 7-5 (page 217)

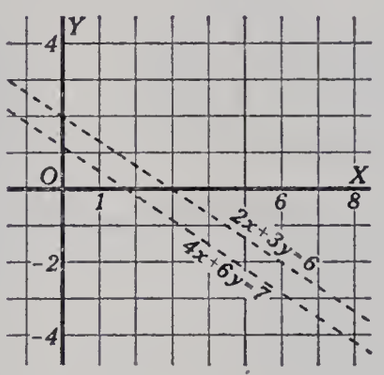
1. 38, 17
2. 14, 8
3. 8, 5
4. $4\frac{1}{3}$, $13\frac{1}{3}$
5. 12, 26
6. 4, 8
7. \$6, \$9
8. \$1.10, \$.85
9. \$1.20, \$.90
10. 15 doz., 20 doz.
11. 12 ft. $\times$ 15 ft.
12. 5 in., 12 in., 13 in.
13. 20 tons, 60 tons
14. \$4,000, \$2,000
15. \$13,600
16. \$4,000, \$8,000
17. 260 m.p.h., 40 m.p.h.
18. 4 m.p.h., 2 m.p.h.
19. 24
20. 74
21. 63
22. 82
23. 72
24. 54
25. $\frac{45}{56}$
26. 35

Exercise 7-6 (page 221)

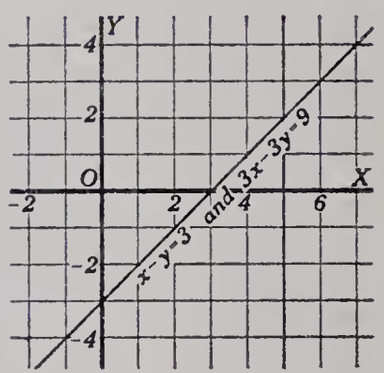
1.



2.



3.



Practice Exercise 7-7 (page 223)

1. (r, s)
2. $\left(\frac{c(a+b)}{a+2b}, \frac{c}{a+2b}\right)$
3. $(2n, 2m)$
4. $(a+b, a-b)$
5. (k, h)
6. $\left(\frac{c(a-b)}{a^2+b^2}, \frac{c(a+b)}{a^2+b^2}\right)$
7. $\left(\frac{c_1-b_1}{a_1-b_1}, \frac{a_1-c_1}{a_1-b_1}\right)$
8. $\frac{1}{a}, \frac{1}{b}$
9. (b, a)
10. (a, b)

Practice Exercise 7-8 (page 223)

1. (2, 1)
2. (2, 3)
3. (7, 9)
4. (3, 2)
5. (5, 3)
6. (9, 6)
7. $\{(-2, 1)\}$
8. $\{(-9, -2)\}$
9. $\left\{\left(\frac{3}{4}, \frac{2}{3}\right)\right\}$
10. $\{(7, 3)\}$
11. $\{(2, -2)\}$
12. $\{(7, 19)\}$

Practice Exercise 7-9 (page 224)

1. (11, 9)
2. (2, 3)
3. (−1, 3)
4. (3, 7)
5. (6, 4)
6. (12, 6)
7. $\{(13, 5)\}$
8. $\left\{\left(\frac{1}{2}, 4\right)\right\}$
9. $\{(16, 35)\}$
10. $\{(5, 6)\}$
11. $\{(9, 3)\}$
12. $\{(5, -9)\}$

Practice Exercise 7-10 (page 224)

1. (3, 2)
2. $\left(\frac{3}{7}, \frac{7}{3}\right)$
3. (−1, 1)
4. (5, −2)
5. (6, 9)
6. (20, 15)
7. (−8, 10)
8. (3, 8)
9. $\left\{\left(-\frac{1}{2}, 3\right)\right\}$
10. {(−15, 8)}
11. {(−2, 3)}
12. $\left\{\left(\frac{1}{3}, \frac{1}{2}\right)\right\}$
13. {(20, 15)}
14. {(66, 32)}
15. (5, −9)
16. (13, 7)
17. (30, 6)
18. (10, 16)
19. (5, 3)

Practice Exercise 7-11 (page 225)

1. (7, 5, 2)
2. (3, 2, 1)
3. (1, 0, −1)
4. (1, 2, 2)
5. $(5\frac{1}{2}, -6\frac{1}{2}, 8\frac{2}{3})$
6. (1, 3, 2)
7. (5, 6, 7)
8. (2, 3, 4)
9. (1, −2, 2)
10. (2, −1, 1)

Practice Exercise 7-12 (page 226)

1. 9, 14
2. \$1.20, \$.10
3. 60, 24
4. \$1,500, \$1,000
5. 60¢, 80¢
6. 100, 120
7. 30, 18
8. \$8, \$6
9. 15, 12
10. 42, 36
11. $6\frac{1}{2}$ m.p.h., $2\frac{1}{2}$ m.p.h.
12. 54

Discovery Exercise C-1 (page 234)

1.

	(i)	(ii)
(a)	geometries	in a plane
(b)	coins	10 cents
(c)	writing instruments	lead
(d)	relations	=
(e)	relations	≠
(f)	polynomials	one term

2.

	(i)	(ii)
(a)	point set	The point set consisting of two distinct points on a line and all the points between them is a line segment.
(b)	point set	The point set consisting of a point on a line and all the points on the line on one side of the given point is a ray.
(c)	point set	The point set consisting of all points on a line which are on one side of a given point on the line is a half-line.
(d)	point set	The set of points on two distinct rays with a common end point is an angle.
(e)	point set	The set of all points on the line segments determined by three points which are not all on one line is a triangle.

3. (a) (i) triangles (ii) two equal sides
   (iii) An isosceles triangle is a triangle which has two of its sides equal.
   (iv) A triangle having two of its sides equal is an isosceles triangle.
   (b) (i) triangles (ii) no sides equal
   (iii) A scalene triangle is a triangle, no two sides of which have the same length.
   (iv) A triangle, none of whose sides have the same length, is a scalene triangle.

(c) (i) triangles (ii) three sides equal
(iii) An equilateral triangle is a triangle all of whose sides have the same length.
(iv) A triangle, all of whose sides have the same length, is an equilateral triangle.

4. (i) $\{A\}$ (ii) $\varnothing$

5. (i) rays AB, AC, AD, AE (ii) half-lines AB, AC, AD, AE
(iii) $\angle BAC$, $\angle CAD$, $\angle DAE$, $\angle EAB$, $\angle BAD$, $\angle EAC$
(iv) (a) $\{A\}$ (b) $\varnothing$ (v) $\{l_1, l_2\}$
(vi) (a) $\{\text{ray } AB\}$ (b) $\{A\}$ (vii) (a) $\{\text{half-line } AD, l_2\}$ (b) $\{l_1, l_2\}$
(viii) the set of points on l_1 and l_2, except A

6. (i) $\{G, H\}$ (ii) $\{A, B\}$ (iii) $\{A, B\}$
(iv) $\{A\}$ (v) $\varnothing$ (vi) $\{A, B, E, D\}$ (vii) $\{A, B, C\}$

7. (i) the set of points on l; the set of points on one side of l; the set on the other side of l
(ii) A half-plane is the point set consisting of all points on one side of a given line in a given plane.

8. (i) $\{l\}$ (ii) $\varnothing$
(iii) Dihedral angle is the point set consisting of the union of a line and two non-co-planar half-planes having the line as a common edge.
(iv) 4 (v) $\{l\}$
(vi) $\{P_1, P_2\}$ (vii) (a) $\{l, H_1\}$ (b) $\{l\}$
(viii) (a) $\{P_2, H_2\}$ (b) $\{P_1, P_2\}$ (ix) $\varnothing$

Discovery Exercise C-2 (page 237)

1. (iii) any number of lines
(v) The members of this family are said to lie on A, intersect at A, or to be concurrent at A.
(vi) No. (vii) one, two.

3. (i) parallel

4. (i) AB, AC, AD, AE, BC, BD, BE, CD, CE, DE
(ii) A, B, C, D, E
(iii) Collinear points are points on the same straight line.
(iv) $\{A, B, C\}$, $\{A, G, D\}$, $\{A, F, E\}$, $\{C, G, F\}$, $\{C, D, E\}$, $\{E, G, B\}$
(v) AC, AD, AE; AC, FC, EC; AE, BE, CE; CE, DF, AD; AE, FD, FC; AD, CF, EB

5. (i) one (ii) 3 (iii) three collinear points

6. (i) uncountable (ii) the line (iv) one

9. (i) line AB (ii) AB (iii) ray AB, ray BA (iv) ray AB

10. (i) rays on AB, BA, AC, CA, BC, CB
(ii) rays on AB and AC; rays on BA and BC; rays on CA and CB
(iii) Ray on BA and the ray on AC do not have a common end point.
(iv) $\angle BAC$, $\angle ABC$, $\angle ACB$ (v) $\angle DEF$, $\angle EFD$, $\angle FDE$

Discovery Exercise C-3 (page 242)

1. (i) 3 (ii) $\frac{3}{4}$ (iii) 5
(iv) 6.1 (v) 2.725 (vi) 8.167
(vii) $x_1 - x_2$, if $x_1 > x_2$ or $x_2 - x_1$, if $x_2 > x_1$
(viii) $8a$, if $a > 0$ or $-8a$ if $a < 0$ (ix) 6

2. (i) $-1, 5$ (ii) No. (iii) Yes. (iv) one.

3. B; A is to the left of B and B is to the left of C.

4. Since $a > b$, M is to the right of N, $b > c$, P is to the left of N.

5. (i) 2, 2, 4, 5 is between 3 and 7.
(ii) 5, 8, 13, 5 is between 10 and -3.

6. (i) $MN < MP, NP < MP$ (ii) $NM < NP, MP < NP$
(iii) $MP < MN, PN < MN$

7. (i) $MN + NP = MP$ (ii) $NM + MP = NP$

8. (ii) A circle is a point set such that each member of the set is the same distance (radius) from a fixed point (centre).

9. (iii) A circle is closed, i.e., it has a finite length and no end points.

10. (i) 1, 11 (ii) Q is the midpoint of PR. (iii) No.
(v) If PR is a line segment and Q is a point on the segment. then Q is the midpoint if $PQ = QR$.
(vi) Bisect: to divide (separate) into two equal sets.

11. (i) an uncountable number (ii) at least one

12. (ii) If A is 7 miles from C, A is between B and C.

Discovery Exercise C-4 (page 245)

1. (i) completion postulate (ii) measurements of angles
(iii) $\angle ABC - \angle ABP = \angle PBC$, $\angle ABC - \angle PBC = \angle ABP$
(iv) an indefinite number

2. (i) one

3. (i) 180° (ii) (c), (e) (iii) 145°, 108°, 75°, 96°, 88°, 65°, 50°

4. (i) one (ii) 90° (iii) A right angle is an angle whose measurement is 90°.

5. (i) 90° (ii) (a), (b), (e) (iii) 20°, 28°, 36°, 48°, 18°, 54°, 72°

6. (i) The right bisector of a line segment is the line containing the midpoint of the segment such that the line and segment determine a right angle. (ii) one

7. (a) (i) triangles (ii) acute angles
(iii) An acute triangle is a triangle determining three acute angles.
(b) (i) triangles (ii) right angle
(iii) A right triangle is a triangle one angle of which has a measurement of 90°
(c) (i) triangles (ii) obtuse angle
(iii) An obtuse triangle is a triangle one of whose angles is an obtuse angle.

Exercise D-1 (page 252)

4. 180°

6. Draw any circle. Draw any two perpendicular diameters for this circle. Bisect the angles so formed and extend the bisectors to intersect the circle. Join the 8 points on the circle in order.

Exercise D-2 (page 255)

5. 3 in.

6. The circumscribed circle of a triangle is the circle which contains the three vertices of the triangle.

Exercise D-3 (page 256)

1. The three altitudes of any triangle are concurrent.

2. The three medians of any triangle are concurrent.

3. The bisectors of the angles of any triangle are concurrent.

4. The right bisectors of the sides of any triangle are concurrent.

5. Construct the right bisectors of two sides of a triangle. Their point of intersection is the centre of the circle which circumscribes the triangle. The radius of the circle is the distance from the centre to any of the vertices of the triangle.
6. The orthocentre, centroid, and circumcentre of any triangle lie on a straight line. This straight line is called the *Euler line* for the triangle.
7. The exterior angle of any triangle is equal to the sum of the two interior remote angles.
8. If a transversal intersects two parallel lines:
   (i) alternate angles are equal;
   (ii) corresponding angles are equal;
   (iii) pairs of interior angles on the same side of the transversal are supplementary.

Exercise 8-1 (page 261)

18. and **19.** or
20. if . . . then . . . **21.** not
22. and **23.** if . . . then . . .
24. if . . . then . . ., and **25.** not
26. if . . . then . . ., not **27.** if . . . then . . ., and
28. If $2x + 4 = 8$, then $x = 2$; $P \rightarrow Q$.
29. The rooms are warm, and the food is excellent; A and B.
30. It is not easy to bowl a perfect game; $\sim Q$.
31. $a > b$ or $a = b$ or $a < b$; P or Q or R. **32.** $x + 3 \neq 6$; $\sim S$
33. He turned around and looked at us; D and E.
34. disjunction **35.** conjunction **36.** disjunction **37.** conjunction **38.** implication
39. disjunction **40.** disjunction **41.** negation **42.** implication **43.** implication
44. If the team wins today, then the team will be in the finals.
45. If $x + 3 = 7$, then $x = 4$. **46.** If $x + b = a$, then $a - b = x$.
47. If $x \times b = a$, then $a \div b = x$. **48.** If a day is cloudy, then it is not warm.
49. If it rains, then the crops will grow. **50.** If $2x + 4 = 14$, then $x = 5$.
51. If an argument is good, then it is logical.
52. If an argument is logical, then it is a good argument.
53. If a person understands logic, then he will understand geometry.

Exercise 8-2 (page 264)

28. none **29.** $\sim B$ **30.** T **31.** none **32.** $\angle ADC = 90°$ **33.** none
34. $AB \parallel DC$ and $AD \parallel BC$ **35.** $\angle 1 = \angle 2$ **36.** none **37.** $AB = DC$
38. none **39.** none **40.** $l_1 \parallel l_3$ **41.** $l_1 \parallel l_2$ **42.** none
43. Two circles touch externally. **44.** $DE \parallel BC$

Exercise 8-5 (page 272)

18. valid **19.** not valid **20.** not valid **21.** not valid
22. valid **23.** A monkey has four legs. **24.** $\angle ABC$ is a straight angle.
25. $\angle DEF$ has a measurement less than 90°. **26.** none **27.** $\angle ABC = \angle DEF$
28. A single point is a geometric figure. **29.** A square is a rhombus.
30. $\dfrac{-2}{-3} + \dfrac{-4}{+5} = \dfrac{(-2)(+5) + (-3)(-4)}{(-3)(+5)}$

Review Exercise 9-3 (page 294)

5. $x = 21$, Vertical Angle Theorem
6. $y = 152$, Supplementary Angle Theorem

7. $x = 50$, $y = 80$, Complementary Angle Theorem
8. $a = 45$, Supplementary Angle Theorem
9. $a = 100$, Vertical Angle Theorem
10. $x = 90$, Supplementary Angle Theorem
11. $x = 90$, Supplementary Angle Theorem
12. $x = 50$, Vertical Angle Theorem $y = 130$, Supplementary Angle Theorem
$z = 130$, Vertical Angle Theorem

Exercise 11-1 (page 332)

11. If two triangles are congruent, then two sides and the contained angle of one triangle are respectively equal to two sides and the contained angle of the other triangle. True.
12. If two angles are equal, then they are the supplements of the same angle. True.
13. If a triangle has two unequal angles, then it has two unequal sides. True.
14. If an angle is less than a given angle, then it is part of the given angle. False.
15. No converse.
16. If the sum of two angles is 90°, then they are complementary angles. True.
17. *Hypothesis:* $BC = AC = AB$
Conclusion: ABC is a triangle with $\angle A = \angle B = \angle C$. True.
18. *Hypothesis:* $\angle AED = \angle CEB$
Conclusion: AB and CD are lines intersecting in E. False.

Exercise 12-2 (page 353)

5. (i) $\angle PAB$, $\angle QAC$, $\angle ABW$, $\angle TBC$, $\angle BCS$, $\angle ACR$
(ii) The remote interior angles of:
$\angle PAB$ are $\angle ACB$ and $\angle ABC$; $\angle QAC$ are $\angle ACB$ and $\angle ABC$;
$\angle ABW$ are $\angle BAC$ and $\angle ACB$; $\angle TBC$ are $\angle BAC$ and $\angle ACB$;
$\angle BCS$ are $\angle BAC$ and $\angle ABC$; $\angle ACR$ are $\angle BAC$ and $\angle ABC$.
(iii) $\angle ABW$, $\angle TBC$, $\angle BCS$, $\angle ACR$

Exercise 13-5 (page 377)

7. 45°, 135° **8.** 105°

Review Exercise 13-10 (page 389)

1. (b) **2.** (c) **3.** (d) **4.** (a) **5.** (d)
6. (a) **7.** (d) **8.** (d) **9.** (c) **10.** (a)

Exercise 13-11 (page 390)

1. $2\frac{3}{8}$ in. **2.** $5\frac{1}{8}$ in., $3\frac{3}{8}$ in., $3\frac{1}{4}$ in. **3.** $\frac{7}{8}$ in.
4. $1\frac{5}{8}$ in. **5.** $7\frac{7}{16}$ in. **6.** $3\frac{1}{4}$ in.
7. 5.2 cm., 11.4 sq. cm. **8.** 12.4 cm., 4.0 cm., 2.8 cm., 8.7 cm. **9.** 5.3 cm.

Exercise 14-4 (page 405)

3. 12,000 sq. ft. **4.** 2,400 sq. cm. **7.** 80 sq. cm. **8.** 700 sq. in.
9. (i) mn sq. in. (ii) The area is doubled.
(iii) The area is tripled. (iv) The area is four times as great.

Exercise 14-5 (page 408)

4. 96 sq. in. **5.** 225 sq. in. **6.** 141 sq. units **7.** 24 sq. cm.

Exercise 14-6 (page 415)

4. 15 ft. **5.** 60 rods **6.** 29 rods
7. 20 ft. **8.** 75 miles **9.** 63 in.
10. 100 cm. **16.** $x = \dfrac{b^2 + a^2 - c^2}{2a}$ **17.** $x = \dfrac{c^2 - b^2 - a^2}{2a}$
23. (i) 26 ft. (ii) 27.5 ft.
24. (i) 10 ft. (ii) 24 ft. (iii) 24.6 ft.

Exercise 14-7 (page 420)

11. $\angle ABC > 90°$

Exercise 14-8 (page 422)

5. 850 sq. ft. **6.** 25,600 sq. ft. **7.** 62 sq. units
8. 17 sq. units **9.** 56 sq. in. **10.** 1.4 acres
11. 28 sq. in.

Exercise 14-9 (page 424)

1. 12 cm. **2.** 240 sq. units **3.** 14 sq. in.
4. (i) 5 sq. cm. (ii) 10 sq. cm. **5.** 24 sq. in.
6. 100 sq. in. **7.** 30 sq. units **8.** 8 in.
9. 32 sq. in. **10.** 18 sq. in. **11.** (i) 404 sq. in. (ii) 160 sq. in.

Exercise 15-1 (page 428)

3. $5 : 7 : 9 = \dfrac{10}{3} : \dfrac{14}{3} : 6, k = \dfrac{2}{3}$; $12 : 18 : 24 = 2 : 3 : 4, k = \dfrac{1}{6}$
4. 4 : 5 **5.** 18 inches, 12 inches **6.** 5 : 3
7. (i) 2 : 3 (ii) 4 : 9

Exercise 15-2 (page 430)

14. 30 **15.** (i) 7 (ii) y^2 (iii) $x^2 - y^2$ (iv) $\dfrac{a+b}{a}$
16. ± 10 **17.** (i) ± 12 (ii) $\pm \dfrac{3}{2}$ (iii) ± 1 (iv) $\pm ar$
(v) $\pm(x^2 - y^2)$ **18.** 4
19. (i) 27 (ii) $\dfrac{4}{3}$ (iii) ar^2 (iv) y^2
20. (i) 7 : 5 (ii) 3 : 2 (iii) 2 : 1 (iv) $b : a$
(v) $\dfrac{12}{5}, \dfrac{5}{2}, \dfrac{3}{1}, \dfrac{b+a}{a}$ (vi) $\dfrac{2}{5}, \dfrac{1}{2}, \dfrac{1}{1}, \dfrac{b-a}{a}$
21. (i) $\dfrac{3}{2}$ (ii) $\dfrac{5}{3}$ (iii) $\dfrac{2}{5}$
22. 42 cm. **23.** (i) $\dfrac{2}{5}$ (ii) $\dfrac{3}{5}$
24. (i) $\dfrac{1}{5}$ (ii) $\dfrac{4}{5}$ **25.** (i) $\dfrac{5}{7}$ (ii) $\dfrac{2}{7}$

26. (i) $\frac{m}{m+n}$ (ii) $\frac{n}{m+n}$ **27.** (i) $\frac{2m}{2m+3s}$ (ii) $\frac{3s}{2m+3s}$

28. (i) $\frac{a+b}{2a}$ (ii) $\frac{a-b}{2a}$

Test Papers 1-4

Test Paper 1 (page 456)

2. 90° **3.** 70° **4.** 100° **8.** $1\frac{1}{2}$ in.

Test Paper 2 (page 457)

2. 180° **3.** 2 cm. **4.** 135° **5.** 40°

Test Paper 3 (page 457)

2. 6 cm. **3.** 55°, 55° **4.** 90° **5.** 115° **9.** $5\frac{3}{4}$ in.

Test Paper 4 (page 458)

2. 80° **3.** 2 cm., 3 cm., 6 cm. **4.** 65°

5. $(165 - x)°$

Test Papers 5-8

Test Paper 5 (page 459)

2. 9.6 cm. **3.** 8 in., 8 in., $8\sqrt{2}$ in. **4.** 4 cm.

5. 12 sq. in. **7.** 5° **9.** 9.4 cm.

Test Paper 6 (page 459)

2. 24 ft. **3.** 52 sq. cm. **4.** $12\sqrt{2}$ in.

5. 7.5 sq. cm.

Test Paper 7 (page 460)

2. 3 sq. cm., 4 sq. cm. **3.** 11.0 cm. **4.** 15 sq. cm.

5. 15 sq. in. **9.** 180°

Test Paper 8 (page 461)

2. 40 sq. cm. **3.** 50 cm. **4.** 17 cm.

5. 48 sq. cm.

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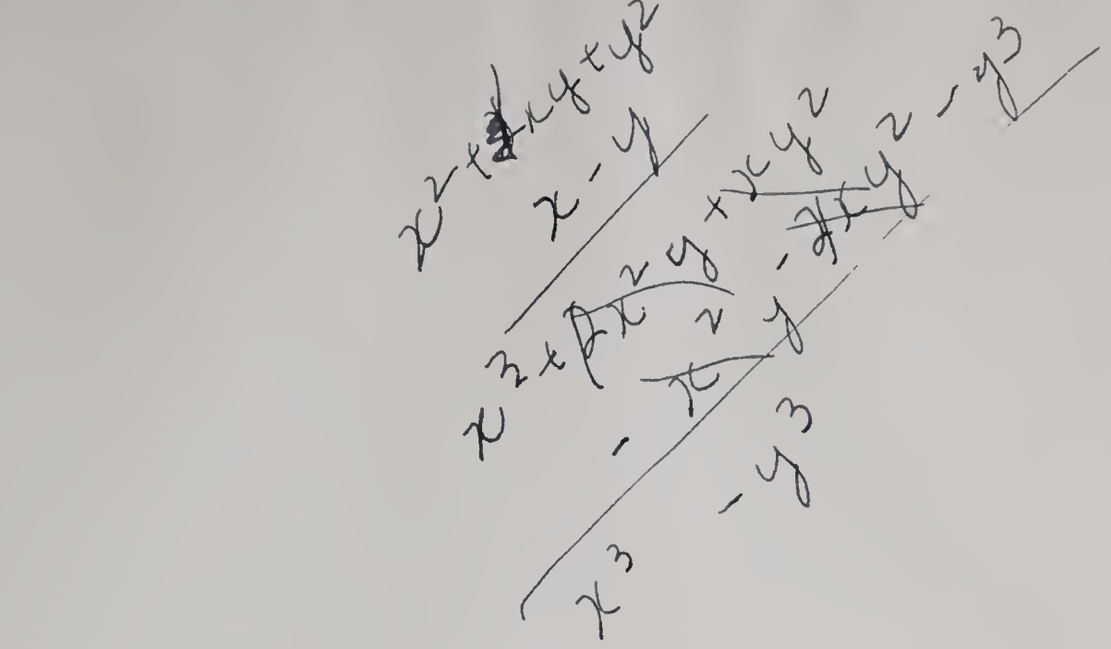

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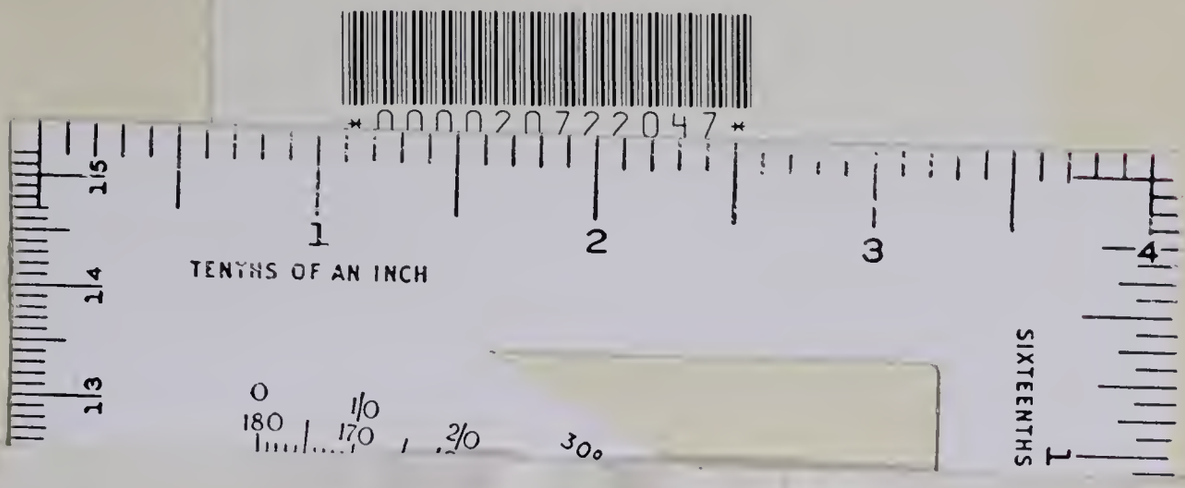

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